Shellable drawings
and the crossing number of the complete graph

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Introduction

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  * all crossings are **proper** (no tangents).

$$cr(K_5) = 1$$
The crossing number of a graph

Finding the crossing number of a graph is hard:

- Computing $cr(G)$ is NP-hard.
The crossing number of a graph

Finding the crossing number of a graph is hard:

- Computing \( cr(G) \) is NP-hard.

- If we add a single edge \( e \) to a plane graph \( G \), computing \( cr(G \cup \{e\}) \) is also NP-hard.

[Cabello-Mohar, 2010]
A brief history of $\text{cr}(K_n)$

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The number of crossings in these drawings is $Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n - 1}{2} \right\rfloor \left\lfloor \frac{n - 2}{2} \right\rfloor \left\lfloor \frac{n - 3}{2} \right\rfloor$

Zarankiewicz number
A brief history of $\text{cr}(K_n)$

* Conjecture [Harary-Hill (1963), Guy (1962)]: $\text{cr}(K_n) = \mathbb{Z}_n$. 
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* Some known results for small \( n \):
  
  - \( cr(K_n) = Z(n) \) si \( n \leq 10 \) [Guy, 1971]
  
  - \( n = 11, n = 12 \) [Pan-Richter, 2007]
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* Assymptotics:

  $\text{cr}(K_n) \geq 0.8594 Z(n)$ [de Klerk-Pasechik-Schrijver, 2007]
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* Some known results for small $n$:
  
  $\diamond$ $\text{cr}(K_n) = Z(n)$ si $n \leq 10$  [Guy, 1971]

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* Assymptotics:

  $\text{cr}(K_n) \geq 0.8594 Z(n)$  [de Klerk-Pasechik-Schrijver, 2007]

* This was the situation, till a new tool was borrowed from the rectilinear case.
Rectilinear crossing number

* The rectilinear crossing number of $G$, $\text{cr}(G)$, is the smallest number of crossings in drawings of $G$ in which edges are segments. (Vertices in general position).
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* For $\text{cr}(K_n)$, there is an equivalent formulation: $\square(S)$: number of convex quadrilaterals in $S$.

$$\text{cr}(K_n) = \min_{|S|=n} \square(S)$$
Rectilinear crossing number

* The rectilinear crossing number of $G$, $\overline{cr}(G)$, is the smallest number of crossings in drawings of $G$ in which edges are segments. (Vertices in general position).

* For $\overline{cr}(K_n)$, there is an equivalent formulation: $\square(S)$: number of convex quadrilaterals in $S$.

\[ \overline{cr}(K_n) = \min_{|S|=n} \square(S) \]
Rectilinear crossing number

* Until 2004, the status of the rectilinear problem was similar to that of the general case.
  * small values of $n$
  * far from tight asymptotics
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  * far from tight asymptotics


Relation between $\square(S')$ and the number of $j$-edges of $S$. 
$j$-edges

* Let $S$ be a set of $n$ points in the plane in general position. Given $p, q \in S$, we say that $pq$ is an (oriented) $j$-edge if there are $j$ points of $S$ in the right half-plane defined by $pq$. 
$j$-edges

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3-edge
\( j \)-edges

* Let \( S \) be a set of \( n \) points in the plane in general position. Given \( p, q \in S \), we say that \( pq \) is an (oriented) \( j \)-edge if there are \( j \) points of \( S \) in the right half-plane defined by \( pq \).

* \( e_j(S) := \# j \)-edges of \( S \).
**j-edges**

* Let $S$ be a set of $n$ points in the plane in general position. Given $p, q \in S$, we say that $pq$ is an (oriented) $j$-edge if there are $j$ points of $S$ in the right half-plane defined by $pq$.

![Diagram of j-edge](image)

* $e_j(S) := \# j$-edges of $S$.

* If $pq$ is a $j$-edge, then $qp$ is a $n - j - 2$-edge.

