Shellable drawings

and the crossing number of the complete graph

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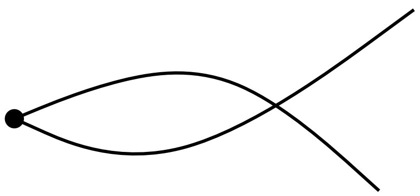

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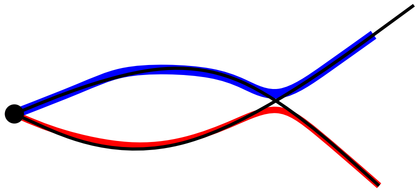

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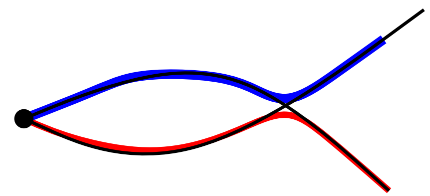

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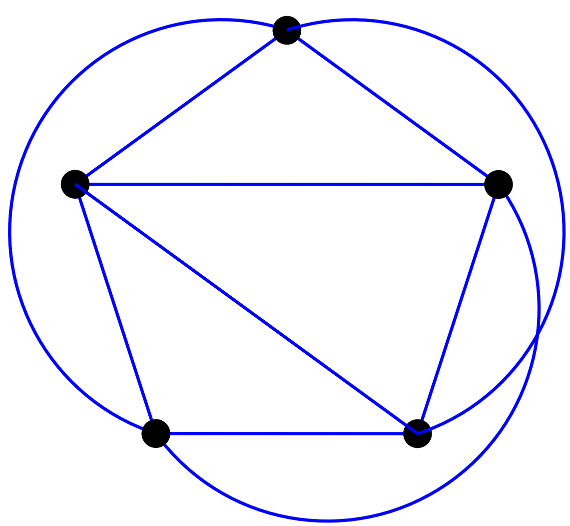

$$\mathrm{cr}(K_5) = 1$$

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  * If we add a single edge e to a plane graph G, computing $\mathrm{cr}(G \cup \{e\})$ is also NP-hard. [Cabello-Mohar, 2010]

A brief history of $\mathrm{cr}(K_n)$

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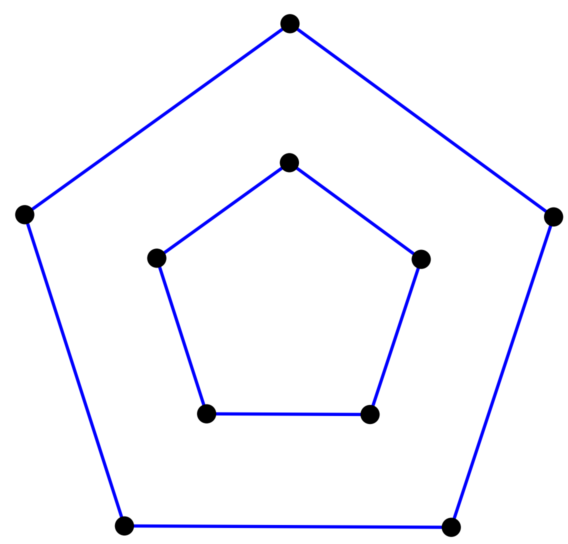

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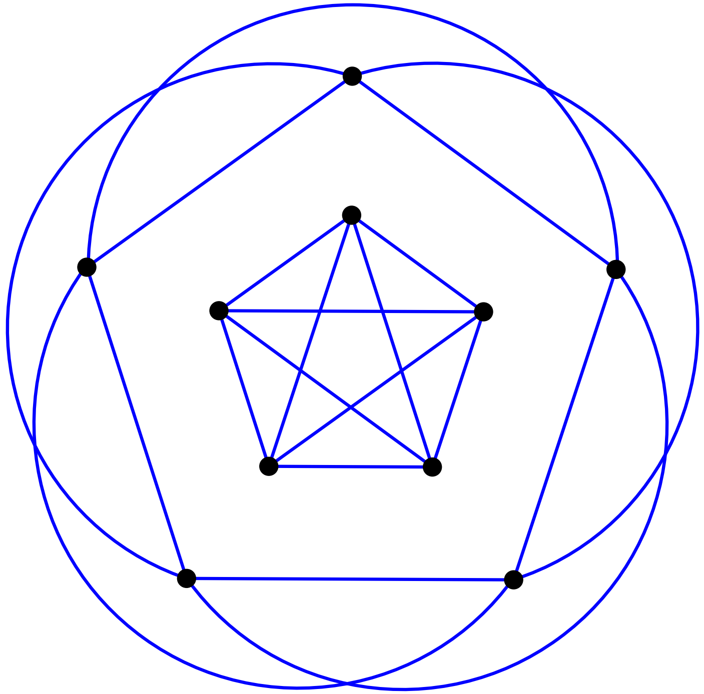

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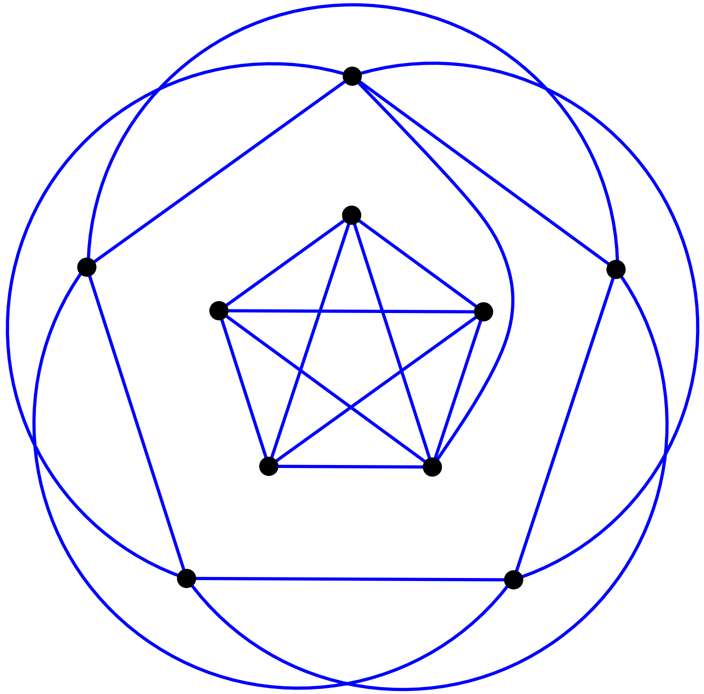

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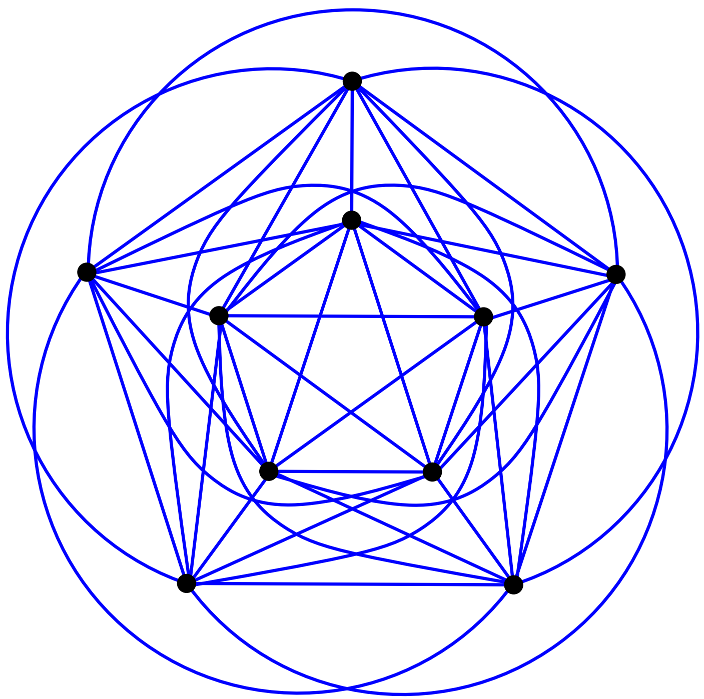

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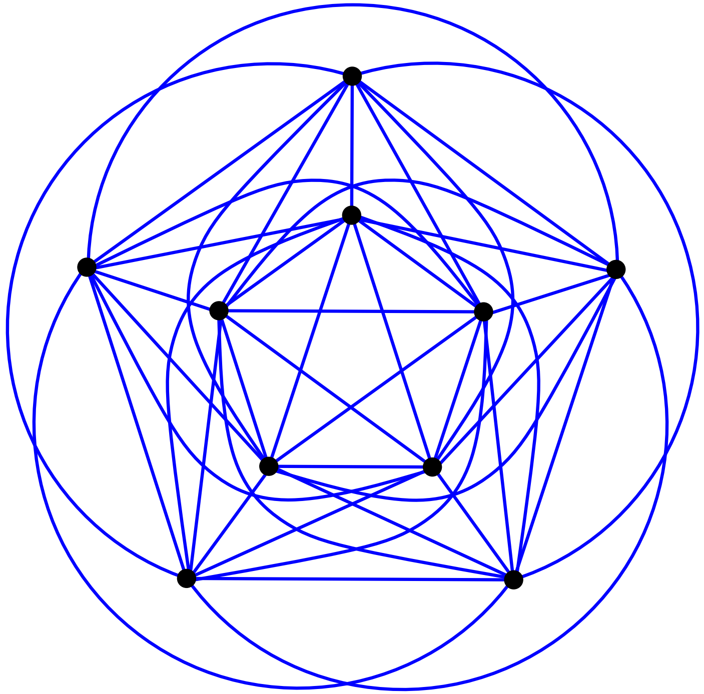

The number of crossings in these drawings is

$$Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

Zarankiewicz number

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* Some known results for small n:
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* This was the situation, till a new tool was borrowed from the rectilinear case.

Rectilinear crossing number

* The rectilinear crossing number of G, $\overline{\mathrm{cr}}(G)$, is the smallest number of crossings in drawings of G in which edges are segments. (Vertices in general position).

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  $\square(S)$: number of convex quadrilaterals in S.

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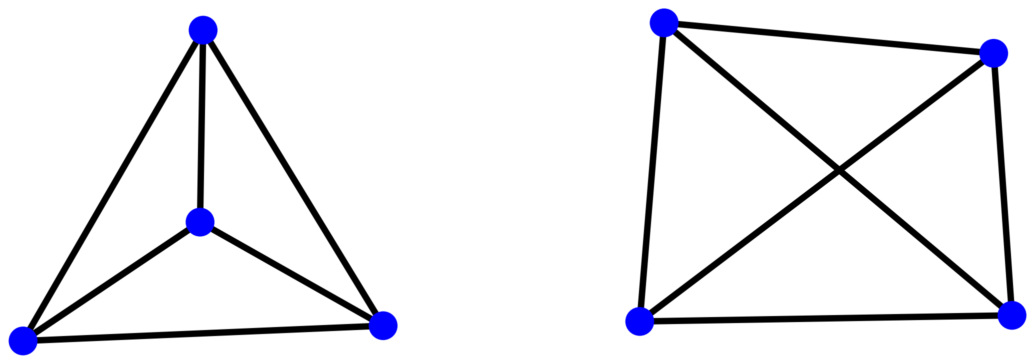

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* 2004: Ábrego - Fernández-Merchant, Lovász-Vesztergombi-Wagner-Welzl

  Relation between $\square(S)$ and the number of j-edges of S.

j-edges

* Let S be a set of n points in the plane in general position. Given $p, q \in S$, we say that pq is an (oriented) j-edge if there are j points of S in the right halpf-plane defined by pq.

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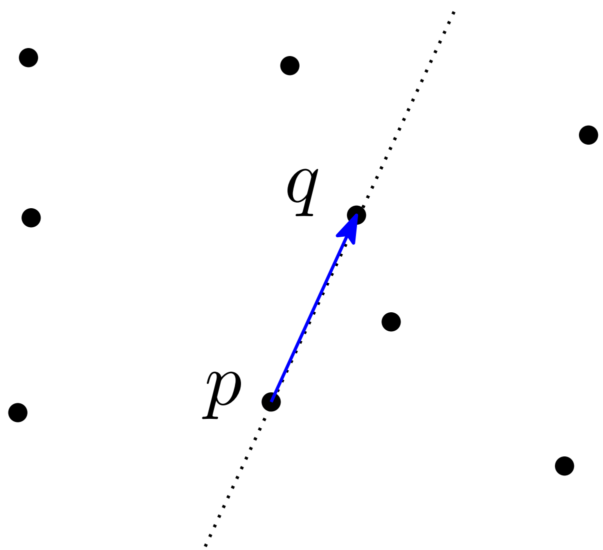

3-edge

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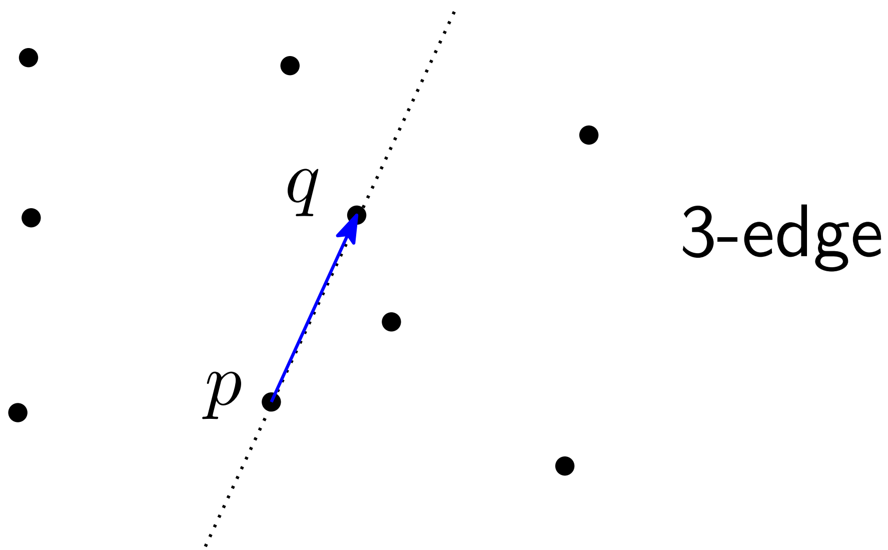

