A Polyhedral Proof of the Matrix-Tree Theorem

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joint work with

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The Matrix-Tree Theorem

- $G = (V, E)$ connected graph on $n$ vertices; undirected

![Graph diagram with vertices 1, 2, and 3 connected by edges]

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The Matrix-Tree Theorem

- $G = (V, E)$ connected graph on $n$ vertices; directed arbitrarily

![Diagram](1-2-3)
The Matrix-Tree Theorem

- $G = (V, E)$ connected graph on $n$ vertices; directed arbitrarily

![Directed Graph Diagram]

- Incidence matrix

$$
N = \begin{bmatrix}
1 & e_1 & e_2 \\
2 & -1 & 1 \\
3 & 1 & -1 \\
3 & 1 & 1
\end{bmatrix}
$$
The Matrix-Tree Theorem

- \( G = (V, E) \) connected graph on \( n \) vertices; directed arbitrarily

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
& \rightarrow & \\
& 3 &
\end{array}
\]

- Incidence matrix \( N = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \); depends on orientation
The Matrix-Tree Theorem

- \( G = (V, E) \) connected graph on \( n \) vertices; directed arbitrarily

\[ 1 \rightarrow 2 \rightarrow 3 \]

- Incidence matrix \( N = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \); depends on orientation

- Laplacian Matrix

\[
L = NN^\top = \begin{bmatrix}
\deg v_1 & -1 & \cdots & -1 \\
-1 & \deg v_2 & \cdots & -1 \\
-1 & \cdots & \deg v_n
\end{bmatrix} = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix}
\]
The Matrix-Tree Theorem

- $G = (V, E)$ connected graph on $n$ vertices; directed arbitrarily

![Graph](image)

- Incidence matrix $N = \begin{bmatrix} -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$; depends on orientation

- Laplacian Matrix $L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$; independent of orientation
The Matrix-Tree Theorem

- \( G = (V, E) \) connected graph on \( n \) vertices; directed arbitrarily

\[ \begin{array}{ccc}
1 & \rightarrow & 2 \\
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Theorem (Kirchhoff 1847)

- If \( G \) has \( s \) spanning trees, and
- \( \text{Spec } L = \{0, \lambda_1, \ldots, \lambda_{n-1}\} \),

then \( n \cdot s = \lambda_1 \cdots \lambda_{n-1} \).
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- \( G = (V, E) \) connected graph on \( n \) vertices; directed arbitrarily

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In the example: \( 3 \cdot 1 = 1 \cdot 3 \)
A Geometric Reformulation: Zonotopes

Definition (The zonotope generated by the columns of a matrix)

\[ Z(A) = \sum_v \text{column of } A \text{ conv}\{0, v\} \]

The segments \( \text{conv}\{0, N_i(K_3)\} \)
A Geometric Reformulation: Zonotopes

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The cells corresponding to bases of \( N \)
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The zonotope of N and of L
A Geometric Reformulation: Zonotopes

Definition (The zonotope generated by the columns of a matrix)
\[ Z(A) = \sum_v \text{column of } A \ \text{conv} \{0, v\} \]

Theorem (Polyhedral Matrix Tree Theorem)
\[ G \text{ connected on } n \text{ vertices} \implies \text{vol} Z(L) = n \text{vol} Z(N). \]
From Graphs to Matroids

- **Graphs**
  - spanning forest
  - spanning tree
  - cycle
  - minimal cut

- **Matroids**
  - independent set
  - basis (max. independent set)
  - circuit (min. dependent set)
  - cocircuit

Oriented Graphs
- \( \ker N \)
  - cycle space
- \( \text{im} N \top \)
  - cut space

Oriented Matroids
- \( R \)-span of (oriented) circuits
- \( R \)-span of (oriented) cocircuits

oriented matroid orthogonality
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- \( \ker \mathbf{N} = \) cycle space

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We want matroids where these notions of orthogonality coincide.
Regular Matroids

For a rank $d$ matroid $\mathcal{M}$, the following are equivalent:

- $\mathcal{M}$ regular matroid
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- $\exists$ a T.U. matrix $\mathbf{M}$ with $d$ rows that represents $\mathcal{M}$ over $\mathbb{R}$
Regular Matroids

For a rank $d$ matroid $\mathcal{M}$, the following are equivalent:

- $\mathcal{M}$ regular matroid
- $\exists$ a T.U. matrix $\mathbf{M}$ with $d$ rows that represents $\mathcal{M}$ over $\mathbb{R}$
- $\mathbf{Z}(\mathbf{M})$ tiles $\mathbb{R}^d$