It is also possible to work with unoriented $j$-edges.
$j$-edges and convex quadrilaterals (crossings)

\[ \Delta(S') + \square(S') = \binom{n}{4} \]  \hspace{1cm} (1)
$j$-edges and convex quadrilaterals (crossings)

* $\Delta(S) + \Box(S) = \binom{n}{4}$ (1)

* Another relation: double counting of separations. A separation is a 4-tuple $\{p, q, u, v\}$ where the ordered pair $p, q$ leaves $u$ to the right and $v$ to the left.
$j$-edges and convex quadrilaterals (crossings)

* $\triangle(S) + \square(S) = \binom{n}{4}$ (1)

* Another relation: double counting of separations. A separation is a 4-tuple \( \{p, q, u, v\} \) where the ordered pair \( p, q \) leaves \( u \) to the right and \( v \) to the left.

\[ \text{six separations} \]
\( j \)-edges and convex quadrilaterals (crossings)

* \( \Delta(S) + \Box(S) = \binom{n}{4} \) \hspace{2cm} (1)

* Another relation: double counting of separations. A separation is a 4-tuple \( \{p, q, u, v\} \) where the ordered pair \( p, q \) leaves \( u \) to the right and \( v \) to the left.

\[ \text{six separations} \quad \text{four separations} \]
$j$-edges and convex quadrilaterals (crossings)

* $\Delta(S) + \Box(S) = \binom{n}{4}$ \hspace{1cm} (1)

* Another relation: double counting of separations. A separation is a 4-tuple \{p, q, u, v\} where the ordered pair $p, q$ leaves $u$ to the right and $v$ to the left.

\begin{align*}
\sum_{j=0}^{n-2} j(n - j - 2) e_j(S) \hspace{1cm} (2)
\end{align*}
\( j \)-edges and convex quadrilaterals (crossings)

* From these equations (and the relations \( e_j = e_{n-j-2} \) and \( \sum_{j=0}^{n-2} e_j = n(n - 1) \)) we get

\[
\square(S) = \sum_{j<\frac{n-2}{2}} \left( \frac{n-2}{2} - j \right)^2 e_j(S) - \frac{3}{4} \binom{n}{3}
\]
* From these equations (and the relations $e_j = e_{n-j-2}$ and $\sum_{j=0}^{n-2} e_j = n(n-1)$) we get

$$\square(S) = \sum_{j < \frac{n-2}{2}} \left( \frac{n-2}{2} - j \right)^2 e_j(S) - \frac{3}{4} \binom{n}{3}$$

* Considering $E_{\leq k}(S) = \sum_{j=0}^{k} e_j(S)$

$$\square(S) = \sum_{k < \frac{n-2}{2}} (n - 2k - 3) E_{\leq k}(S) + c_n$$
\( j \)-edges and convex quadrilaterals (crossings)

* From these equations (and the relations \( e_j = e_{n-j-2} \) and \( \sum_{j=0}^{n-2} e_j = n(n-1) \)) we get

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* Considering \( E_{\leq k}(S') = \sum_{j=0}^{k} e_j(S') \)

\[
\square(S) = \sum_{k < \frac{n-2}{2}} (n - 2k - 3) E_{\leq k}(S') + c_n
\]
Lower bounds for $\overline{cr}(K_n)$

* [AF - LVWW, 2004] \[ E_{\leq k}(S) \geq 3 \binom{k + 2}{2} \]
Lower bounds for $\overline{\text{cr}}(K_n)$

* [AF - LVWW, 2004]  

\[
E_{\leq k}(S) \geq 3 \binom{k + 2}{2} \\
\downarrow \\
\overline{\text{cr}}(K_n) \geq 0.375 \binom{n}{4} \approx Z(n)
\]
Lower bounds for $\overline{\text{cr}}(K_n)$

* [AF - LVWW, 2004] \[ E_{\leq k}(S) \geq 3 \binom{k + 2}{2} \]
  \[ \Downarrow \]
  \[ \overline{\text{cr}}(K_n) \geq 0.375 \binom{n}{4} \approx Z(n) \]

* LVWW use an improved bound for $E_{\leq k}$ (for $k$ close to $n/2$), to show that
  \[ \overline{\text{cr}}(K_n) \geq 0.37501 \binom{n}{4} \]
Bounds for $\overline{cr}(K_n)$

* 2006 – 2010  Series of improvements on the lower bound for $E_{\leq k}(S)$. (And on the lower bound for $\overline{cr}(K_n)$)

  ★ [Balogh-Salazar’06]
  ★ [Aichholzer-García-Orden-R.’07]
  ★ [Ábrego,Cetina,Fernández-Merchant,Leaños,Salazar’11].

