Introduction to Straight Skeletons
Spatial Decompositions and Graphs (VORONOI)

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Introduction

Straight Skeletons of Simple Polygons were introduced by Aichholzer, Aurenhammer, Alberts, Gaertner in the year 1995. [AAAG95]

- Defined by *shrinking process*
- Each edge moves inwards at the same speed

**Figure**: Straight Skeleton [AAAG95]

**Events**
- edge event
- split event

**Complexity**
- $n$ faces
- $n - 2$ nodes
- $2n - 3$ arcs
Notice

- Straight Skeleton is unique for a given polygon
- Its structure may be interpreted as plane tree
- All structural combinations in width and height of the tree possible
- *Bisector graph* is **not** unique

**Figure:** Bisector graph [AAAG95]
Construction

- Straight Skeleton is interpreted as 2-dim. projection of a 3-dim. roof model
- A horizontal plane $\Pi$ moves upwards
- Events are stored in a priority queue
- Priority reflects the height of $\Pi$

Figure: Roof model [AAAG95]
Generalized Straight Skeletons for General Polygonal Figures in the Plane were introduced by Aichholzer and Aurenhammer in the year 1996. [AA96]

**Complexity**

$2n + t - 2$ nodes

Roof model gives landscape (rivers, coasts, ...)

**Figure:** Generalized Straight Skeleton [AA96]
Construction

- Initial wavefronts are generated (duplicating vertices)
- Area is triangulated
- Area where the wavefront already swept over is not triangulated
- During wavefront propagation flip, edge and split events occur
- Priority queue for events (& triangles)
- Proven upper bound for flip events: $O(n^3)$, likely to be $O(n^2)$
Eppstein and Erickson extracted the main problem: The Motorcycle Graph [EE99]

- Each reflex vertex emanates a motorcycle
- Speed of the motorcycle determined by angle
- Used to calculate priorities of reflex vertices

Figure: Motorcycle Graph [CV07]
Special Cases

Figures found on Jeff Erickson’s homepage\(^1\)

\(^1\)http://theory.cs.uiuc.edu/~jeffe/open/cycles.html
Known lower and upper bounds

- The Straight Skeleton of a convex polygon is equal to its Medial Axis. It can be constructed in $\Theta(n)$ time. [AGSS87]
- Lower bound for polygons with holes: $\Omega(n \log n)$

Figure: Lower bound for polygons with holes [Hub11]
## Overview of existing algorithms

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Time</th>
<th>Space</th>
<th>Complexity²</th>
</tr>
</thead>
<tbody>
<tr>
<td>[AAAG95]</td>
<td>$O(nr \log n)$</td>
<td>$O(n)$</td>
<td>+</td>
</tr>
<tr>
<td>[AA96]</td>
<td>$O(n^3 \log n)$ (pract. $O(n \log n)$)</td>
<td>$O(n)$</td>
<td>++</td>
</tr>
<tr>
<td>[EE99]</td>
<td>$O(n^{1+\varepsilon} + n^{8/11+\varepsilon} r^{9/11+\varepsilon})$</td>
<td>ditto</td>
<td>++++++++</td>
</tr>
<tr>
<td>[CV07]</td>
<td>exp. $O(n^{\sqrt{h} + 1} \log^2 n + r^{\sqrt{r}} \log r)$</td>
<td>$O(n)$</td>
<td>++++</td>
</tr>
</tbody>
</table>

$n$ ... total number of vertices  
$r$ ... number of reflex vertices  
$h$ ... number of holes

²suitable for implementation
Structure of Straight Skeletons

Vyatkina separated the sticking events from the split events. [Vya08]

Sticking event

- Does not divide the polygon into 2 parts
- No priority calculation of reflex vertices necessary

Refined events

- sticking event
- vertex event

Figure: Structure of Straight Skeleton [Vya08]

- Sticking events only:
  Construction in $O(n \log n)$ time possible
Monotone polygons

G. Das et al. showed that the straight skeleton of a monotone polygon can be computed in $O(n \log n)$ time [DMN$^+$10].

- Decompose the polygon into a right and a left monotone chains
- The reflex vertex and the edge being split always belongs to different monotone chains
- The ordering of vertices stays the same during the shrinking process
- No priority calculation of reflex vertices necessary. (No Motorcycle Graph needs to be solved.)
Comparison to Medial Axis

M. Tanase et al. wrote a paper titled “Straight Line Skeleton in Linear Time, Topologically Equivalent to the Medial Axis” [TV04].

- In case the medial axis of a polygon is topologically equivalent to the straight skeleton, it can be computed in $O(n)$ time.
- No sharp reflex vertices may occur.

Figure: Approximation [TV04]
Known implementations

Computational Geometry Algorithms Library (CGAL)\(^3\)

- Open source
- Can not handle general polygonal figures
- \(O(n^2 \log n)\) time and \(O(n^2)\) space [Hub11]

Figure: CGAL’s Sample 3

Huber’s BONE [Hub11]

- \(O(n^2 \log n)\) time and \(O(n)\) space
- In practice: \(O(n \log n)\) time

\(^3\)http://www.cgal.org/
Conclusion

- Straight Skeleton is a challenging task
- with hidden traps
- General setting hard to attack
- Promising special cases

Planned Work

- Investigate special cases
  - Some structure, if only sticking events?
  - If only obtuse-angled reflex vertices are allowed?
  - Possible to concatenate monotone polygons?
- Implementation
  - Create a fast and less complex (suitable for implementation) algorithm
  - Implement this algorithm for demonstrational purposes
- Investigate Straight Skeleton in $\mathbb{R}^3$
Oswin Aichholzer and Franz Aurenhammer. 
Straight skeletons for general polygonal figures in the plane.

Oswin Aichholzer, Franz Aurenhammer, David Alberts, and Bernd Gärtner. 
Straight skeletons of simple polygons.

Alok Aggarwal, Leonidas Guibas, James Saxe, and Peter Shor. 
A linear time algorithm for computing the voronoi diagram of a convex polygon.

Siu-Wing Cheng and Antoine Vigneron. 
Motorcycle graphs and straight skeletons.

Computing the straight skeleton of a monotone polygon in $O(n \log n)$ time.
References II

David Eppstein and Jeff Erickson.
Raising roofs, crashing cycles, and playing pool: Applications of a data structure for finding pairwise interactions.

Stefan Huber.
*Computing Straight Skeletons and Motorcycle Graphs: Theory and Practice.*
PhD thesis, Faculty of Natural Sciences, University of Salzburg, June 2011.

Mirela Tanase and Remco C. Veltkamp.
Straight line skeleton in linear time, topologically equivalent to the medial axis.

Kira Vyatkina.
On the structure of straight skeletons.