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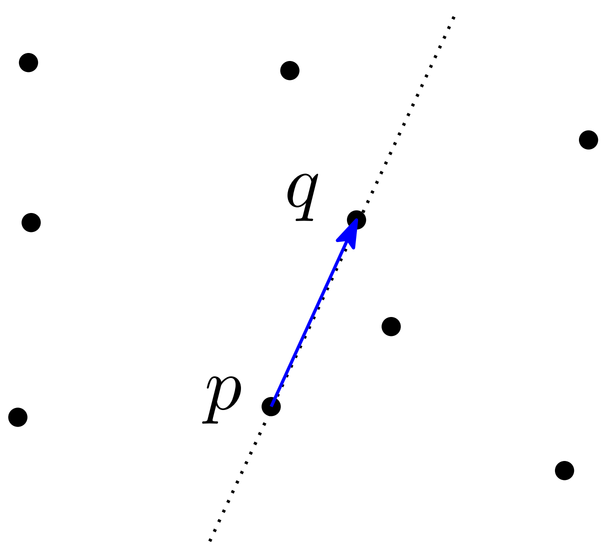


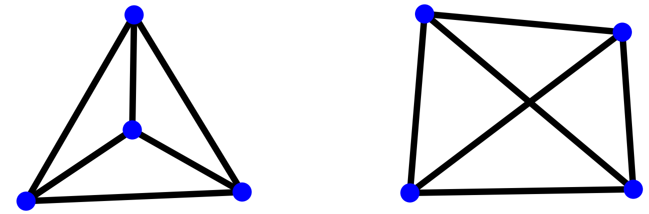
3-edge

* $e_j(S)$:= # j-edges of S.

* If pq is a j-edge, then qp is a $n - j - 2$-edge.
  It is also possible to work with unoriented j-edges.

j-edges and convex quadrilaterals (crossings)

$$* \quad \triangle(S) + \square(S) = \binom{n}{4} \qquad (1)$$



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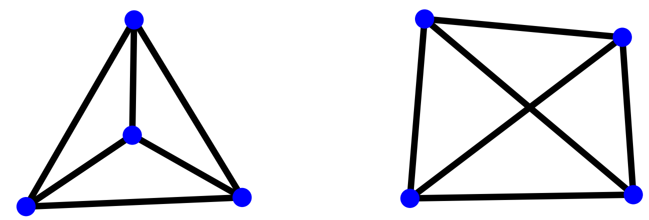
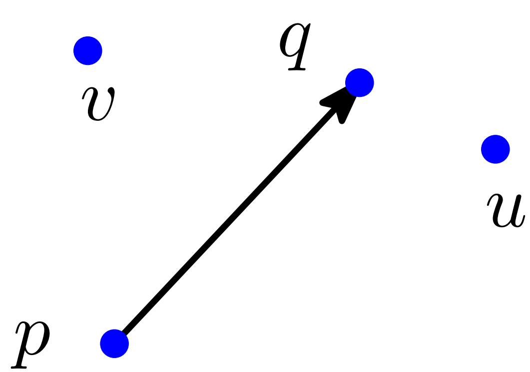


* Another relation: double counting of separations.
  A separation is a 4-tuple $\{p, q, u, v\}$ where the ordered pair p, q leaves u to the right and v to the left.



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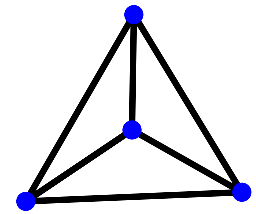

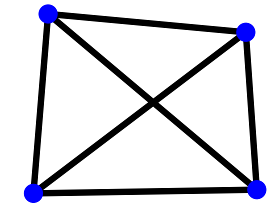

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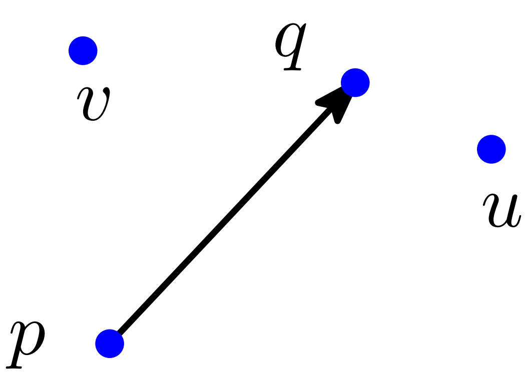

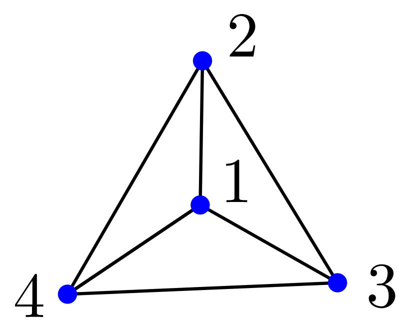

six separations

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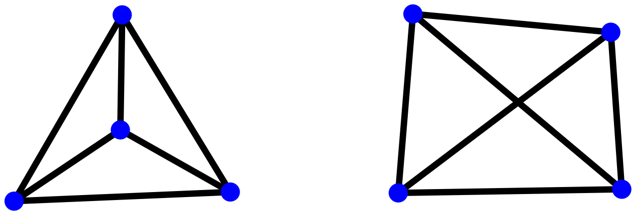

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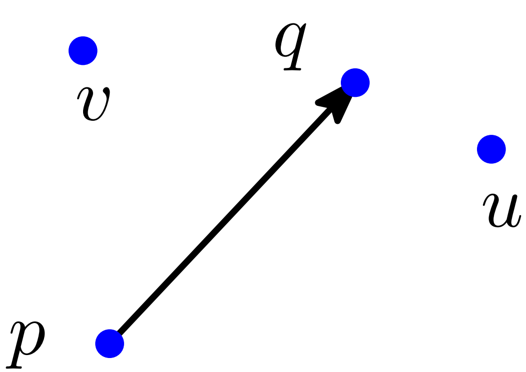

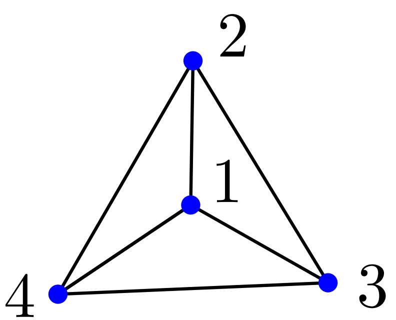

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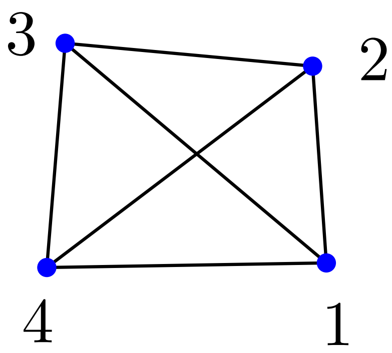

four separations

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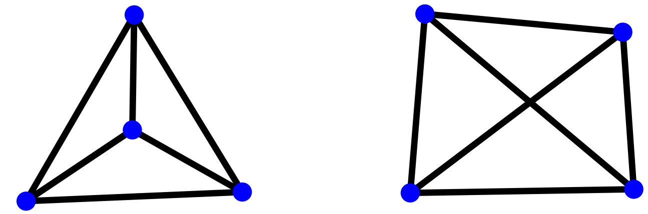
* $\triangle(S) + \square(S) = \binom{n}{4}$ (1)



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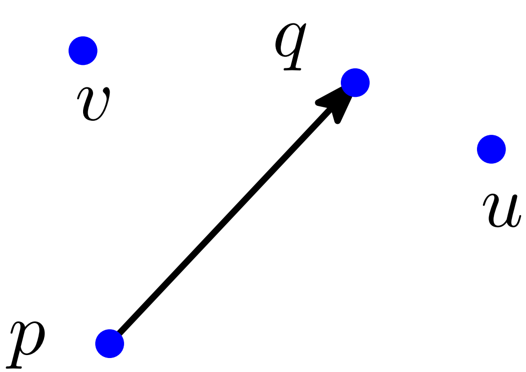

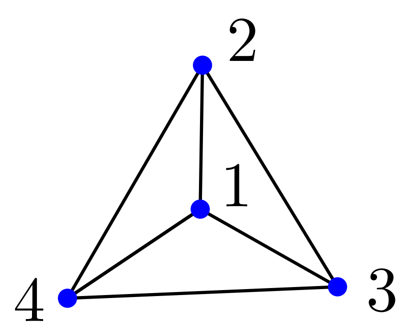

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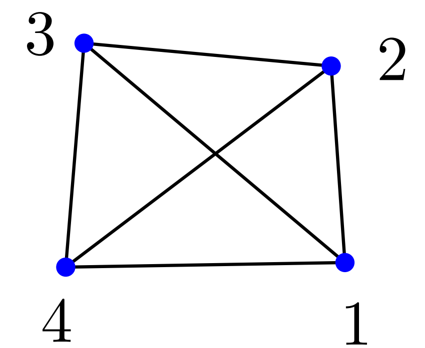

four separations

* $6\,\triangle(S) + 4\,\square(S) = \sum_{j=0}^{n-2} j(n-j-2)\,e_j(S)$ (2)

j-edges and convex quadrilaterals (crossings)

* From these equations (and the relations $e_j = e_{n-j-2}$ and $\sum_{j=0}^{n-2} e_j = n(n-1)$) we get

$$\square(S) = \sum_{j<\frac{n-2}{2}} \left(\frac{n-2}{2} - j\right)^2 e_j(S) - \frac{3}{4}\binom{n}{3}$$

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Lower bounds for $\overline{\mathrm{cr}}(K_n)$

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* LVWW use an improved bound for $E_{\leq k}$ (for k close to $n/2$), to show that

$$\overline{\text{cr}}(K_n) \geq 0.37501\binom{n}{4}$$

Bounds for $\overline{\text{cr}}(K_n)$

* 2006 – 2010 Series of improvements on the lower bound for $E_{\leq k}(S)$. (And on the lower bound for $\overline{\text{cr}}(K_n)$)
  * [Balogh-Salazar'06]
  * [Aichholzer-García-Orden-R.'07]
  * [Ábrego,Cetina,Fernández-Merchant,Leaños,Salazar'11].

  Current bounds:

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[Aurenhammer-Aichholzer-Krasser]
[Ábrego,Cetina,Fernández-Merchant,Leaños,Salazar]

General (topological) drawings

* BIRS - Crossing numbers turn useful. (August 2011)

  If in the formula

$$\square(S) = \sum_{k < \frac{n-2}{2}} (n - 2k - 3)\, E_{\leq k}(S) + c_n$$

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$$||$$

$$Z(n)$$

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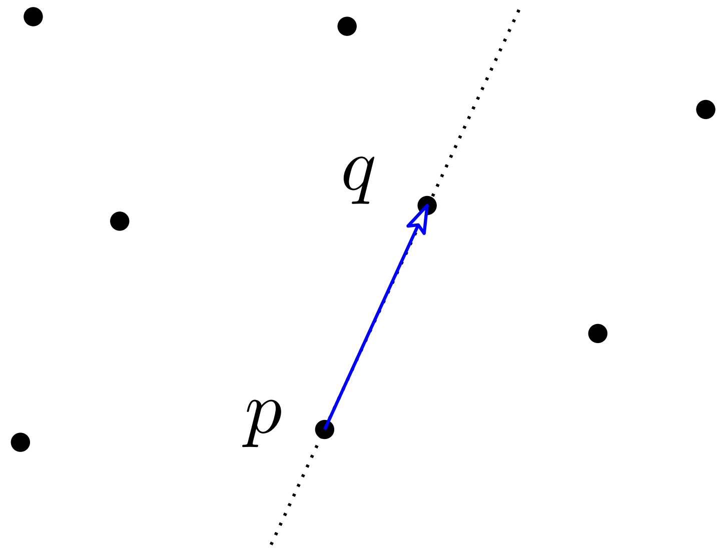
$$\sum_{k < \frac{n-2}{2}} (n-2k-3)\, 3\binom{k+2}{2} + c_n = \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

$$||$$

$$Z(n)$$

* Is that a coincidence?

j-edges in topological drawings



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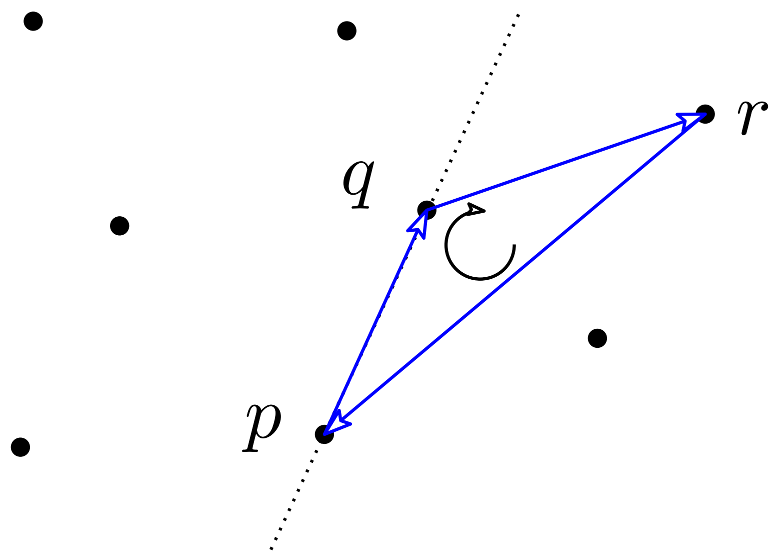

Consider the triangles!