Current bounds:

$$0.37968 \binom{n}{4} + O(n^3) \leq \overline{cr}(K_n) \leq 0.380488 \binom{n}{4} + O(n^3)$$
Bounds for $\overline{cr}(K_n)$

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  * [Balogh-Salazar’06]
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[Aurenhammer-Aichholzer-Krasser]
[Ábrego,Cetina,Fernández-Merchant,Leaños,Salazar]
General (topological) drawings

* BIRS - Crossing numbers turn useful. (August 2011)

If in the formula

\[ \square(S') = \sum_{k < \frac{n-2}{2}} (n - 2k - 3) E_{\leq k}(S) + c_n \]

we write \(3 \binom{k+2}{2}\) in the place of \(E_{\leq k}(S')\) we get
General (topological) drawings

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If in the formula

\[ \Box(S) = \sum_{k < \frac{n-2}{2}} (n - 2k - 3) E \leq_k (S) + c_n \]

we write \(3\binom{k+2}{2}\) in the place of \(E \leq_k (S)\) we get

\[ \sum_{k < \frac{n-2}{2}} (n - 2k - 3) 3\binom{k+2}{2} + c_n = \frac{1}{4} \floor{\frac{n}{2}} \floor{\frac{n-1}{2}} \floor{\frac{n-2}{2}} \floor{\frac{n-3}{2}} \]
General (topological) drawings

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If in the formula

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we write \[ 3\binom{k + 2}{2} \] in the place of \( E_{\leq k}(S) \) we get

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\sum_{k < \frac{n-2}{2}} (n-2k-3) 3\binom{k + 2}{2} + c_n = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor \]

\[ Z(n) \]
General (topological) drawings

* BIRS - Crossing numbers turn useful. (August 2011)

If in the formula

$$\square(S) = \sum_{k < \frac{n-2}{2}} (n - 2k - 3) E_{\leq k}(S) + c_n$$

we write $3\left(\frac{k}{2} + 2\right)$ in the place of $E_{\leq k}(S)$ we get

$$\sum_{k < \frac{n-2}{2}} (n - 2k - 3) 3\left(\frac{k}{2} + 2\right) + c_n = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor \parallel Z(n)$$

* Is that a coincidence?
$j$-edges in topological drawings
$j$-edges in topological drawings

Consider the triangles!

$$\sigma(pqr) = +$$
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\[ \sigma(pqr) = + \]
\[ \sigma(pqs) = - \]
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Let \( D \) be a good drawing of \( K_n \). We say that \( r \) is to the right of \( pq \) if \( pqr \) is oriented clockwise.
Consider the triangles!

\[ \sigma(pqr) = + \]

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* Let \( D \) be a good drawing of \( K_n \). We say that \( r \) is to the right of \( pq \) if \( pqr \) is oriented clockwise.
\(j\)-edges in topological drawings

Consider the triangles!

\[
\sigma(pqr) = +
\]
\[
\sigma(pqs) = -
\]

* Let \(D\) be a good drawing of \(K_n\). We say that \(r\) is to the right of \(pq\) if \(pqr\) is oriented clockwise.

