Planar Packing of Binary Trees

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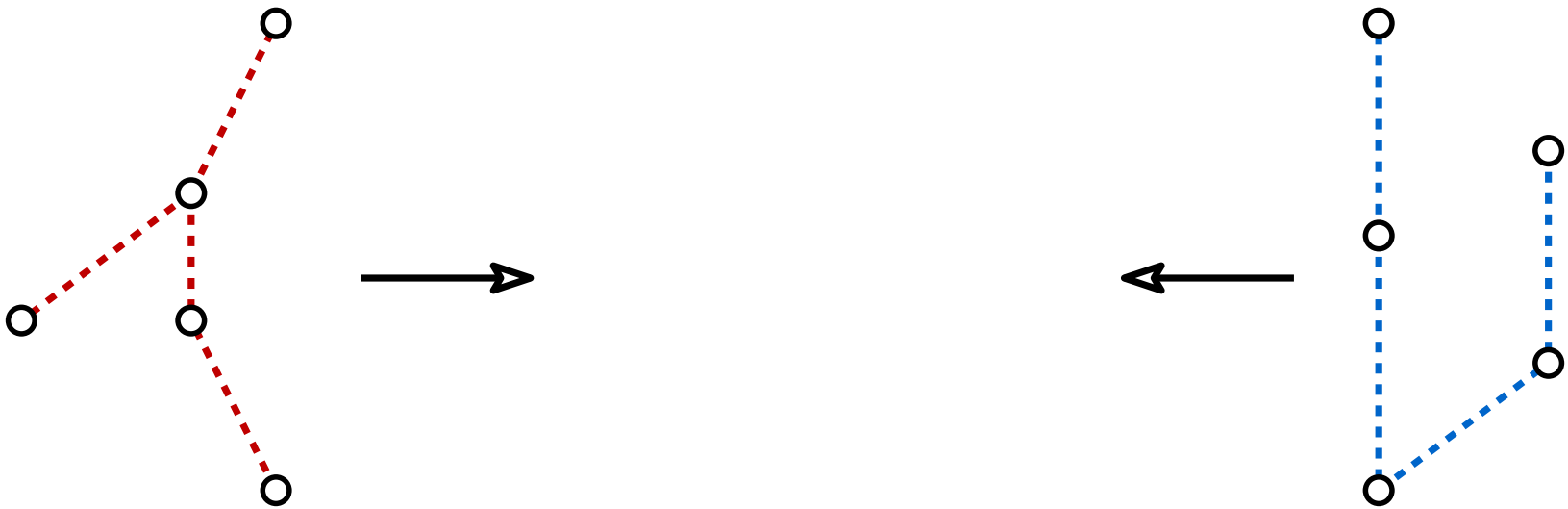
Introduction

Graph packing: given k graphs on n vertices, is there a graph on n vertices that contains them as edge-disjoint subgraphs?



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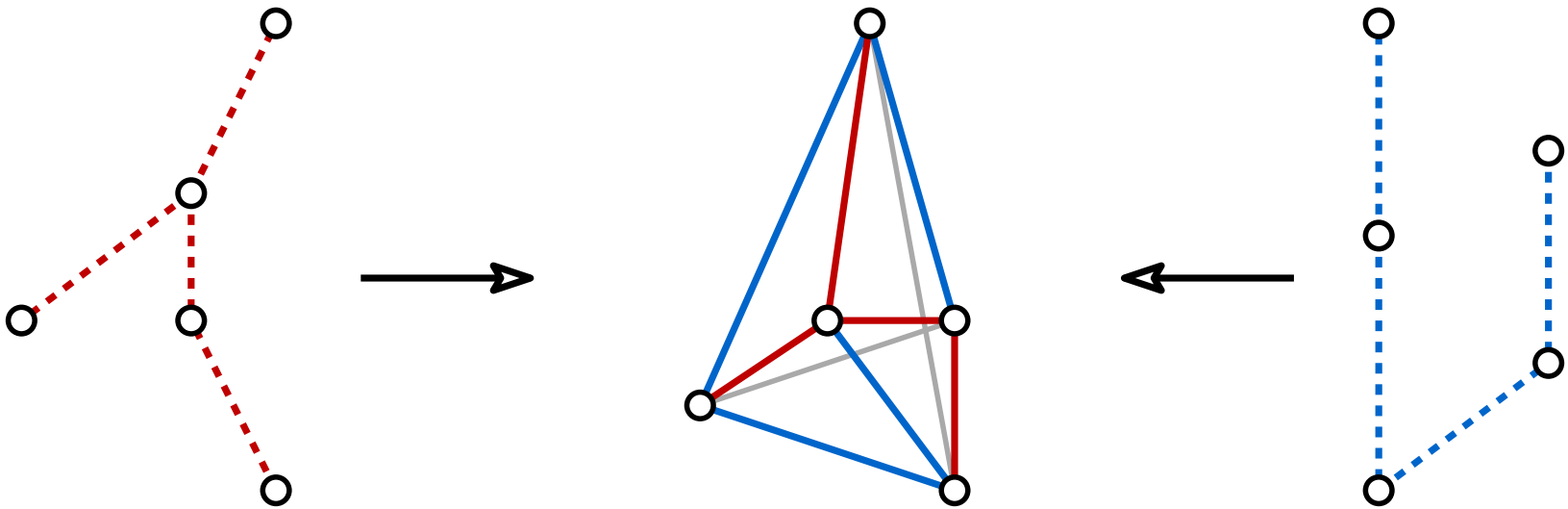

