Introduction to Straight Skeletons

Spatial Decompositions and Graphs (VORONOI)

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Contents

Introduction

Straight Skeletons of Simple Polygons were introduced by Aichholzer, Aurenhammer, Alberts, Gaertner in the year 1995. [AAAG95]

- Defined by *shrinking process*
- Each edge moves inwards at the same speed

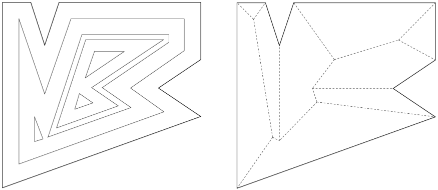


Figure: Straight Skeleton [AAAG95]

Events

- edge event
- split event

Complexity

- n faces
- $n - 2$ nodes
- $2n - 3$ arcs

Notice

- Straight Skeleton is unique for a given polygon
- Its structure may be interpreted as plane tree
- All structural combinations in width and height of the tree possible
- *Bisector graph* is **not** unique

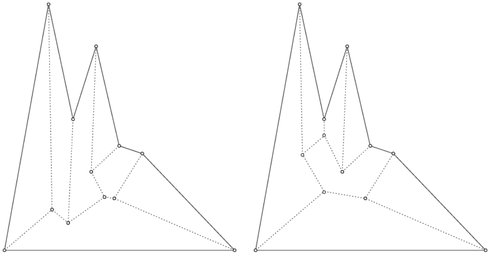


Figure: Bisector graph [AAAG95]

Construction

- Straight Skeleton is interpreted as 2-dim. projection of a 3-dim. roof model
- A horizontal plane Π moves upwards
- Events are stored in a priority queue
- Priority reflects the height of Π

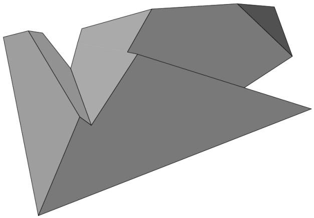


Figure: Roof model [AAAG95]

Generalized

Straight Skeletons for General Polygonal Figures in the Plane were introduced by Aichholzer and Aurenhammer in the year 1996. [AA96]

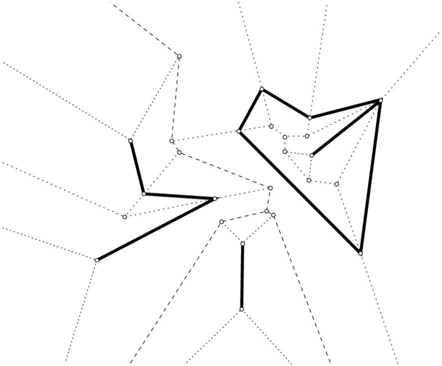


Figure: Generalized Straight Skeleton [AA96]

Complexity

- $2n + t - 2$ nodes

Roof model gives landscape (rivers, coasts, ...)

Construction

- Initial wavefronts are generated (duplicating vertices)
- Area is triangulated
- Area where the wavefront already swept over is not triangulated
- During wavefront propagation
  flip, edge and split events occur
- Priority queue for events (& triangles)
- Proven upper bound for flip events: $O(n^3)$, likely to be $O(n^2)$

Motorcycle Graph

Eppstein and Erickson extracted the main problem:
The Motorcycle Graph [EE99]

- Each reflex vertex emanates a motorcycle
- Speed of the motorcycle determined by angle
- Used to calculate priorities of reflex vertices

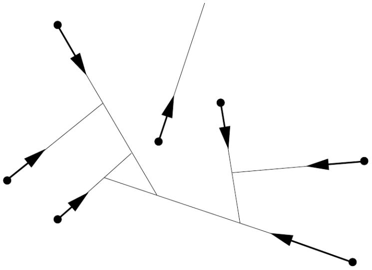
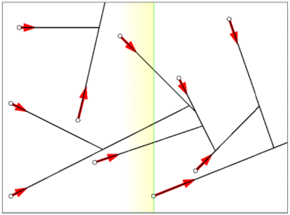
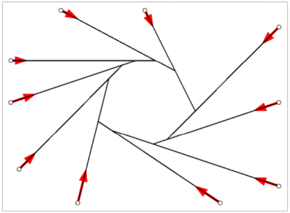