$\mathbf{Z}(\mathbf{M})$
Regular Matroids

For a rank $d$ matroid $\mathcal{M}$, the following are equivalent:

- $\mathcal{M}$ is a regular matroid
- $\exists$ a T.U. matrix $\mathbf{M}$ with $d$ rows that represents $\mathcal{M}$ over $\mathbb{R}$
- $\mathbf{Z}(\mathbf{M})$ tiles $\mathbb{R}^d$

twice the centers of facets of $\mathbf{Z}(\mathbf{M})$
Regular Matroids

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For a rank $d$ matroid $\mathcal{M}$, the following are equivalent:

- $\mathcal{M}$ regular matroid
- $\exists$ a T.U. matrix $\mathbf{M}$ with $d$ rows that represents $\mathcal{M}$ over $\mathbb{R}$
- $\mathbb{Z}(\mathbf{M})$ tiles $\mathbb{R}^d$
- the lattice generated by twice the barycenters of facets of $\mathbb{Z}(\mathbf{M})$ coincides with $\mathbb{Z} \langle L \rangle$

The lattice $\mathbb{Z} \langle L \rangle$
Generalization to Regular Matroids

- $\mathcal{M}$ rank $d$ regular (oriented) matroid
Generalization to Regular Matroids

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  - $\mathbf{M}$ a rank $d$ totally unimodular representation of $\mathcal{M}$ over $\mathbb{R}$
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  - $\mathbf{L} = \mathbf{M}\mathbf{M}^T$
Generalization to Regular Matroids

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Theorem (Regular Matroid MTT)

If $\mathcal{M}$ has $b$ bases, and

- $\text{Spec } \mathbf{L} = \{\lambda_1, \ldots, \lambda_d\}$,

then $b = \lambda_1 \cdots \lambda_d$. 
Generalization to Regular Matroids

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  - $\mathbf{M}$ a rank $d$ totally unimodular representation of $\mathcal{M}$ over $\mathbb{R}$
  - $\mathbf{L} = \mathbf{MM}^{\top}$

**Theorem (Regular Matroid MTT)**

If $\mathcal{M}$ has $b$ bases, and $\text{Spec } \mathbf{L} = \{\lambda_1, \ldots, \lambda_d\}$, then $b = \lambda_1 \cdots \lambda_d$.

**Theorem (Polyhedral Version)**

$\text{vol}(\mathbb{Z}(\mathbf{M})) = \text{vol}(\mathbb{Z}(\mathbf{L}))$
Proof Outline

1 \( \text{vol } \mathbf{Z}(\mathbf{M}) = b \)
Proof Outline

1. \( \text{vol } Z(\mathbf{M}) = b \)
   
   - Take a maximal cubical subdivision of \( Z(\mathbf{M}) \)

\[
\text{vol } Z(\mathbf{L}) = \lambda_1 \cdots \lambda_d
\]

- The columns of \( \mathbf{L} \) form a basis of \( \mathbb{R}^d \)

\[
\Rightarrow Z(\mathbf{L}) \text{ is a parallelepiped}
\]

\[
\Rightarrow \text{vol } Z(\mathbf{L}) = \det \mathbf{L}
\]

\[
\text{vol } Z(\mathbf{L}) = \text{vol } Z(\mathbf{M}) (= b):	ext{ the hard part}
\]

- "decompose and rearrange" argument
Proof Outline

1. \( \text{vol } Z(M) = b \)
   
   - Take a maximal cubical subdivision of \( Z(M) \)
   - Use total unimodularity of \( M \)
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2. \( \text{vol } \mathbf{Z}(\mathbf{L}) = \lambda_1 \cdots \lambda_d \)
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   - Take a maximal cubical subdivision of $\mathbf{Z}(\mathbf{M})$
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2. $\text{vol } \mathbf{Z}(\mathbf{L}) = \lambda_1 \cdots \lambda_d$
   - the columns of $\mathbf{L}$ form a basis of $\mathbb{R}^d$
Proof Outline

1. \( \text{vol } \mathbb{Z}(\mathbf{M}) = b \)
   - Take a maximal cubical subdivision of \( \mathbb{Z}(\mathbf{M}) \)
   - Use total unimodularity of \( \mathbf{M} \)

2. \( \text{vol } \mathbb{Z}(\mathbf{L}) = \lambda_1 \cdots \lambda_d \)
   - the columns of \( \mathbf{L} \) form a basis of \( \mathbb{R}^d \)
   - \( \Rightarrow \mathbb{Z}(\mathbf{L}) \) is a parallelepiped
Proof Outline