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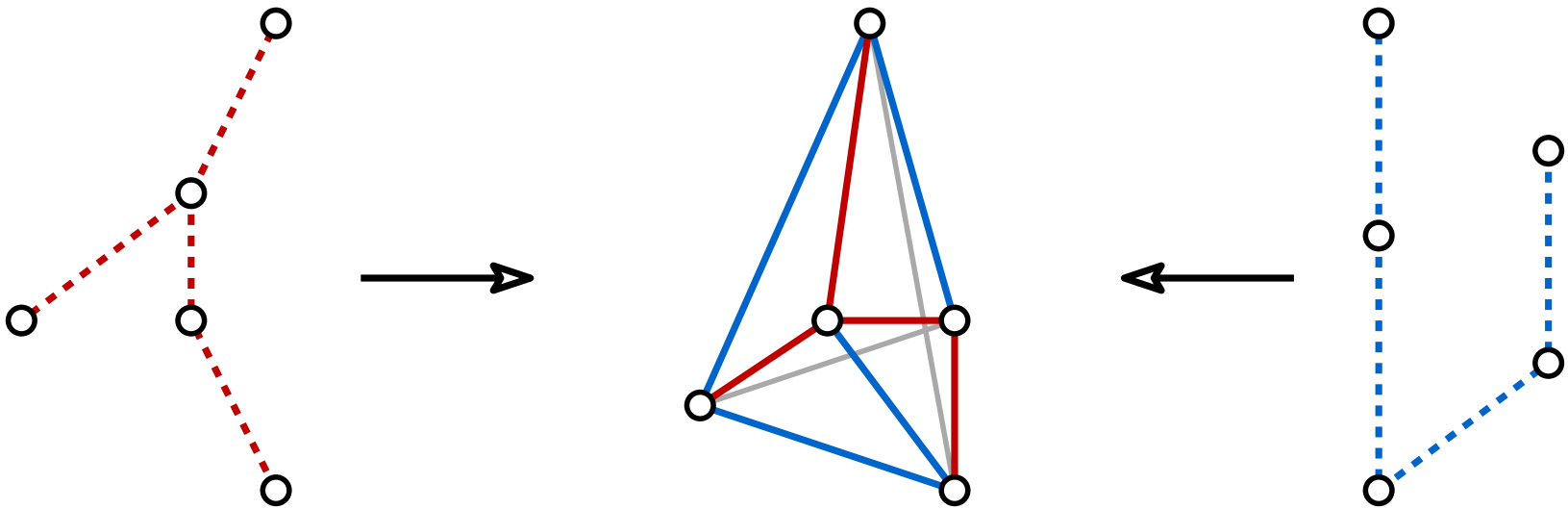

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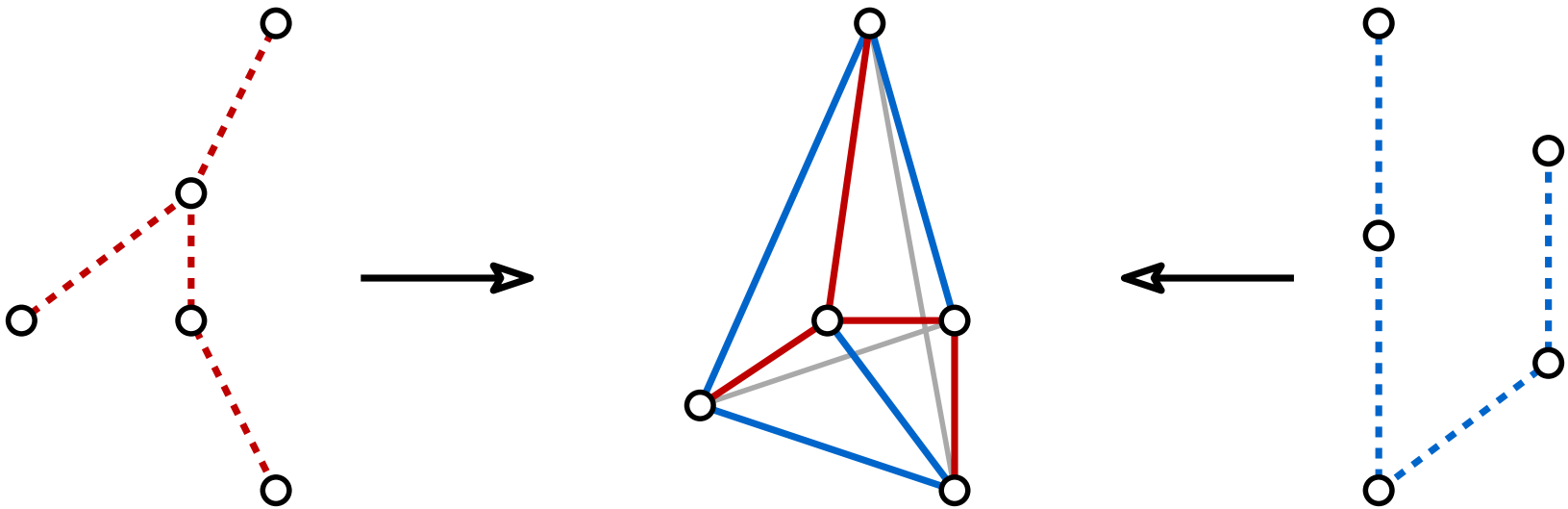