* And now we can define \(j\)-edges exactly as in the geometric setting.
$j$-edges and crossings (in topological drawings)

* Now we need to generalize the relation

$$6 \Delta(S') + 4 \square(S') = \sum_{j=0}^{n-2} j(n - j - 2) e_j(S')$$
$j$-edges and crossings (in topological drawings)

* Now we need to generalize the relation

$$6 \Delta(S) + 4 \Box(S) = \sum_{j=0}^{n-2} j(n - j - 2) e_j(S)$$

* In a good drawing of $K_4$ there is at most one crossing.
\(j\)-edges and crossings (in topological drawings)

* Now we need to generalize the relation

\[
6 \Delta(S) + 4 \Box(S) = \sum_{j=0}^{n-2} j(n - j - 2)e_j(S)
\]

* In a good drawing of \(K_4\) there is at most one crossing.
$j$-edges and crossings (in topological drawings)

* There are three “different” drawings of $K_4$. 

\begin{figure}
\centering
\begin{tikzpicture}
\tikzstyle{vertex}=[circle, fill=black, inner sep=1pt]
\tikzstyle{edge}=[thick, blue]
\node[vertex] (v1) at (0,0) [label=1:1]{};
\node[vertex] (v2) at (3,0) [label=2:2]{};
\node[vertex] (v3) at (2,2) [label=3:3]{};
\node[vertex] (v4) at (0,3) [label=4:4]{};
\draw[edge] (v1) -- (v3);
\draw[edge] (v1) -- (v4);
\draw[edge] (v2) -- (v3);
\draw[edge] (v2) -- (v4);
\end{tikzpicture}
\end{figure}
$j$-edges and crossings (in topological drawings)

* There are three “different” drawings of $K_4$. 

\begin{align*}
\text{1} & \quad 2 & \quad 3 & \quad 4 \\
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\end{align*}
$j$-edges and crossings (in topological drawings)

* There are three “different” drawings of $K_4$. 

![Diagram of $K_4$ drawings](image-url)
$j$-edges and crossings (in topological drawings)

* The relation between $j$-edges and crossings is the same as in the rectilinear case.
\( j \)-edges and crossings (in topological drawings)

* The relation between \( j \)-edges and crossings is \textbf{the same} as in the rectilinear case.
\textbf{$j$-edges and crossings (in topological drawings)}

\begin{itemize}
  \item The relation between $j$-edges and crossings is \textit{the same} as in the rectilinear case.
\end{itemize}

\begin{figure}
\centering
\begin{subfigure}{0.3\textwidth}
\centering
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (2,0) {2};
  \node (3) at (1,1) {3};
  \node (4) at (1,-1) {4};
  \draw (1) -- (2);
  \draw (1) -- (3);
  \draw (1) -- (4);
  \draw (2) -- (3);
  \draw (2) -- (4);
  \node at (1.5,0) {$C_A$};
\end{tikzpicture}
\end{subfigure}
\begin{subfigure}{0.3\textwidth}
\centering
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (2,0) {2};
  \node (3) at (1,1) {3};
  \node (4) at (1,-1) {4};
  \draw (1) -- (3);
  \draw (1) -- (4);
  \draw (2) -- (3);
  \draw (2) -- (4);
  \node at (1.5,0) {6 separations};
\end{tikzpicture}
\end{subfigure}
\begin{subfigure}{0.3\textwidth}
\centering
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (2,0) {2};
  \node (3) at (1,1) {3};
  \node (4) at (1,-1) {4};
  \draw (1) -- (3);
  \draw (1) -- (4);
  \draw (2) -- (3);
  \draw (2) -- (4);
  \node at (1.5,0) {no crossing};
\end{tikzpicture}
\end{subfigure}
\end{figure}
\textit{j}-edges and crossings (in topological drawings)

* The relation between \textit{j}-edges and crossings is \textit{the same} as in the rectilinear case.

\begin{itemize}
\item \(C_A\)
\end{itemize}
$j$-edges and crossings (in topological drawings)

* The relation between $j$-edges and crossings is the same as in the rectilinear case.

\[ C_A \]

\[ C_B \]

no crossing

6 separations
$j$-edges and crossings (in topological drawings)

* The relation between $j$-edges and crossings is the same as in the rectilinear case.

\[ C_A \]

\[ C_B \]

no crossing
6 separations

one crossing $\rightarrow$ 4 separations
$j$-edges and crossings (in topological drawings)

So we have:

1. $|C_B| = \text{cr}(D)$
2. $|C_A| + |C_B| = \binom{n}{4}$
3. $6|C_A| + 4|C_B| = \sum_{j=0}^{n-2} j(n - j - 2) e_j(D)$
$j$-edges and crossings (in topological drawings)

* So we have:

1. $|C_B| = \text{cr}(D)$
2. $|C_A| + |C_B| = \binom{n}{4}$

3. $6|C_A| + 4|C_B| = \sum_{j=0}^{n-2} j(n - j - 2) e_j(D)$

* $\text{cr}(D) = \sum_{j < \frac{n-2}{2}} \left( \frac{n - 2}{2} - j \right)^2 e_j(D) - \frac{3}{4} \binom{n}{3}$
$j$-edges and crossings (in topological drawings)

* So we have:

1. $|C_B| = \text{cr}(D)$
2. $|C_A| + |C_B| = \binom{n}{4}$
3. $6|C_A| + 4|C_B| = \sum_{j=0}^{n-2} j(n - j - 2) e_j(D)$

* $\text{cr}(D) = \sum_{j < \frac{n-2}{2}} \left( \frac{n - 2}{2} - j \right)^2 e_j(D) - \frac{3}{4} \binom{n}{3}$

* Finally, using $(\leq k)$-edges,

$$\text{cr}(D) = \sum_{k < \frac{n-2}{2}} (n - 2k - 3) E_{\leq k}(D) - \frac{3}{4} \binom{n}{3} + c_n$$
$j$-edges and crossings

* If we could prove $E_{\leq k}(D) \geq 3\binom{k + 2}{2}$, we would have $\text{cr}(K_n) \geq Z(n)$. 
\(j\)-edges and crossings

* If we could prove \(E_{\leq k}(D) \geq 3 \binom{k + 2}{2}\), we would have \(\text{cr}(K_n) \geq Z(n)\).

It’s not true 😞
\textbf{\textit{j}-edges and crossings}

* If we could prove $E_{\leq k}(D) \geq 3 \binom{k + 2}{2}$, we would have $\text{cr}(K_n) \geq Z(n)$.

\begin{figure}[h]
\centering
\includegraphics[width=5cm]{example.png}
\caption{Example figure}
\end{figure}

\text{It's not true ☹️}

* First try: is previous lower bound for $E_{\leq k}(D)$ true for any interesting family of drawings of $K_n$?
$j$-edges and crossings

* If we could prove $E_{\leq k}(D) \geq 3\binom{k+2}{2}$, we would have $\text{cr}(K_n) \geq Z(n)$.

It’s not true

* First try: is previous lower bound for $E_{\leq k}(D)$ true for any interesting family of drawings of $K_n$?

* 2-page drawings:
  * vertices on a line
  * edges in one of the halfplanes
\textit{j}-edges and crossings

\begin{itemize}
  \item If we could prove $E_{\leq k}(D) \geq 3\left(\frac{k+2}{2}\right)$, we would have $cr(K_n) \geq Z(n)$. \\

  \begin{itemize}
    \item It’s not true \( \frown \)
  \end{itemize}

  \item First try: is previous lower bound for $E_{\leq k}(D)$ true for any interesting family of drawings of $K_n$?

  \item 2-page drawings:
    \begin{itemize}
      \item vertices on a line
      \item edges in one of the halfplanes
    \end{itemize}

  \item Examples of 2-page drawings of $K_n$ with $Z(n)$ crossings already known.
    [Blažek-Koman, 1964]
\end{itemize}
2-page drawings

* Even for 2-page drawings, it is not true that

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* Idea: average again, and consider \( (\leq \leq k) \)-edges:

\[ E_{\leq \leq k} = \sum_{j=0}^{k} E_{\leq j} \]

\[ \text{cr}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq \leq k}(D) + c_n \]
Optimal lower bounds


\[ E_{\leq k}(D) \geq 3 \binom{k + 3}{3} \implies \nu_2(K_n) = Z(n) \]
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* Soon after, the main idea in the proof was simplified and extended to monotone drawings.