1. \( \text{vol } Z(M) = b \)
   - Take a maximal cubical subdivision of \( Z(M) \)
   - Use total unimodularity of \( M \)

2. \( \text{vol } Z(L) = \lambda_1 \cdots \lambda_d \)
   - the columns of \( L \) form a basis of \( \mathbb{R}^d \)
   - \( \Rightarrow Z(L) \) is a parallelepiped
   - \( \Rightarrow \text{vol}(Z(L)) = \det L \)
Proof Outline

1. $\text{vol } Z(M) = b$
   - Take a maximal cubical subdivision of $Z(M)$
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2. $\text{vol } Z(L) = \lambda_1 \cdots \lambda_d$
   - the columns of $L$ form a basis of $\mathbb{R}^d$
   - $\Rightarrow Z(L)$ is a parallelepiped
   - $\Rightarrow \text{vol}(Z(L)) = \det L$

3. $\text{vol } Z(L) = \text{vol } Z(M) (= b)$: the hard part
Proof Outline

1. \( \text{vol } Z(M) = b \)
   - Take a maximal cubical subdivision of \( Z(M) \)
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2. \( \text{vol } Z(L) = \lambda_1 \cdots \lambda_d \)
   - the columns of \( L \) form a basis of \( \mathbb{R}^d \)
   - \( \implies Z(L) \) is a parallelepiped
   - \( \implies \text{vol}(Z(L)) = \det L \)

3. \( \text{vol } Z(L) = \text{vol } Z(M) (= b) \): the hard part
   - “decompose and rearrange” argument
Proof of the hard part: \( \text{vol} \mathbb{Z}(L) = \text{vol} \mathbb{Z}(M) \)

Proof idea:

- Tile \( \mathbb{R}^d \) with the parallelepiped \( \mathbb{Z}(L) \)
- Cut up \( \mathbb{Z}(M) \) into parts by these cells
- Reassemble the parts inside one of the cells
Decomposition of $\mathbb{Z}_0(M)$ by $\mathbb{Z}(L)$

$\mathbb{Z}_0(M) := \mathbb{Z}(M) - \beta(\mathbb{Z}(M))$
Decomposition of $Z_0(M)$ by $Z(L)$

- $Z_0(M) := Z(M) - \beta(Z(M))$

- Decompose $\mathbb{R}^d$ into cones
  \[\sigma_\varepsilon = \mathbb{R}_+ \langle \varepsilon_i L_i \rangle, \varepsilon \in \{-1, 1\}^d\]
Decomposition of $Z_0(M)$ by $Z(L)$

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- Set $\mu_\varepsilon = \sum_{\varepsilon_i = -1} L_i$

Decompose $\mathbb{R}^d$ with $Z(L)$
Decomposition of $Z_0(M)$ by $Z(L)$

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- Set $P_\varepsilon = Z_0(M) \cap \sigma_\varepsilon$

The $P_\varepsilon$
De composition of $Z_0(M)$ by $Z(L)$

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Theorem (D.-Pfeifle)

$Z(L) = \bigcup_\varepsilon P_\varepsilon + \mu_\varepsilon$
Decomposition of $Z_0(M)$ by $Z(L)$

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Decomposition of $\mathbb{Z}_0(\mathbb{M})$ by $\mathbb{Z}(\mathbb{L})$

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$P_{+-} + \mathbb{L}_2$
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Theorem (D.-Pfeifle)

\[ \mathbf{Z}(\mathbf{L}) = \bigcup_\varepsilon P_\varepsilon + \mu_\varepsilon \]
Decomposition of $Z_0(M)$ by $Z(L)$

- $Z_0(M) := Z(M) - \beta(Z(M))$
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$$P_{--} + L_1 + L_2$$
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Theorem (D.-Pfeifle)

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**Theorem (D.-Pfeifle)**

$\mathbf{Z}(\mathbf{L}) = \bigcup_\varepsilon P_\varepsilon + \mu_\varepsilon$
\textbf{Z(M) for } G = K_4
$\mathbb{Z}(M)$ with the $\pm L_i$
The Polytopes $P_\epsilon$
The Polytopes $P_\epsilon$
The Polytopes $P_e$
Shifting $P_\epsilon$ by $\mu_\epsilon$
Shifting $P_\varepsilon$ by $\mu_\varepsilon$
Shifting $P_\epsilon$ by $\mu_\epsilon$
Shifting $P_\epsilon$ by $\mu_\epsilon$
$Z(L)$ for $G = K_4$
Thank You!