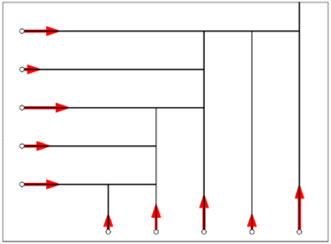
Figure: Motorcycle Graph [CV07]

Special Cases

Figures found on Jeff Erickson's homepage[1]







[1] http://theory.cs.uiuc.edu/~jeffe/open/cycles.html

Known lower and upper bounds

- The Straight Skeleton of a convex polygon is equal to its Medial Axis. It can be constructed in $\Theta(n)$ time. [AGSS87]
- Lower bound for polygons with holes: $\Omega(n \log n)$

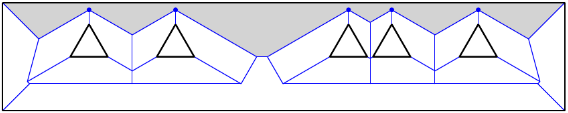


Figure: Lower bound for polygons with holes [Hub11]

Overview of existing algorithms

Algorithm	Time	Space	Complexity[2]
[AAAG95]	$O(nr \log n)$	$O(n)$	+
[AA96]	$O(n^3 \log n)$ (pract. $O(n \log n)$)	$O(n)$	++
[EE99]	$O(n^{1+\varepsilon} + n^{8/11+\varepsilon} r^{9/11+\varepsilon})$	ditto	++++++
[CV07]	exp. $O(n\sqrt{h+1} \log^2 n + r\sqrt{r} \log r)$	$O(n)$	+++++

n … total number of vertices

r … number of reflex vertices

h … number of holes

[2]suitable for implementation

Structure of Straight Skeletons

Vyatkina separated the sticking events from the split events. [Vya08]

Sticking event

- Does not divide the polygon into 2 parts
- No priority calculation of reflex vertices necessary

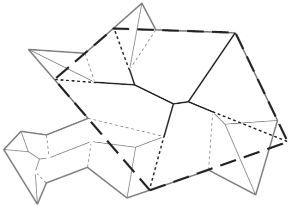


Figure: Structure of Straight Skeleton [Vya08]

Refined events

- sticking event
- vertex event

- Sticking events only:
  Construction in $O(n \log n)$ time possible

Monotone polygons

G. Das et al. showed that the straight skeleton of a monotone polygon can be computed in $O(n \log n)$ time [DMN+10].

- Decompose the polygon into a right and a left monotone chains
- The reflex vertex and the edge being split always belongs to different monotone chains
- The ordering of vertices stays the same during the shrinking process
- No priority calculation of reflex vertices necessary. (No Motorcycle Graph needs to be solved.)

Comparison to Medial Axis

M. Tanase et al. wrote a paper titled "Straight Line Skeleton in Linear Time, Topologically Equivalent to the Medial Axis" [TV04].

- In case the medial axis of a polygon is topologically equivalent to the straight skeleton, it can be computed in $O(n)$ time.
- No sharp reflex vertices may occur

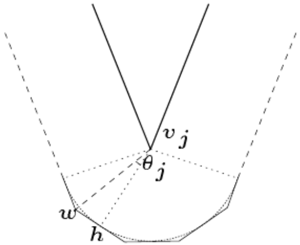


Figure: Approximation [TV04]

Known implementations

Computational Geometry Algorithms Library (CGAL)[3]

- Open source
- Can not handle general polygonal figures
- $O(n^2 \log n)$ time and $O(n^2)$ space [Hub11]

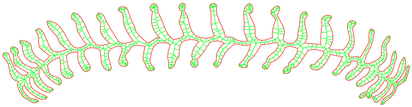


Figure: CGAL's Sample 3

Huber's BONE [Hub11]

- $O(n^2 \log n)$ time and $O(n)$ space
- In practice: $O(n \log n)$ time

[3] http://www.cgal.org/

Conclusion

- Straight Skeleton is a challenging task
- with hidden traps
- General setting hard to attack
- Promising special cases

Planned Work

- Investigate special cases
  - Some structure, if only sticking events?
  - If only obtuse-angled reflex vertices are allowed?
  - Possible to concatenate monotone polygons?
- Implementation
  - Create a fast and less complex (suitable for implementation) algorithm
  - Implement this algorithm for demonstrational purposes
- Investigate Straight Skeleton in $\mathbb{R}^3$

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