$\sigma(pqr) = +$

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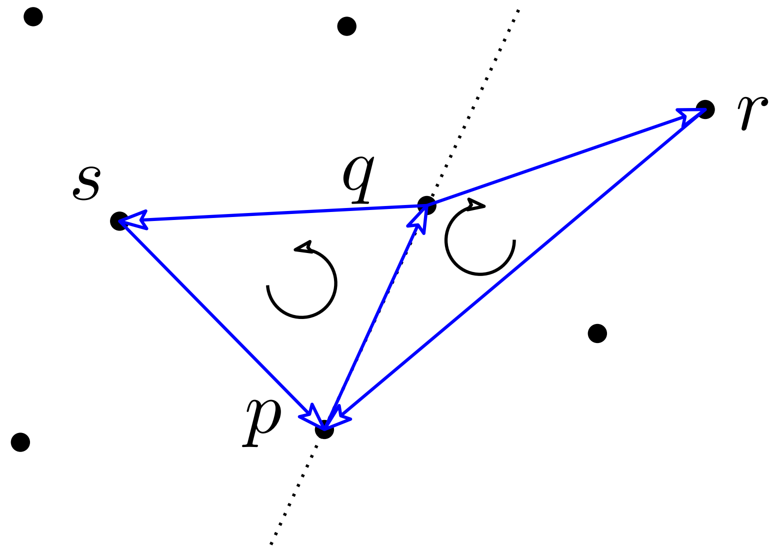

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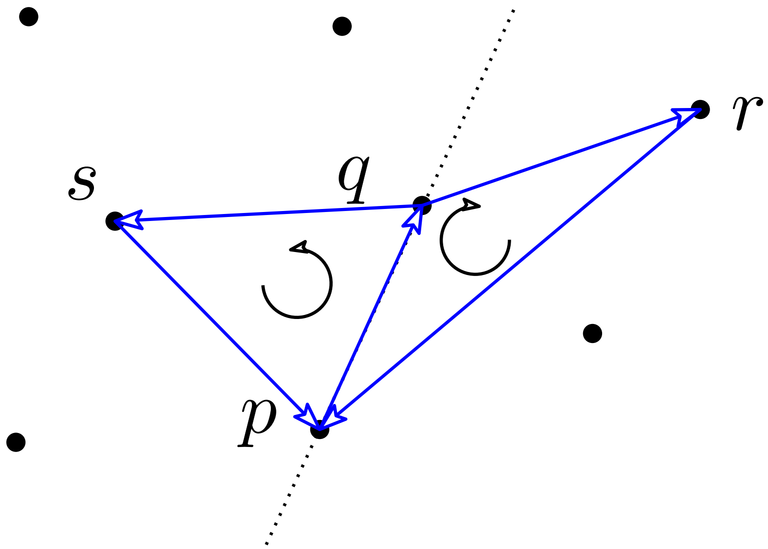

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  We say that r is to the right of pq if pqr is oriented clockwise.

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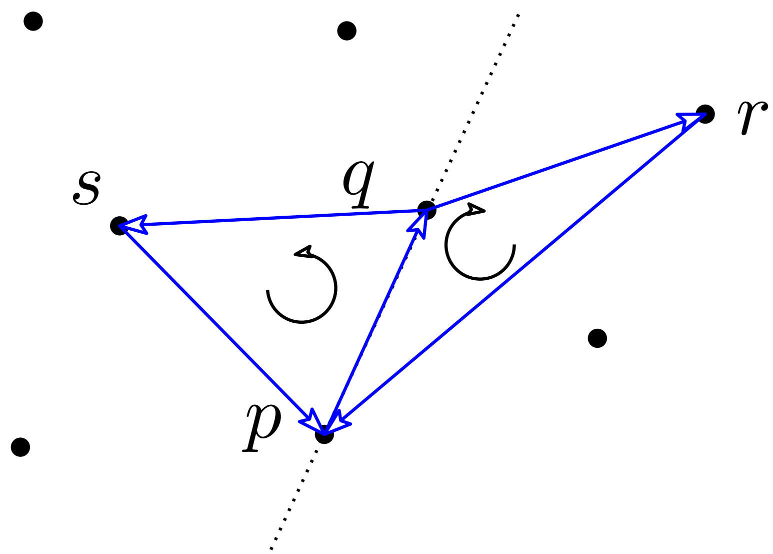

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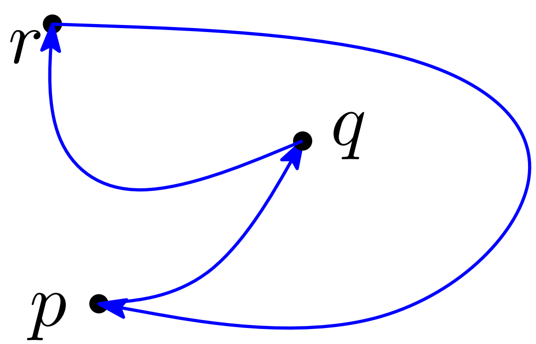

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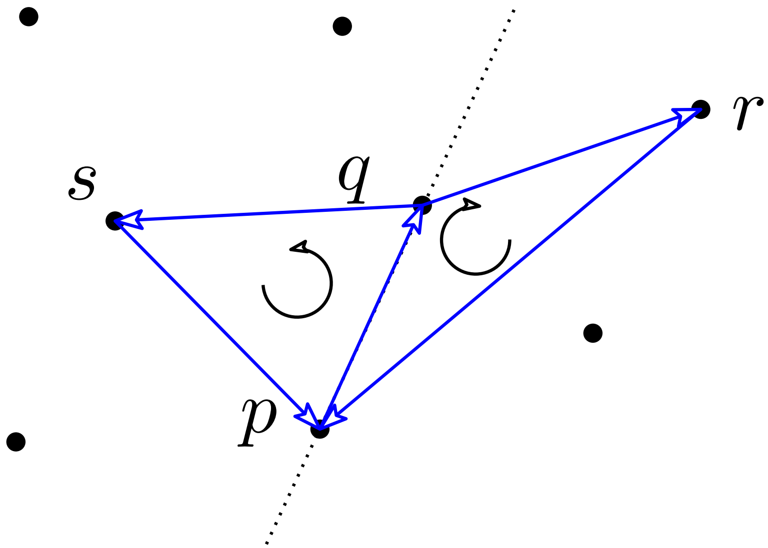

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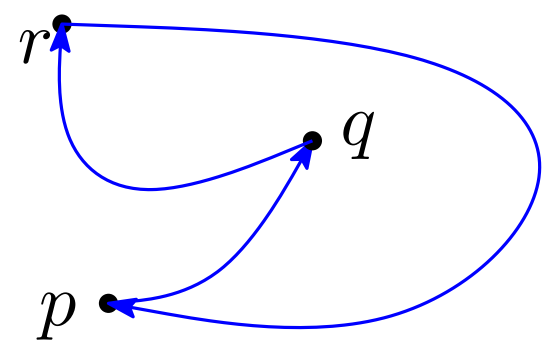

* And now we can define j-edges exactly as in the geometric setting.

j-edges and crossings (in topological drawings)

* Now we need to generalize the relation

$$6\,\triangle(S) + 4\,\square(S) = \sum_{j=0}^{n-2} j(n-j-2)\, e_j(S)$$

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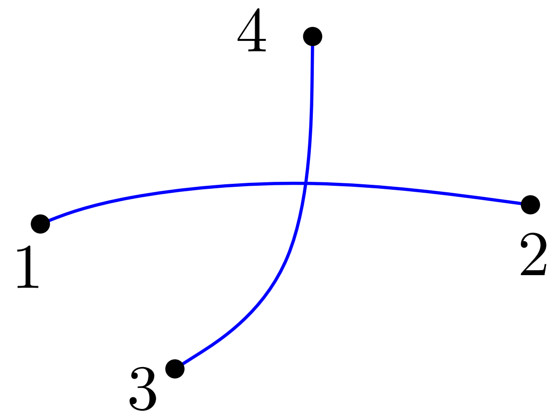

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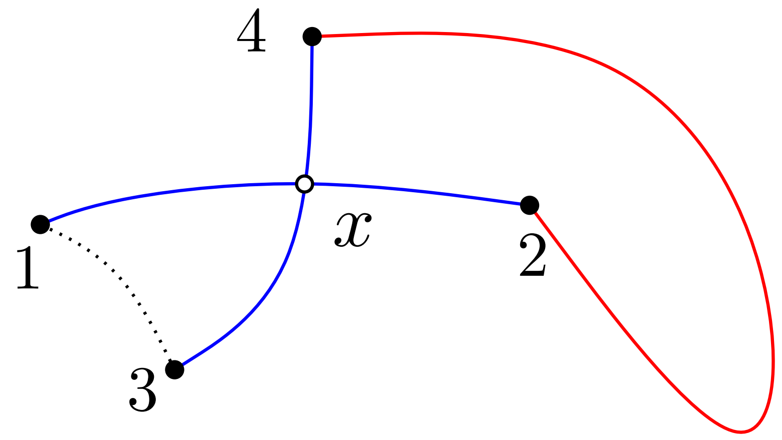
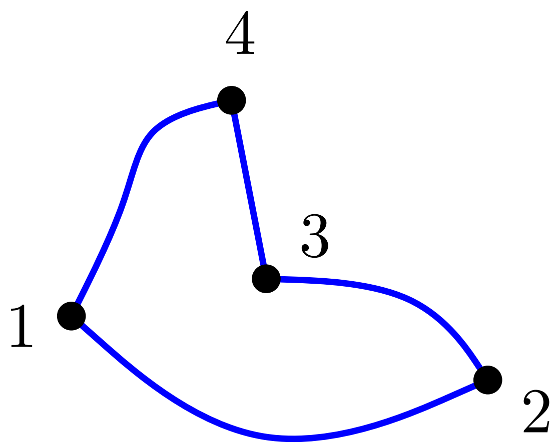
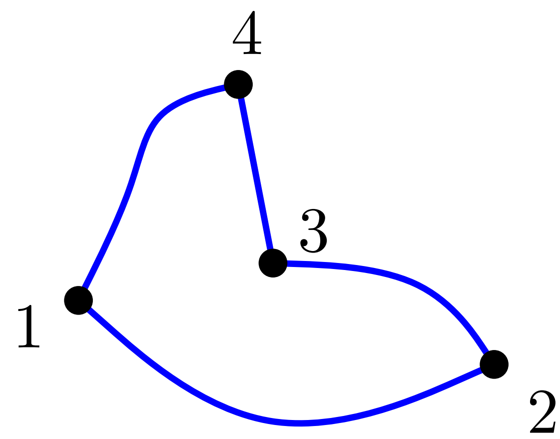
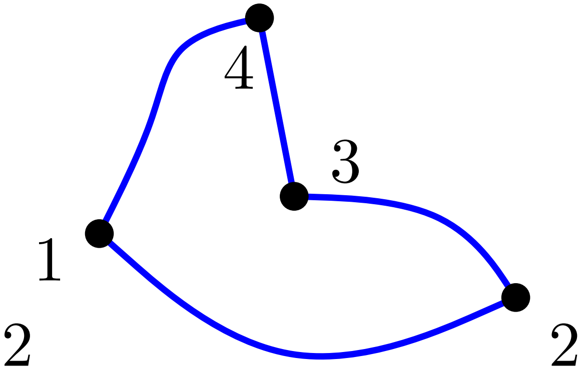


j-edges and crossings (in topological drawings)

* There are three “different” drawings of K_4.