[Ábrego, Aichholzer, Fernández-Merchant, R., Salazar, 2013]
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- **Monotone drawing:**
  - vertices on a line and edges monotone (w.r.t. that line)
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* **Monotone drawing:**
  * vertices on a line and edges monotone (w.r.t. that line)

* In the rest of the talk, sketch of the main ideas and a further extension: **t-shellable drawings.**
Main ideas of the proof (for monotone drawings)

* The proof is by induction.

We remove point $n$, and denote by $D'$ the corresponding drawing of $K_{n-1}$. 
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  We remove point $n$, and denote by $D'$ the corresponding drawing of $K_{n-1}$.

* $E_{\leq k}(D) = E_{\leq k-1}(D') + 2\left(\frac{k}{2} + 2\right) + E_{\leq k}(D, D')$

induction hypothesis
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We remove point $n$, and denote by $D'$ the corresponding drawing of $K_{n-1}$.

$$E_{\leq k}(D) = E_{\leq k-1}(D') + 2\binom{k+2}{2} + E_{\leq k}(D, D')$$

From now on, we consider unoriented $j$-edges.

Consider always the “light” side $(j \leq n/2 - 1)$.
Main ideas of the proof (for monotone drawings)

* The proof is by induction.

We remove point $n$, and denote by $D'$ the corresponding drawing of $K_{n-1}$.

* $E_{\leq k}(D) = E_{\leq k-1}(D') + 2\binom{k+2}{2} + E_{\leq k}(D, D')$

  induction hypothesis

  \begin{align*}
  &j\text{-edges adjacent to } n \quad j = 0, \ldots, k \\
  &\text{invariant } \leq k\text{-edges}
  \end{align*}

* A $j$-edge of $D'$ is an $\leq k$-invariant edge if it is also a $j$-edge of $D$ (and $j \leq k$).

So an edge is invariant if vertex $n$ lies on the “heavy side” of the edge.
Finding invariant edges

* Consider the edges starting at $i$ and order them vertically.

\[ i - 1 \rightarrow i \rightarrow \cdots \rightarrow \leq (i - 1) \text{-edge of } D' \]
Finding invariant edges

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\]
Finding invariant edges

* Consider the edges starting at \( i \) and order them vertically.

* The \( m \text{-th edge} \) in the top-down order, is an \( \leq (i + m - 2) \text{-edge} \) (while \( i + m - 2 \leq n/2 - 1 \)).

The same is true for the bottom-up order.
Finding invariant edges

* Are there enough invariant edges?
Finding invariant edges

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Finding invariant edges

Are there enough invariant edges?

Sweep (top-down and bottom-up) edges starting at $i$: all the edges that we find before reaching $i_{n}$ (or half of the edges) are invariant.
Finding invariant edges

* **Invariant \(\leq k\)-edges** starting at \(i\). At least

\[
\begin{align*}
\text{one } \leq (i - 1)\text{-edge} \\
\text{one } \leq i\text{-edge} \\
\vdots \\
\text{one } \leq k\text{-edge}
\end{align*}
\]

\(k - i + 2\) invariant \(\leq k\)-edges starting at vertex \(i\).
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  one $\leq (i - 1)$-edge

  one $\leq i$-edge

  $\vdots$

  one $\leq k$-edge

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  starting at vertex $i$.

* Considering $i = 1, \ldots, k$, we get

$$E_{\leq k}(D, D') \geq \binom{k + 2}{2}$$
Finding invariant edges

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E_{\leq \leq k}(D) = E_{\leq \leq k-1}(D') + 2 \binom{k + 2}{2} + E_{\leq k}(D, D')
\]

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\geq 3 \binom{k + 2}{3} + 2 \binom{k + 2}{2} + \binom{k + 2}{2}
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  one \( \leq (i - 1) \)-edge  
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  \( \vdots \)  
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E_{\leq k}(D, D') &\geq \binom{k + 2}{2} \\
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&\geq 3\binom{k + 2}{3} + 2\binom{k + 2}{2} + \binom{k + 2}{2}
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* Considering \( i = 1, \ldots, k \), we get \( E_{\leq k}(D, D') \geq \binom{k + 2}{2} \)
Finding invariant edges