- Characterization of the pairs of trees that admit a packing into K_n. (Hedetniemi, Hedetniemi, and Slater 1981)
- Characterization of the triples of trees that admit a packing into K_n. (Maheo, Saclé, and Woźniak 1996)

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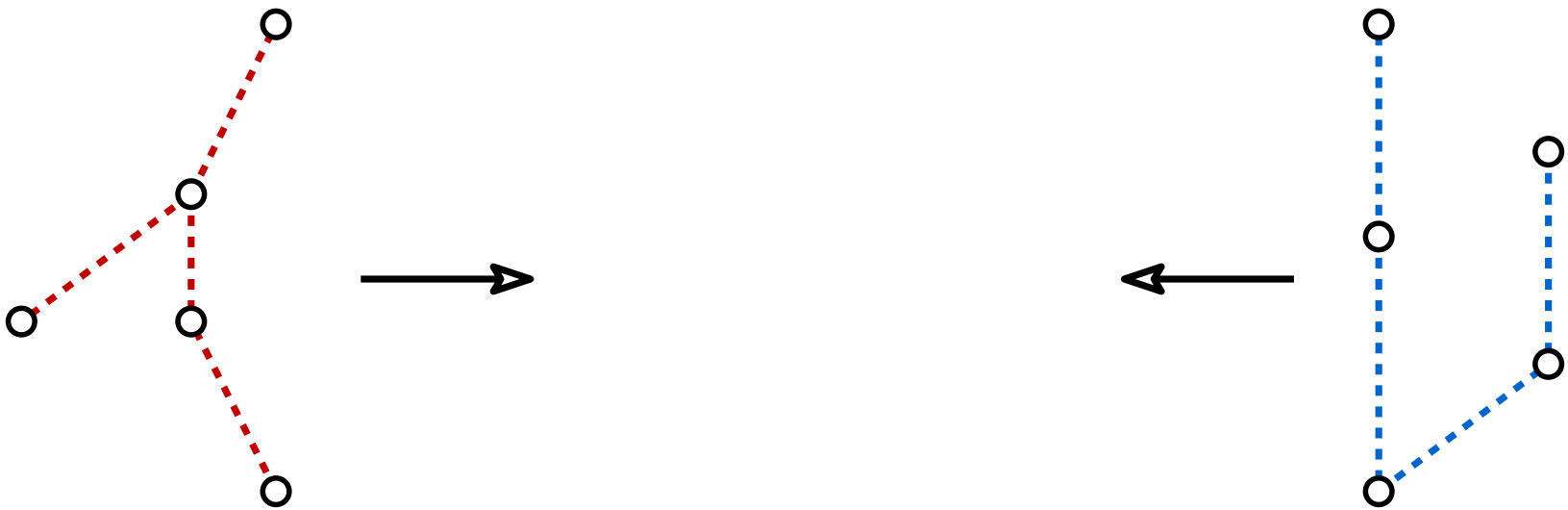