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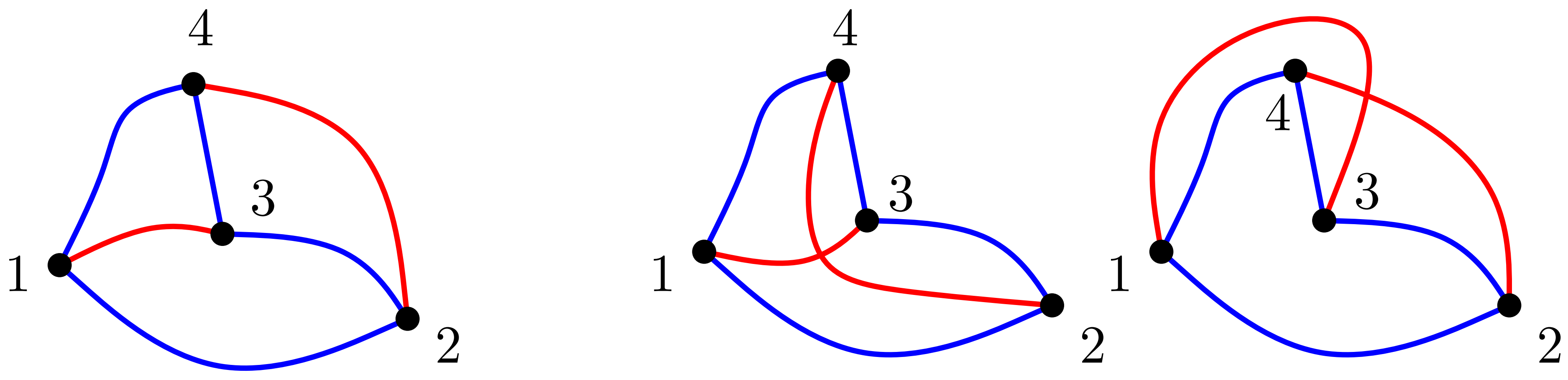

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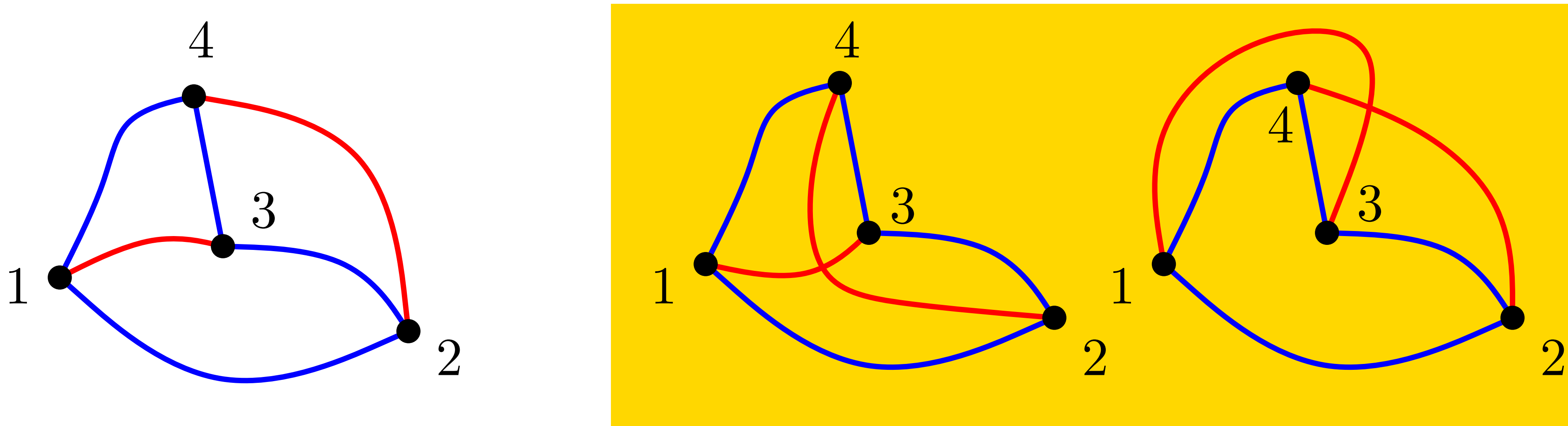

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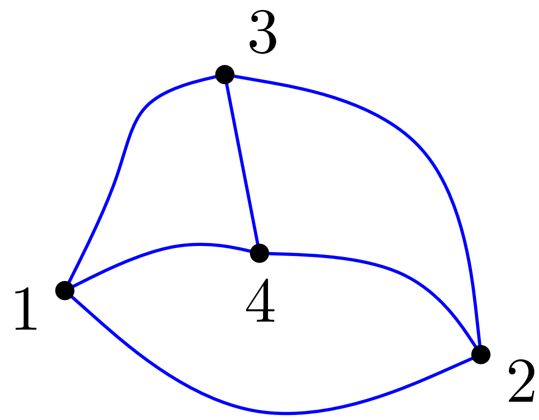
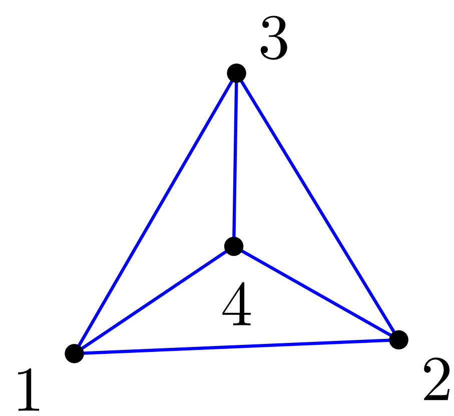


C_A



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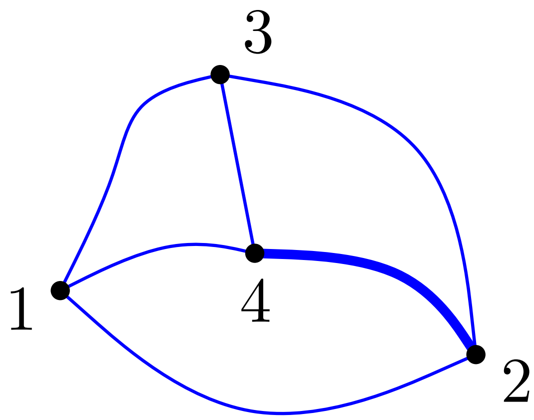
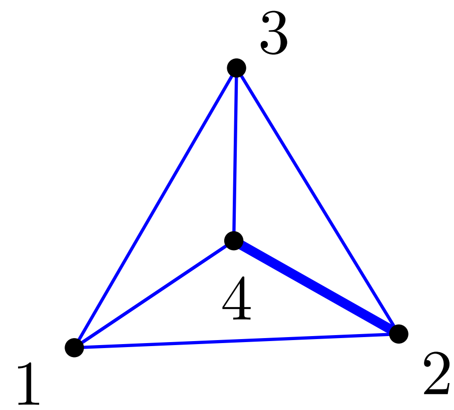


C_A



no crossing

↓

6 separations

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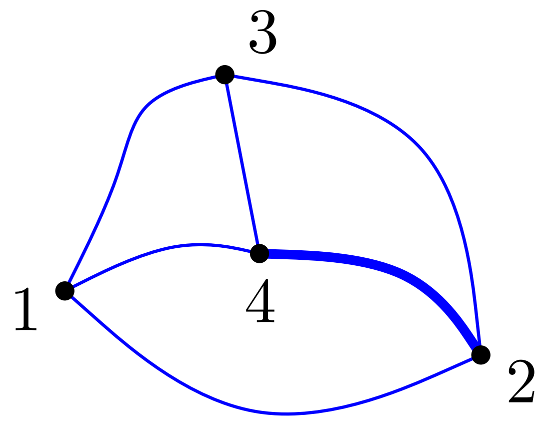
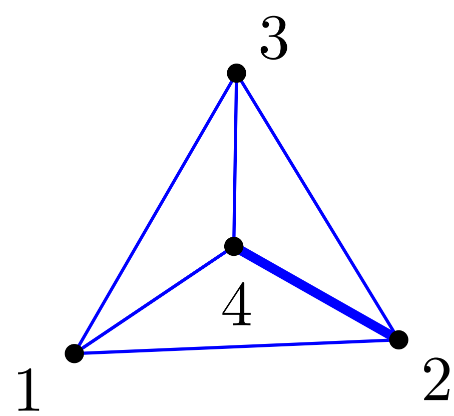


C_A



no crossing

↓

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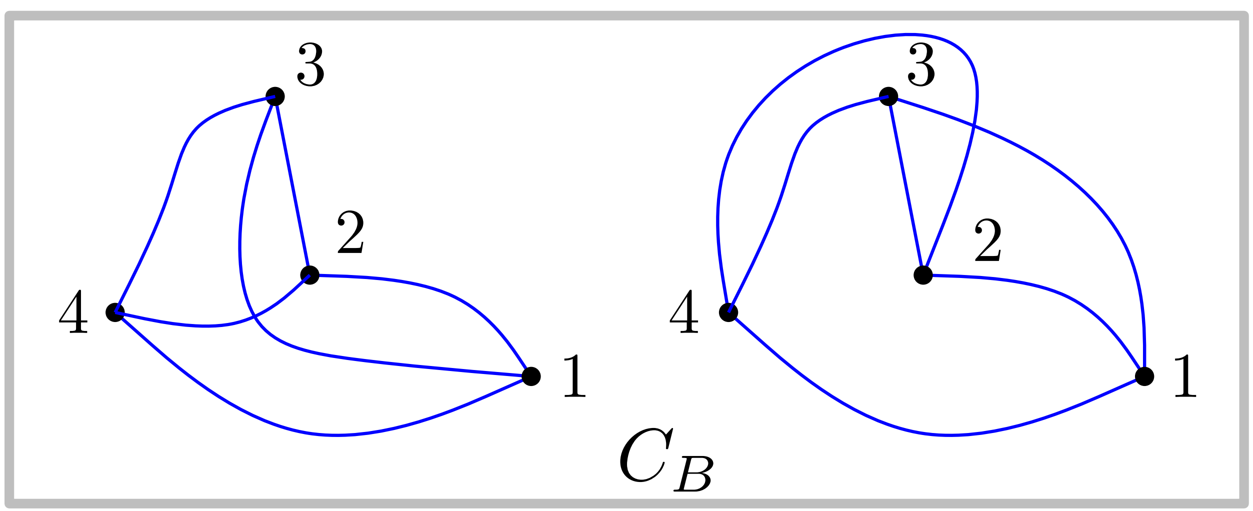
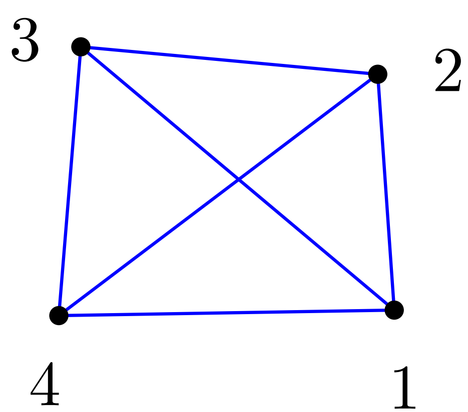


C_B



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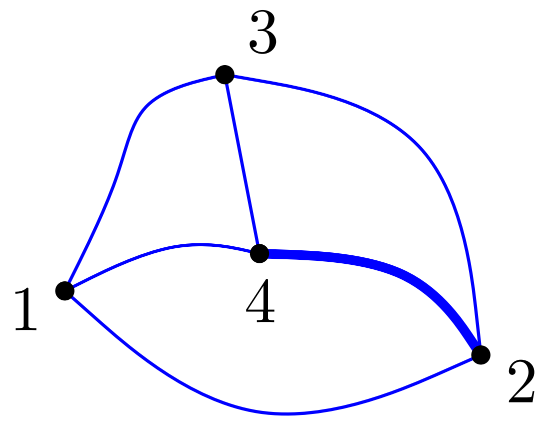
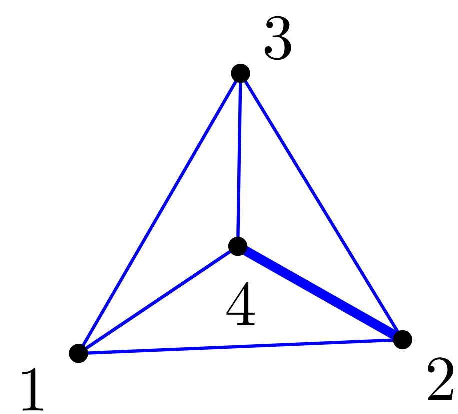


C_A



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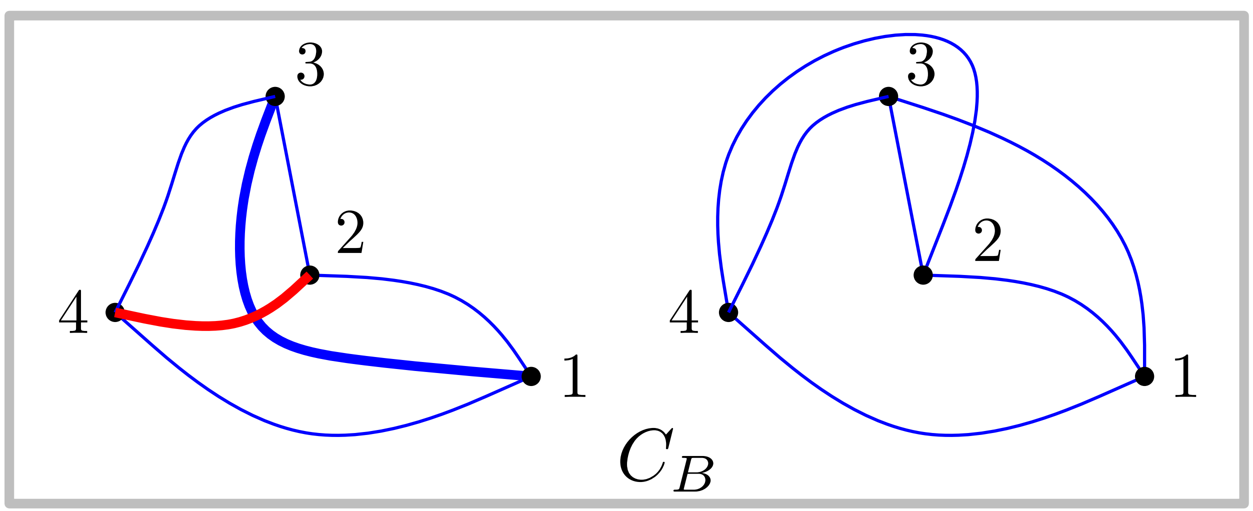