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\geq 3 \binom{k + 2}{3} + 2 \binom{k + 2}{2} + \binom{k + 2}{2} = 3 \binom{k + 3}{3}
\]
Lower bound

* Using \( \text{cr}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq k}(D) + c_n \)

\[
E_{\leq k}(D) \geq 3 \binom{k + 3}{3} \Rightarrow \text{cr}(D) \geq Z(n)
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* What properties of monotone drawings are we really using in the proof?
Shellable drawings

* Only that vertices $i$ and $n$ are on the boundary of the drawing obtained when vertices $1, 2, \ldots, i - 1$ are deleted.

boundary of $D$: boundary of the unbounded face

Vertex $n - 1$ is also on the boundary when vertex $n$ is deleted, and so on.
Shellable drawings

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boundary of $D$: boundary of the unbounded face

* Of course, there is nothing special with the unbounded face: we can take any face of the drawing and convert it the unbounded one.
Shellable drawings

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* Of course, there is nothing special with the unbounded face: we can take any face of the drawing and convert it the unbounded one.

* For $1 \leq i < j \leq t$, let $D_{ij}$ be the drawing obtained from $D$ by removing vertices $\{v_1, \ldots, v_{i-1}, v_{j+1}, \ldots, v_t\}$.

A drawing $D$ of $K_n$ is $t$-shellable if there exists a subset of vertices $S = \{v_1, v_2, \ldots, v_t\}$ and a face $F$ such that for all $1 \leq i < j \leq t$ vertices $v_i$ and $v_j$ are on the boundary of the face of $D_{ij}$ containing $F$. 
$t$-shellable drawings

Examples:
- monotone drawings are $n$-shellable.
- $x$-bounded drawings are $n$-shellable.
$t$-shellable drawings

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- monotone drawings are \( n \)-shellable.
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* **Examples:**
  * monotone drawings are $n$-shellable.
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* **Theorem:** If a drawing $D$ of $K_n$ is $t$-shellable then

\[
E_{\leq k}(D) \geq 3 \binom{k + 3}{3}
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for all $k \leq t - 2$. 
$t$-shellable drawings

* Examples:
  * monotone drawings are $n$-shellable.
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* Theorem: If a drawing $D$ of $K_n$ is $t$-shellable then
  
  $E_{\leq k}(D) \geq 3 \binom{k+3}{3}$

  for all $k \leq t - 2$.

* Theorem: If a drawing $D$ of $K_n$ is $t$-shellable for some $t \geq n/2$ then $cr(D) \geq Z(n)$. 
Cylindrical drawings

A drawing is **cylindrical** if it contains two crossing-free cycles spanning the set of vertices.
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Partial results for equal size sets [Richter-Thomassen’97]
Cylindrical drawings

A drawing is cylindrical if it contains two crossing-free cycles spanning the set of vertices.

Partial results for equal size sets [Richter-Thomassen’97]

* Any cylindrical drawing of $K_n$ is $n/2$-shellable.
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\[ t \geq \frac{n}{2} \]

\* Any cylindrical drawing of \( K_n \) is \( \frac{n}{2} \)-shellable.
Cylindrical drawings

A drawing is \textit{cylindrical} if it contains two crossing-free cycles spanning the set of vertices.

Partial results for equal size sets [Richter-Thomassen’97]

\[ t \geq n/2 \]

* Any cylindrical drawing of \( K_n \) is \( n/2 \)-shellable.

* The number of crossings in any cylindrical drawing of \( K_n \) is at least \( Z(n) \).
Conclusions

* Two known families of optimal drawings:
  ▶ 2-page drawings
  ▶ cylindrical drawings

Lower bound known for those families.
Conclusions

* Two known families of optimal drawings:
  ▶ 2-page drawings
  ▶ cylindrical drawings

  Lower bound known for those families.

* Open problems:
  ▶ other families of optimal drawings?
  ▶ prove that they are really optimal!
Shellable drawings and the crossing number of the complete graph

Thank you for your attention.