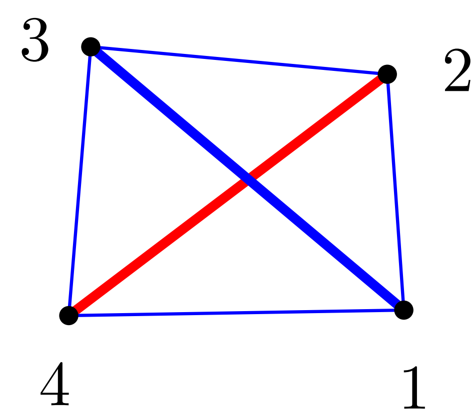

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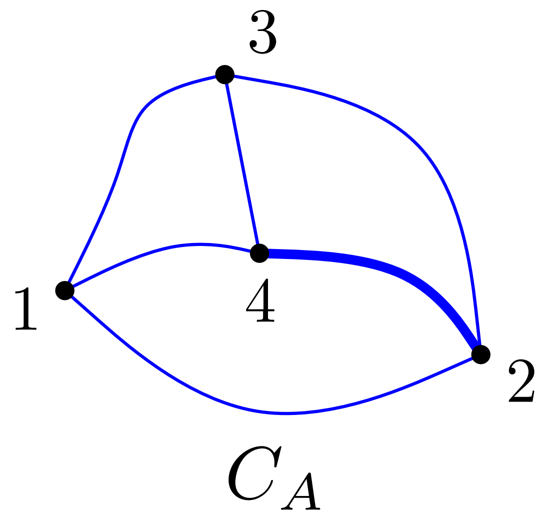

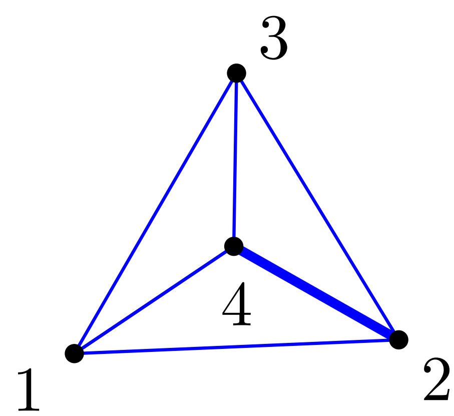

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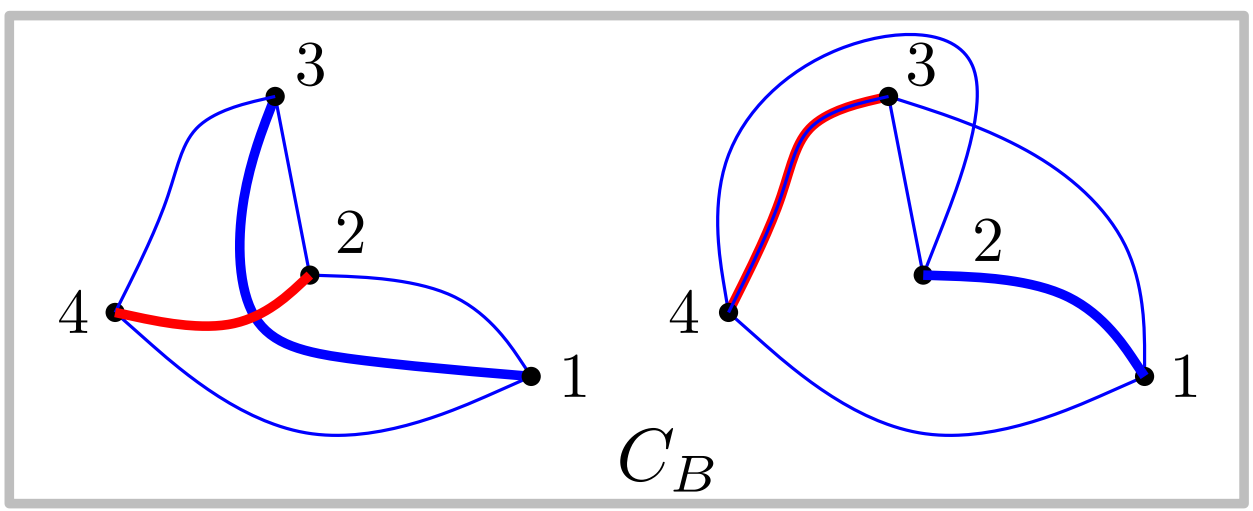

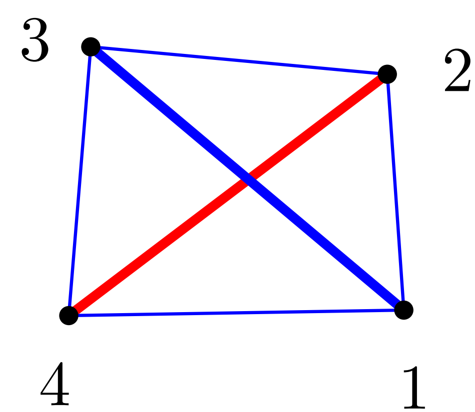

one crossing $\rightarrow$ 4 separations

j-edges and crossings (in topological drawings)

* So we have:

1. $|C_B| = \mathrm{cr}(D)$ 2. $|C_A| + |C_B| = \binom{n}{4}$

3. $6|C_A| + 4|C_B| = \sum_{j=0}^{n-2} j(n-j-2)\, e_j(D)$

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* Finally, using $(\leq k)$-edges,

$$\mathrm{cr}(D) = \sum_{k<\frac{n-2}{2}} (n-2k-3)\, E_{\leq k}(D) - \frac{3}{4}\binom{n}{3} + c_n$$

j-edges and crossings

* If we could prove $E_{\leq k}(D) \geq 3\binom{k+2}{2}$, we would have $\mathrm{cr}(K_n) \geq Z(n)$.

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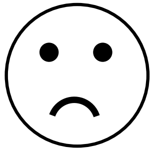

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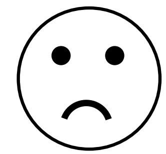
It's not true 

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* 2-page drawings:
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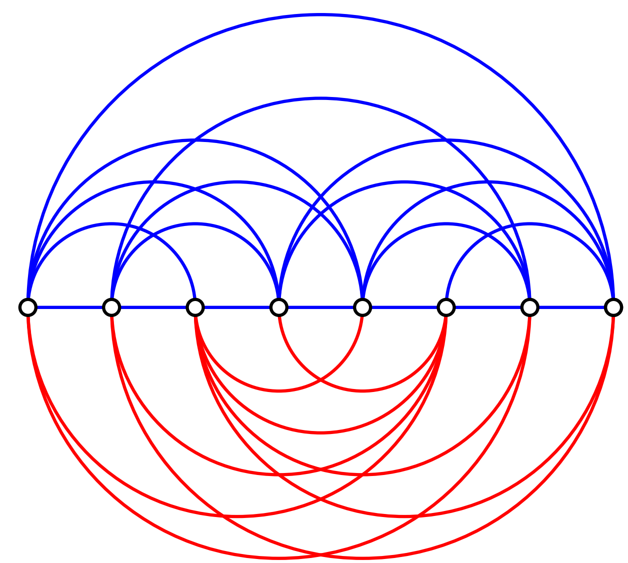
  * edges in one of the halfplanes



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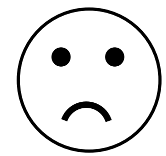
  ↑ It's not true 

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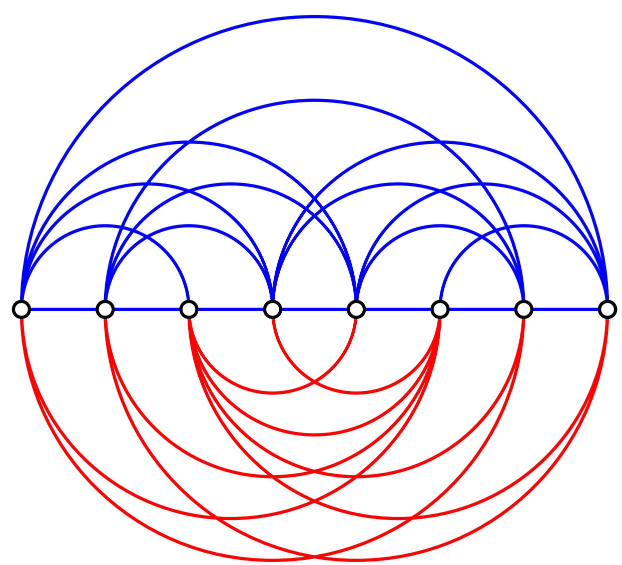


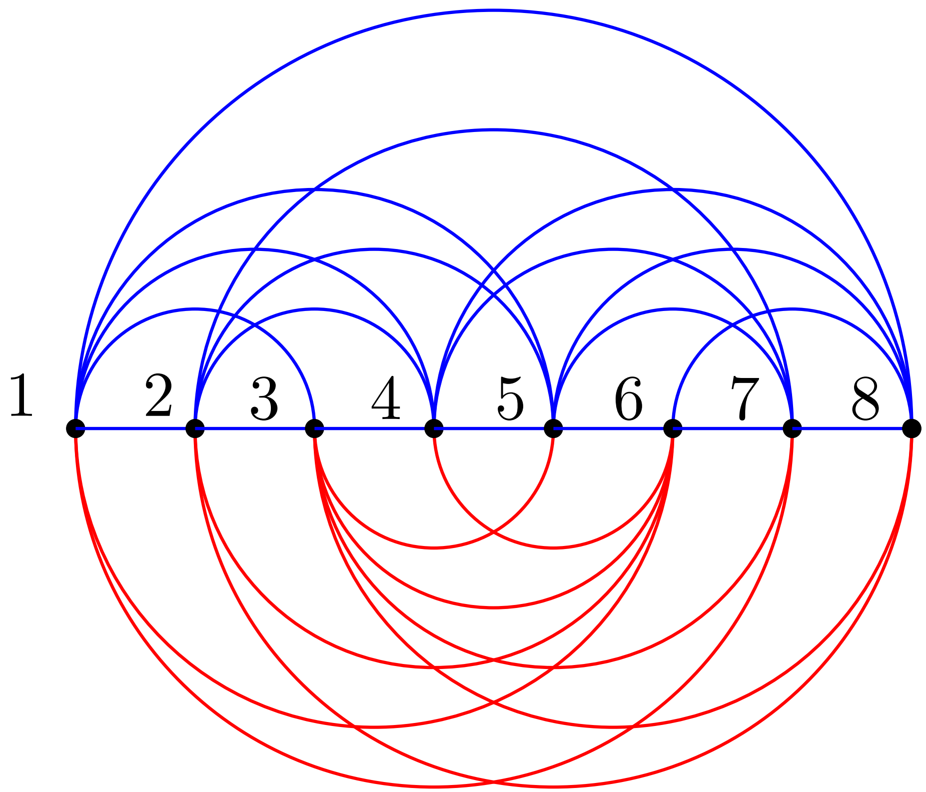
* Examples of 2-page drawings of K_n with $Z(n)$ crossings already known.

  [Blažek-Koman, 1964]

2-page drawings

* Even for 2-page drawings, it is not true that

$$E_{\leq k} \geq 3\binom{k+2}{2}.$$



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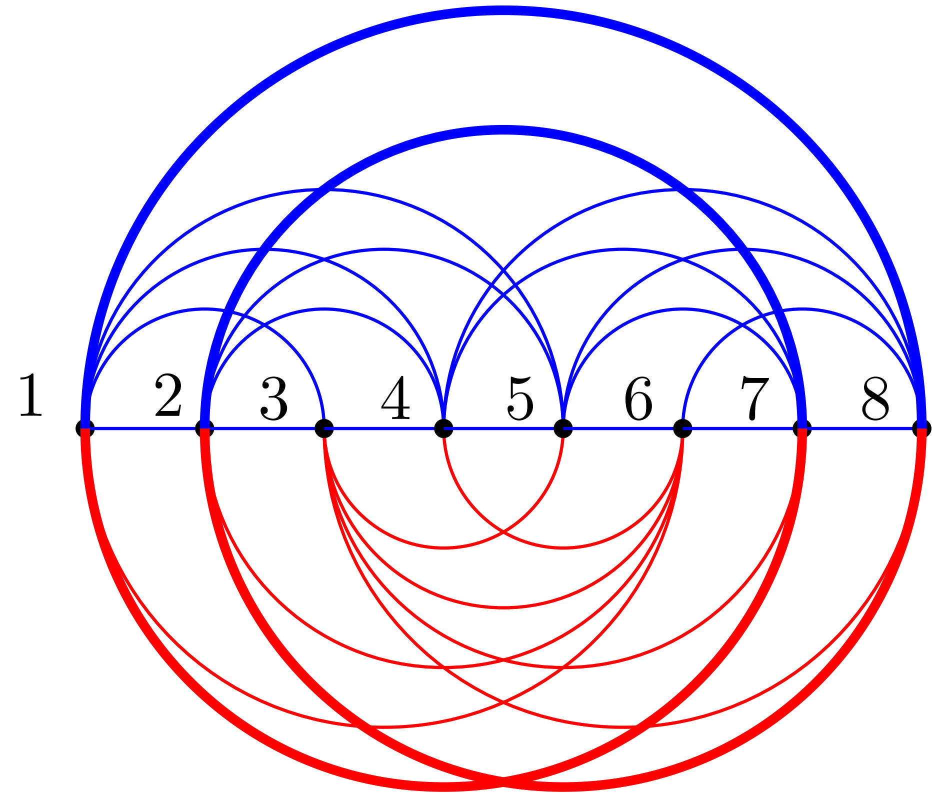

four 0-edges

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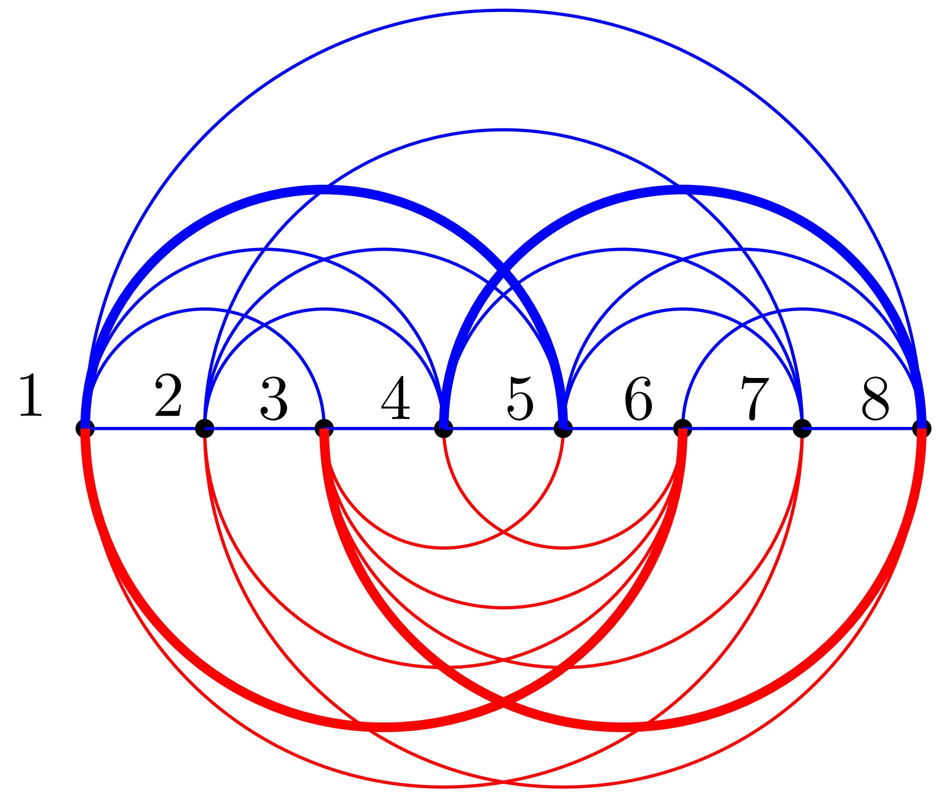

four 1-edges

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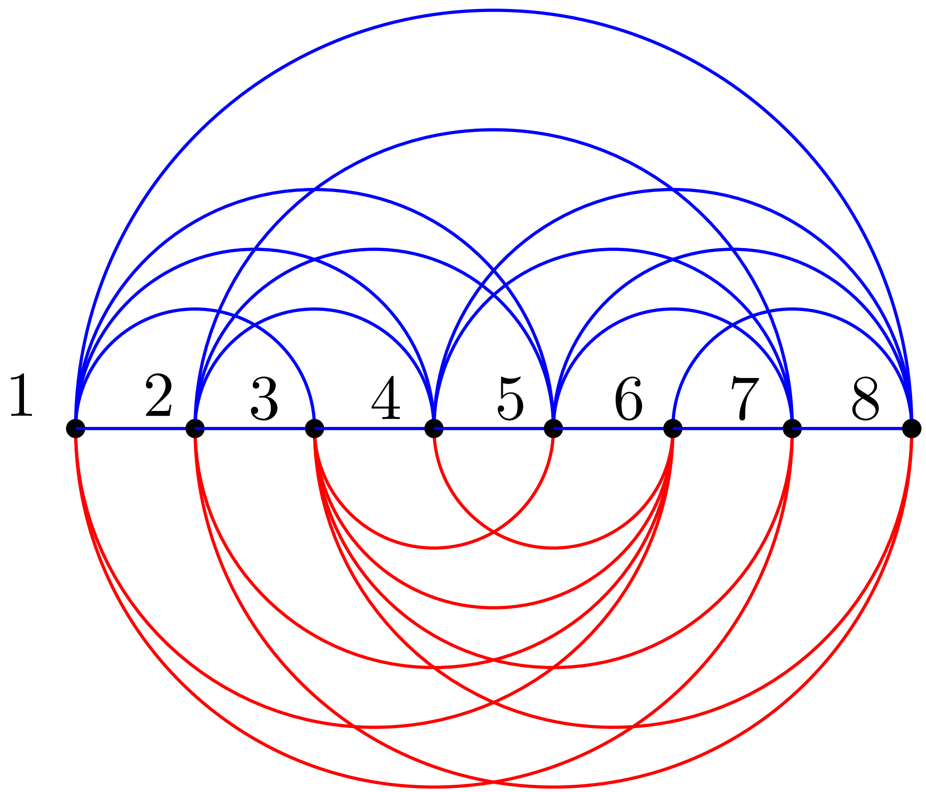

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$$E_{\leq\leq k} = \sum_{j=0}^{k} E_{\leq j}$$

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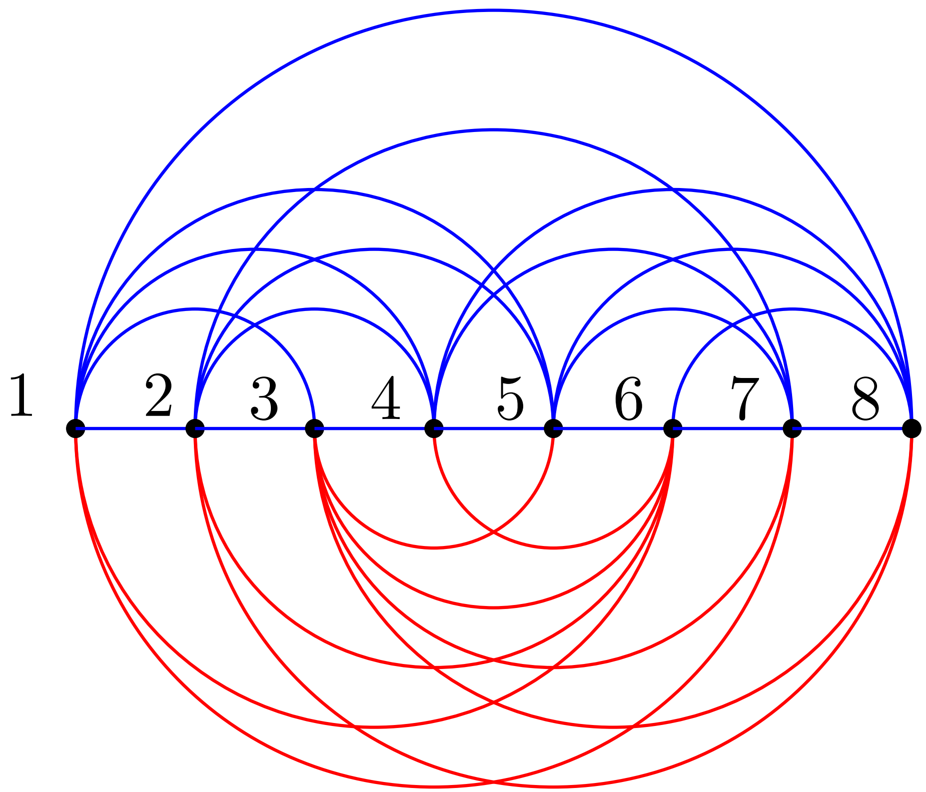

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$$\operatorname{cr}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq\leq k}(D) + c_n$$

Optimal lower bounds

* Ábrego, Aichholzer, Fernández-Merchant, R., Salazar (2012):

$$E_{\leq\leq k}(D) \geq 3\binom{k+3}{3} \Rightarrow \nu_2(K_n) = Z(n)$$

2-page crossing number of K_n

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* Soon after, the main idea in the proof was simplified and extended to monotone drawings.

  [Ábrego, Aichholzer, Fernández-Merchant, R., Salazar, 2013]

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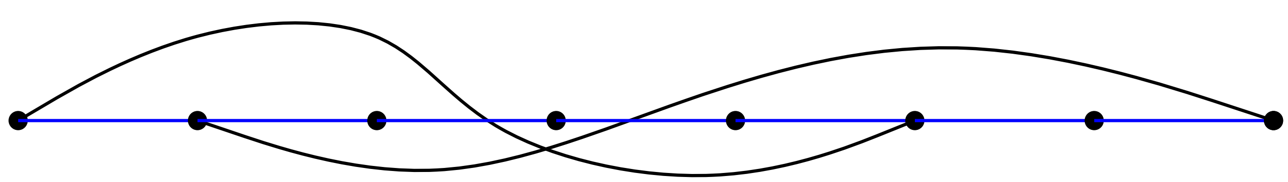

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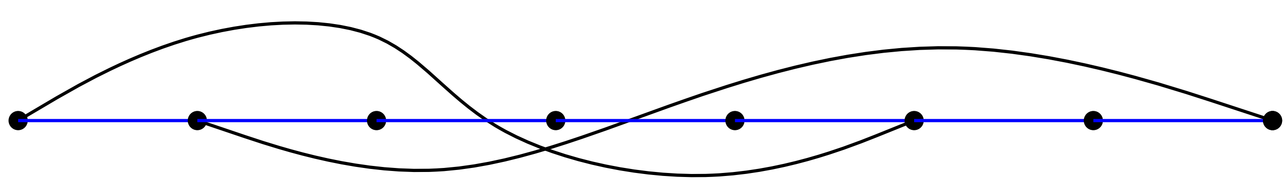

* In the rest of the talk, sketch of the main ideas and a further extension: t-shellable drawings.

Main ideas of the proof (for monotone drawings)

* The proof is by induction.

  We remove point n, and denote by D' the corresponding drawing of K_{n-1}.

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induction hypothesis

j-edges adjacent to n
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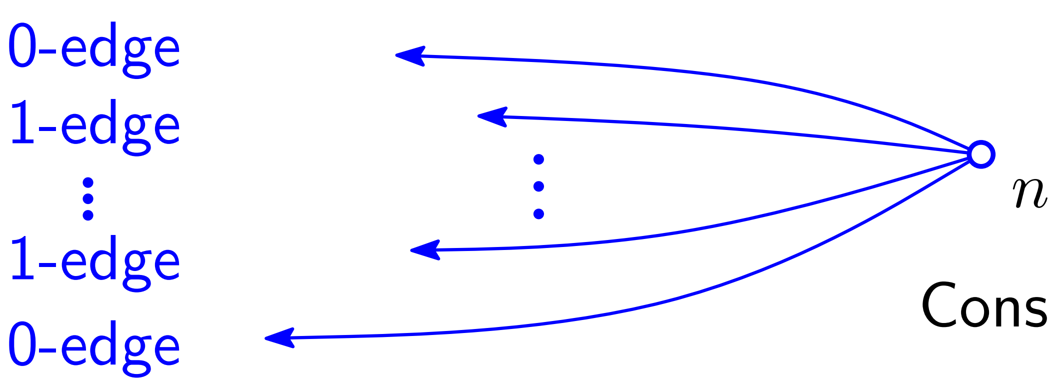

From now on, we consider unoriented j-edges

Consider always the "light" side ($j \leq n/2 - 1$)

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j-edges adjacent to n $j = 0, \ldots, k$

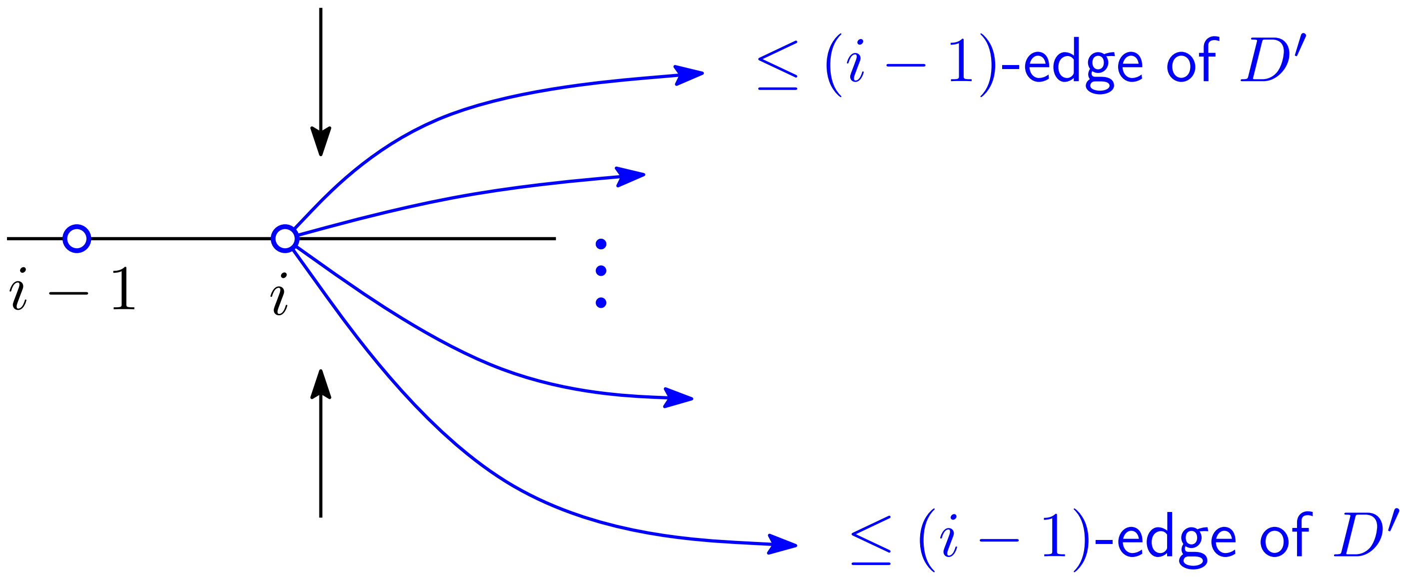
invariant $\leq k$-edges

* A j-edge of D' is an $\leq k$-invariant edge if it is also a j-edge of D (and $j \leq k$).

  So an edge is invariant if vertex n lies on the "heavy side" of the edge.

Finding invariant edges

* Consider the edges starting at i and order them vertically.



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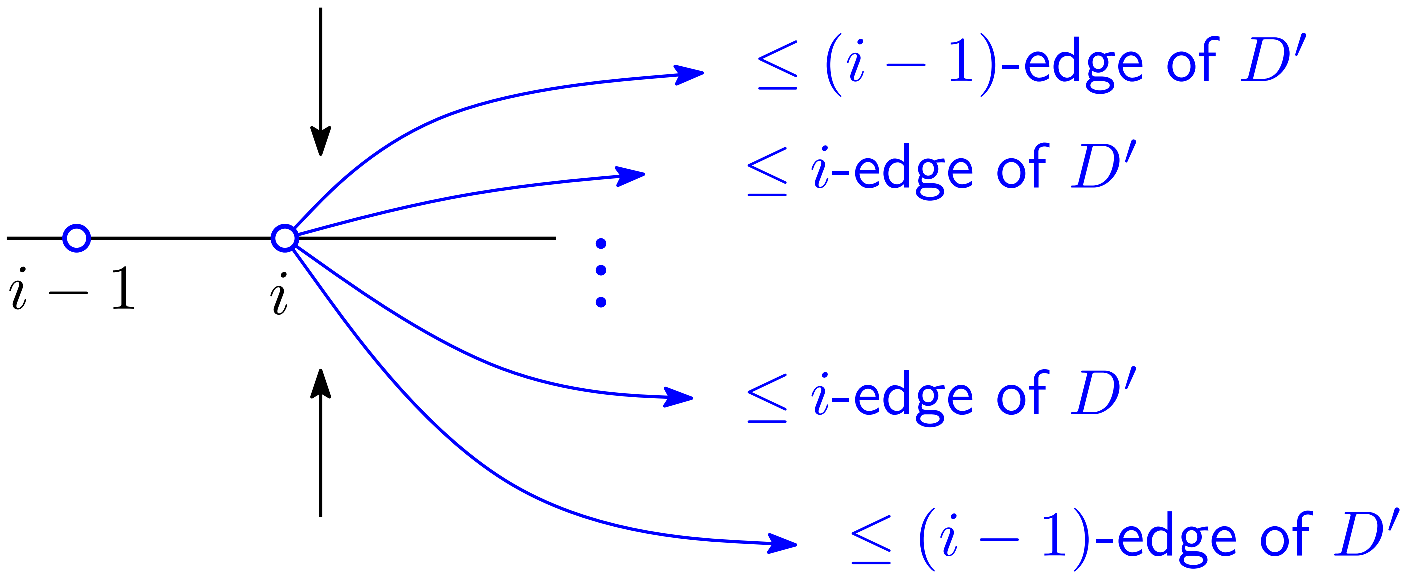

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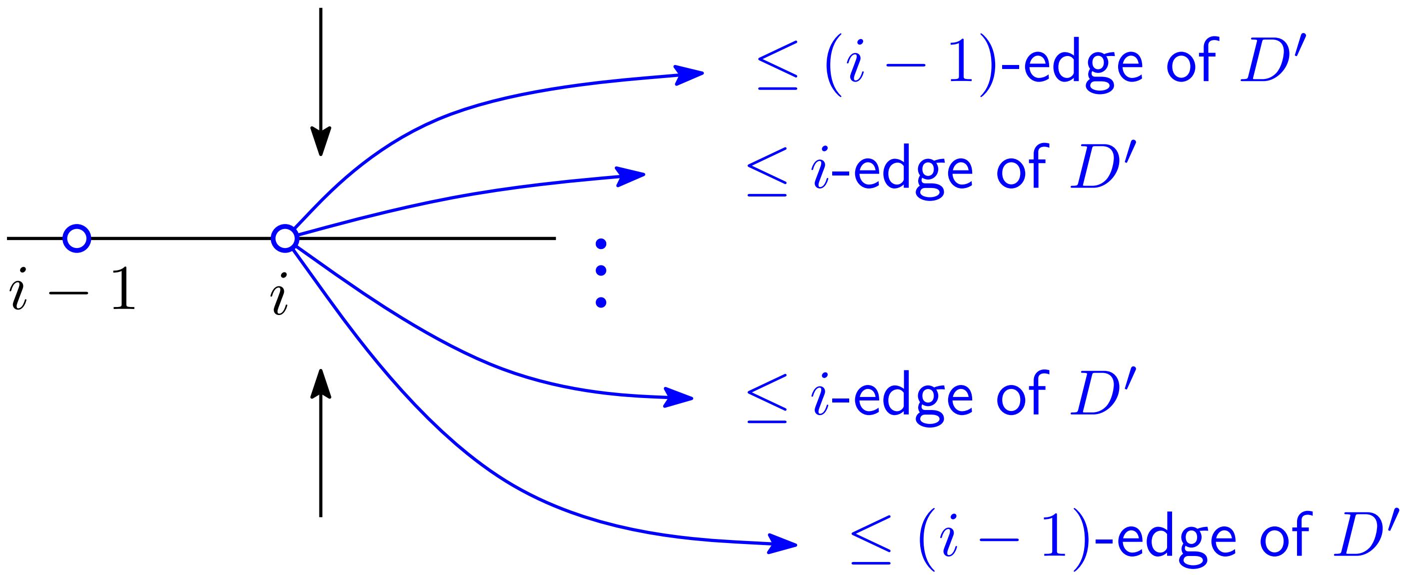
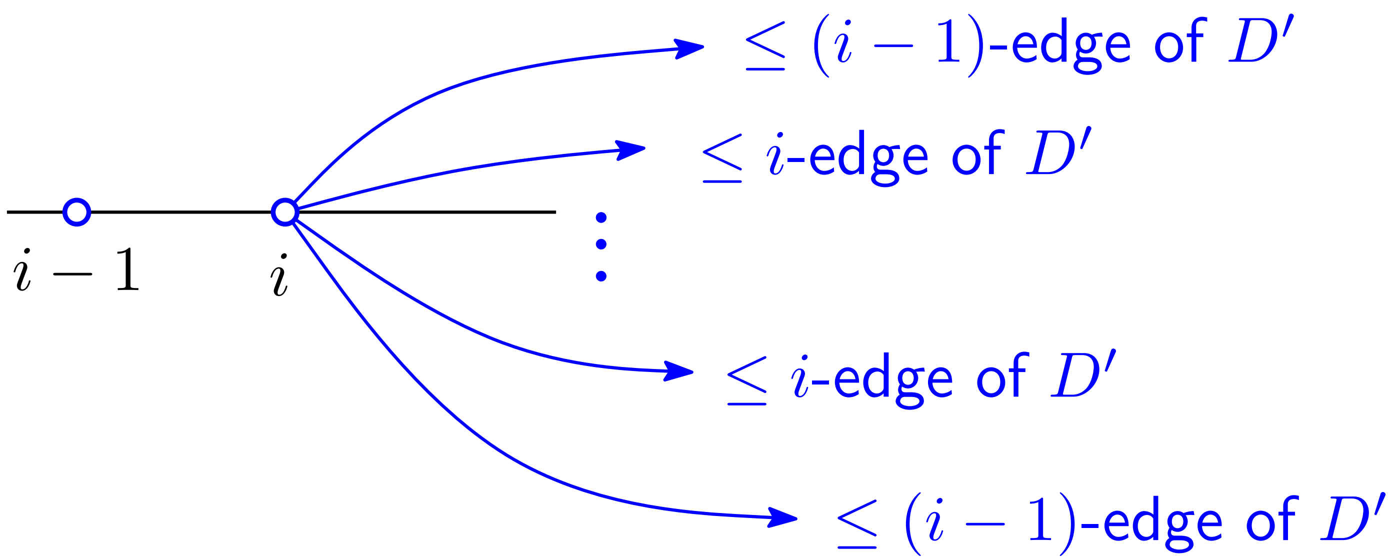


* The m-th edge in the top-down order, is an $\leq (i+m-2)$-edge (while $i+m-2 \leq n/2-1$).

  The same is true for the bottom-up order.

Finding invariant edges



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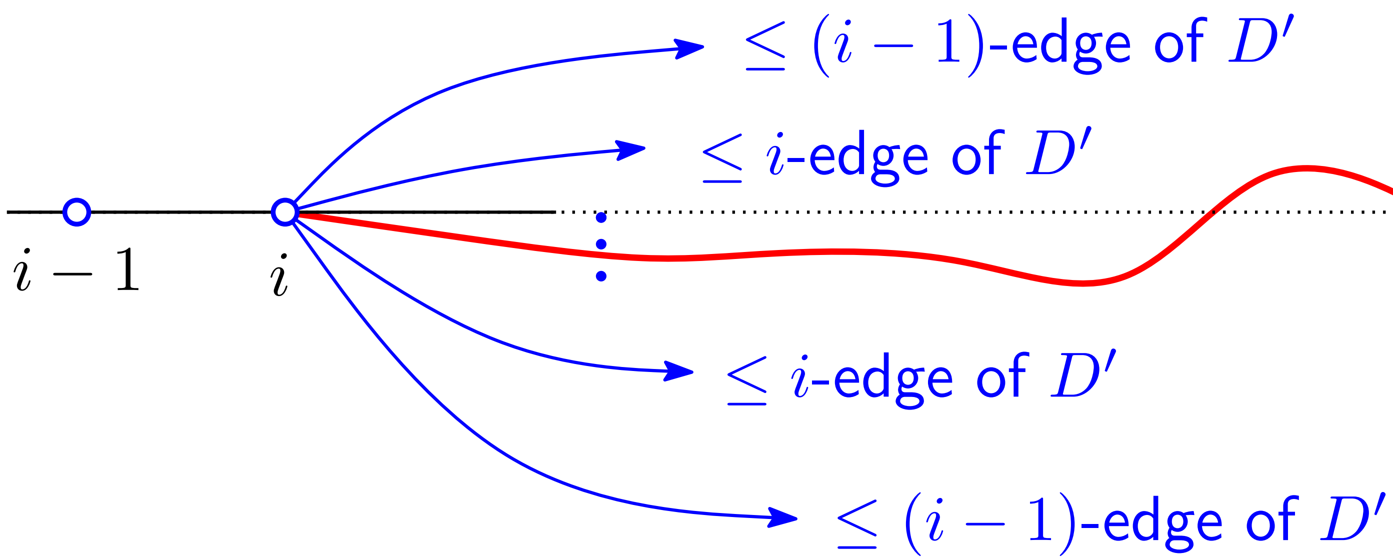

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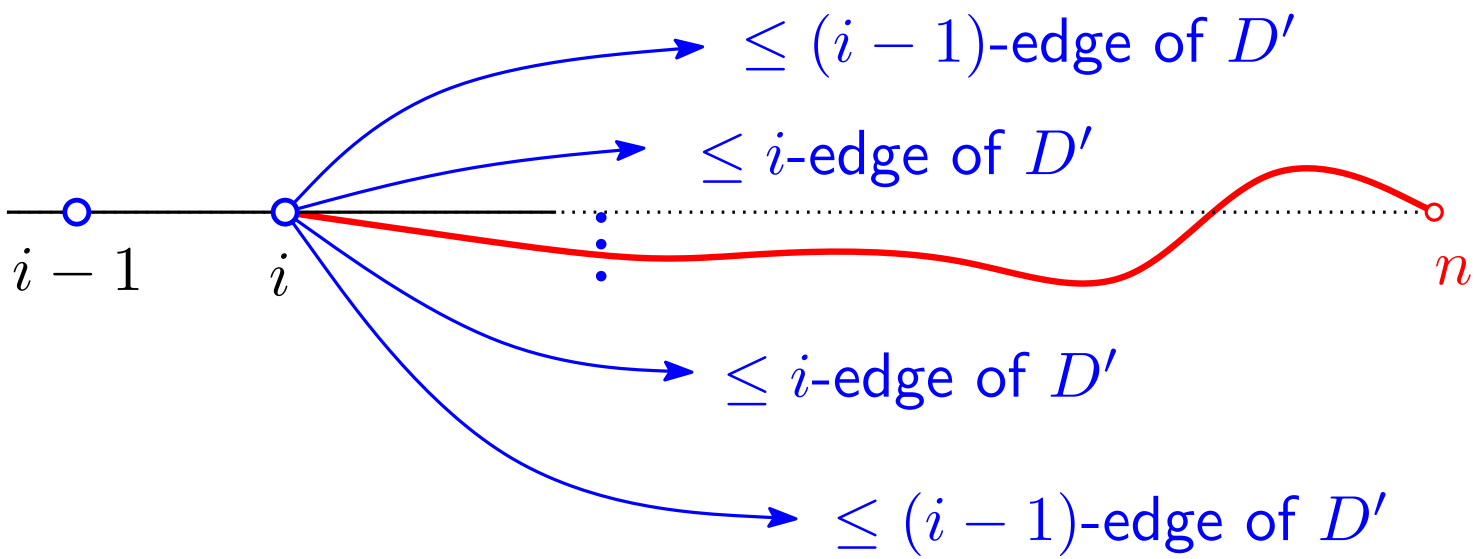

* Are there enough invariant edges?

* Sweep (top-down and bottom-up) edges starting at i: all the edges that we find before reaching in (or half of the edges) are invariant.

Finding invariant edges

* Invariant $\leq k$-edges starting at i. At least

$$\left.\begin{array}{l} \text{one} \leq (i-1)\text{-edge} \\ \text{one} \leq i\text{-edge} \\ \quad\vdots \\ \text{one} \leq k\text{-edge} \end{array}\right\} \quad k-i+2 \text{ invariant} \leq k\text{-edges starting at vertex } i.$$

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Lower bound

* Using $\mathrm{cr}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq\leq k}(D) + c_n$

$$E_{\leq\leq k}(D) \geq 3\binom{k+3}{3} \Rightarrow \mathrm{cr}(D) \geq Z(n)$$

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* What properties of monotone drawings are we really using in the proof?

Shellable drawings

* Only that vertices i and n are on the boundary of the drawing obtained when vertices $1, 2, \ldots, i-1$ are deleted.

  boundary of D: boundary of the unbounded face

  Vertex $n-1$ is also on the boundary when vertex n is deleted, and so on.

Shellable drawings

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* For $1 \leq i < j \leq t$, let D_{ij} be the drawing obtained from D by removing vertices $\{v_1, \ldots, v_{i-1}, v_{j+1}, \ldots, v_t\}$.

  A drawing D of K_n is t-shellable if there exists a subset of vertices $S = \{v_1, v_2, \ldots, v_t\}$ and a face F such that for all $1 \leq i < j \leq t$ vertices v_i and v_j are on the boundary of the face of D_{ij} containing F.

t-shellable drawings

- Examples:
  - monotone drawings are n-shellable.
  - x-bounded drawings are n-shellable.

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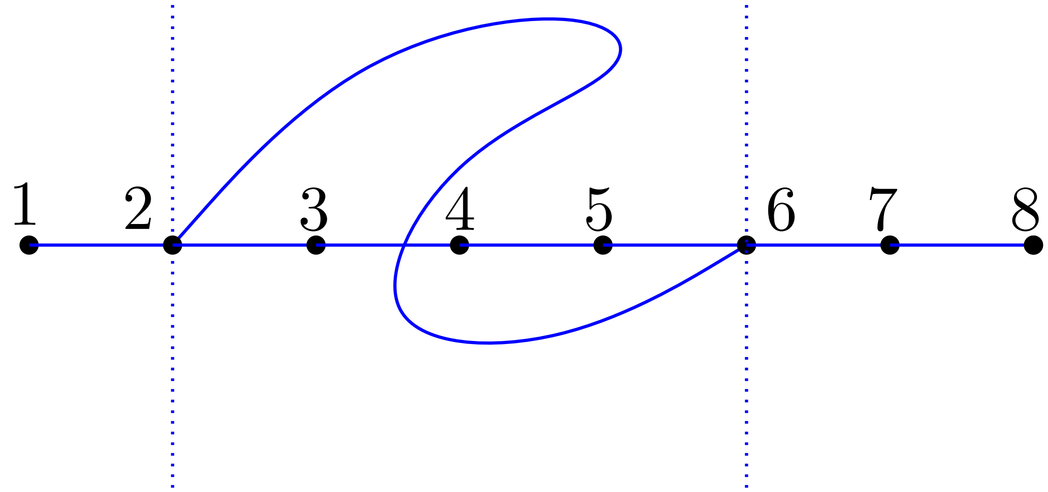

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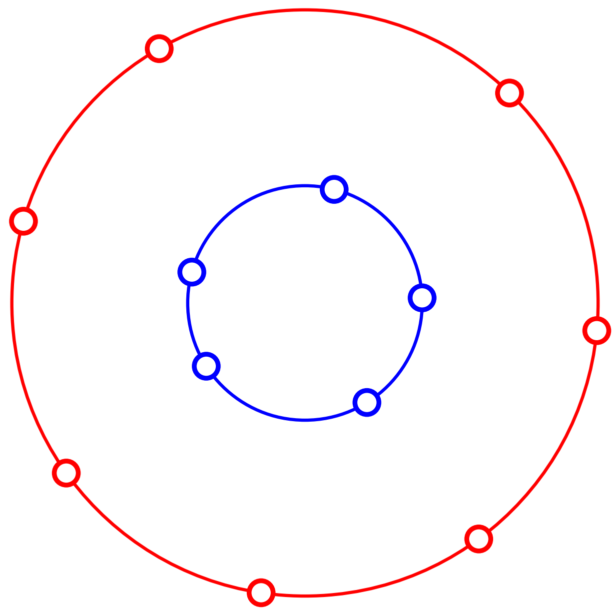
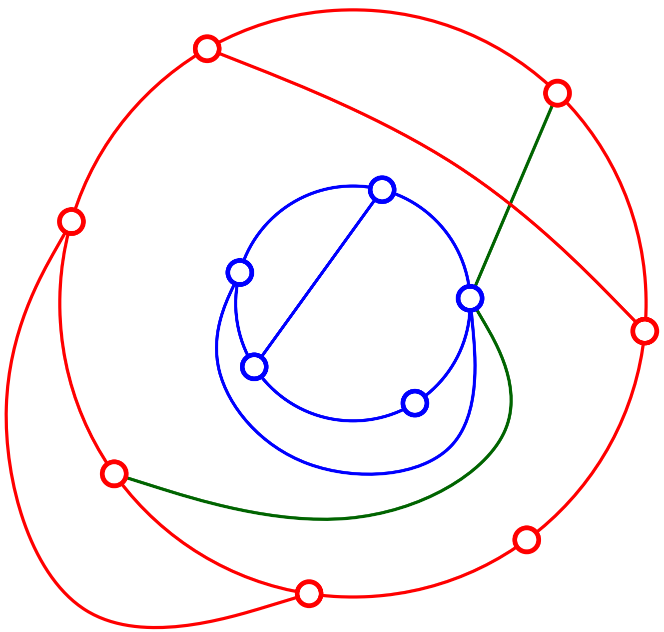
Cylindrical drawings



A drawing is cylindrical if it contains two crossing-free cycles spanning the set of vertices.

Cylindrical drawings



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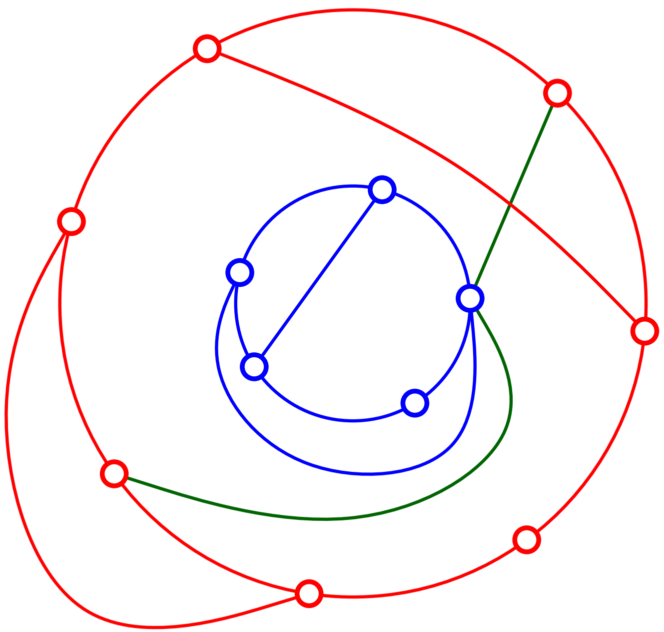

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Partial results for equal size sets [Richter-Thomassen'97]

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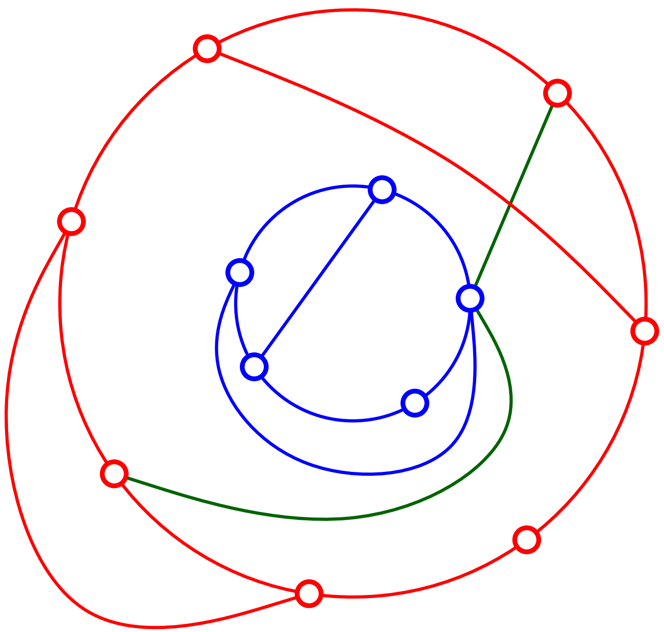

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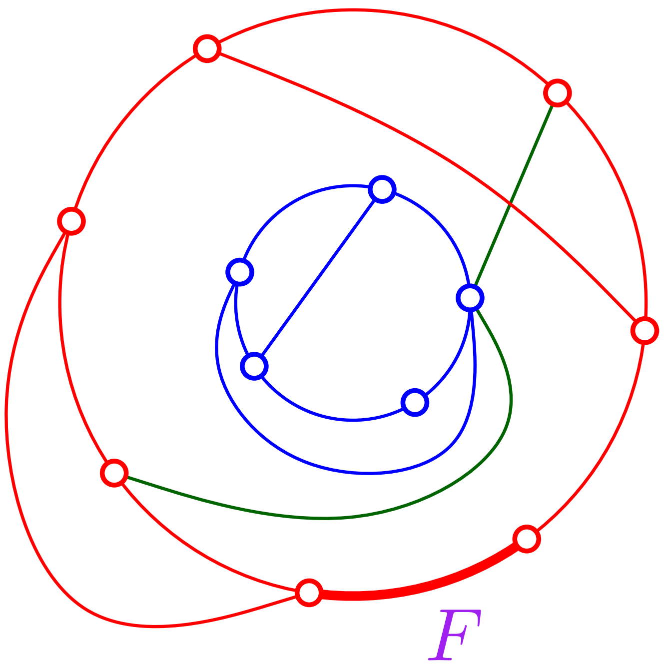

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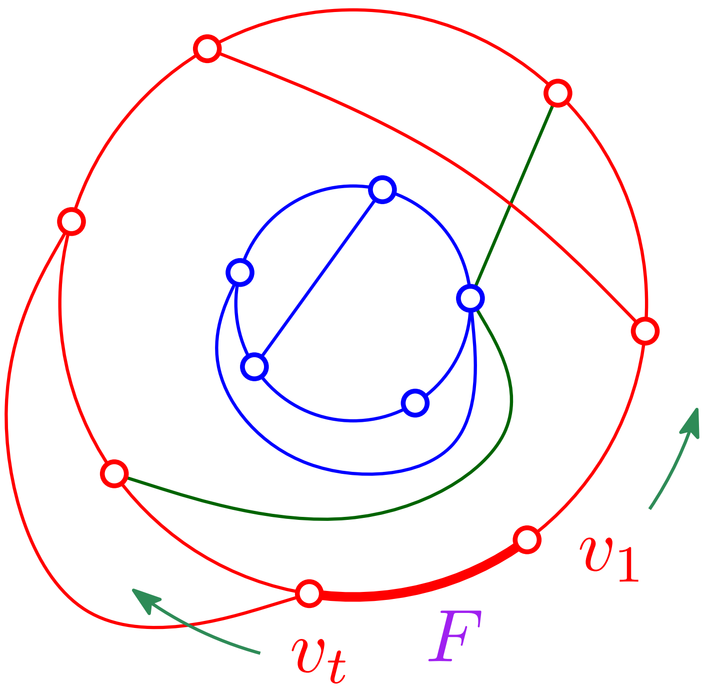

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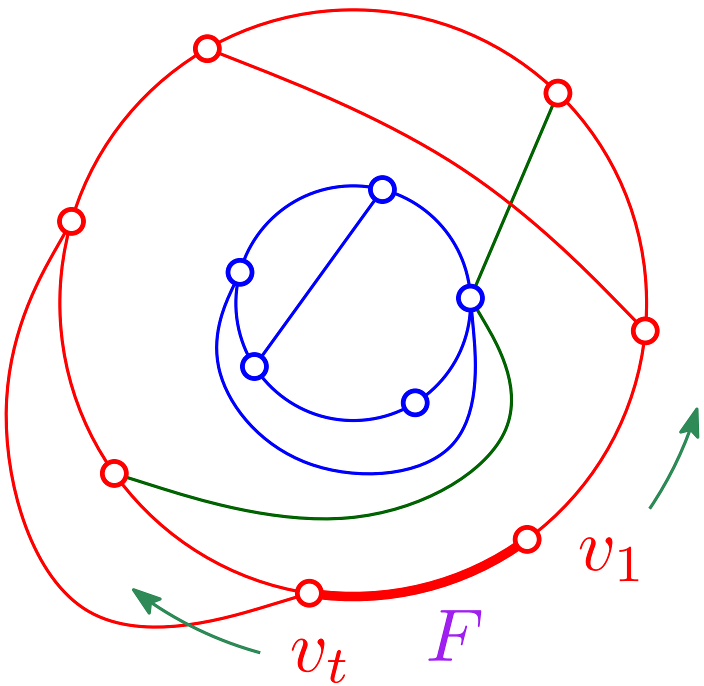

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* Any cylindrical drawing of K_n is $n/2$-shellable.

* The number of crossings in any cylindrical drawing of K_n is at least $Z(n)$.

Conclusions

* Two known families of optimal drawings:
  - ▷ 2-page drawings
  - ▷ cylindrical drawings

  Lower bound known for those families.

Conclusions

* Two known families of optimal drawings:
  - ▷ 2-page drawings
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  Lower bound known for those families.

* Open problems:
  - ▷ other families of optimal drawings?
  - ▷ prove that they are really optimal!

Shellable drawings and the crossing number of the complete graph

Thank you for your attention

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