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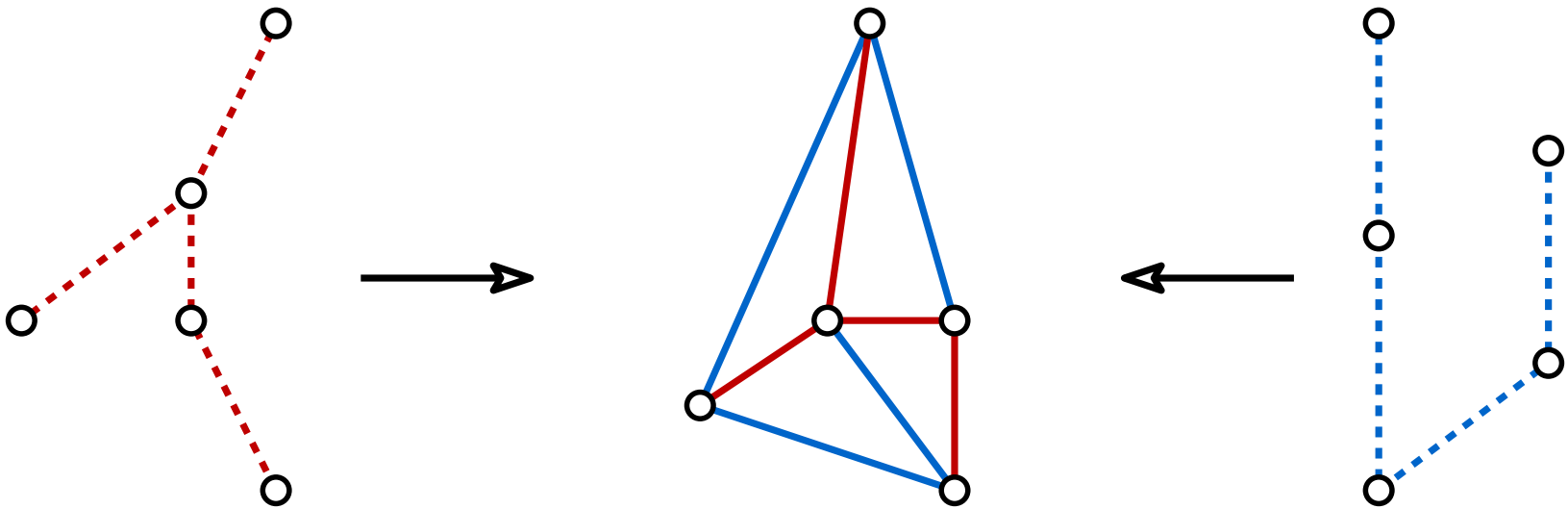

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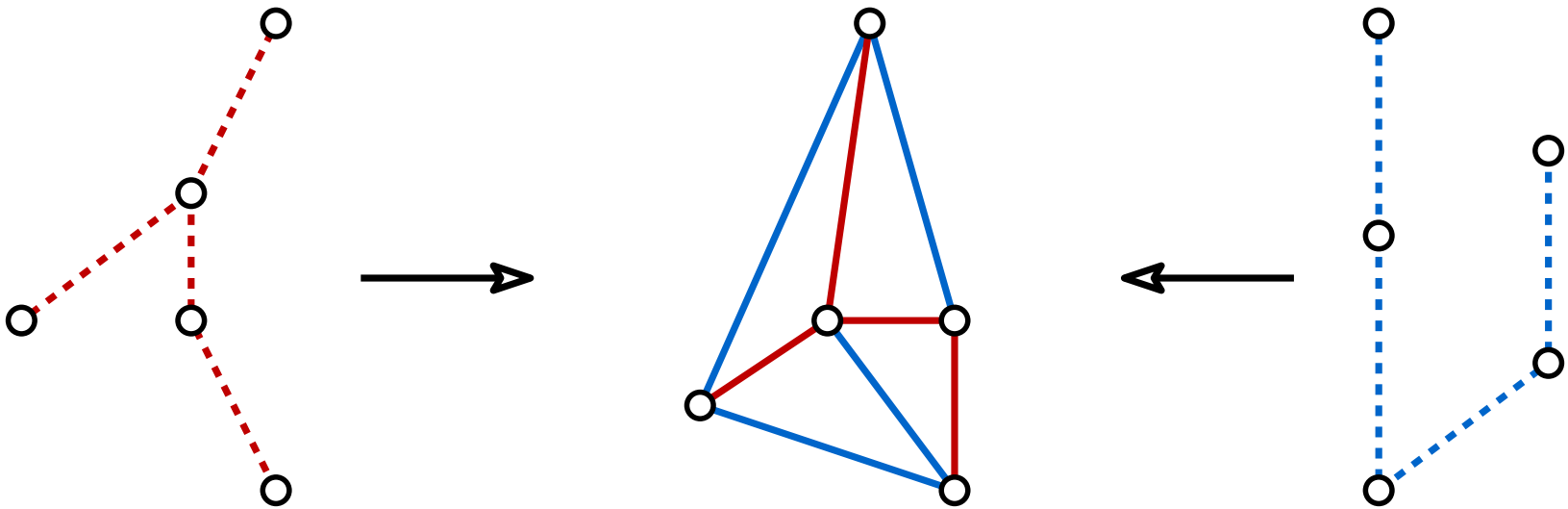

Conjecture (García, Hernando, Hurtado, Noy, and Tejel 2002)

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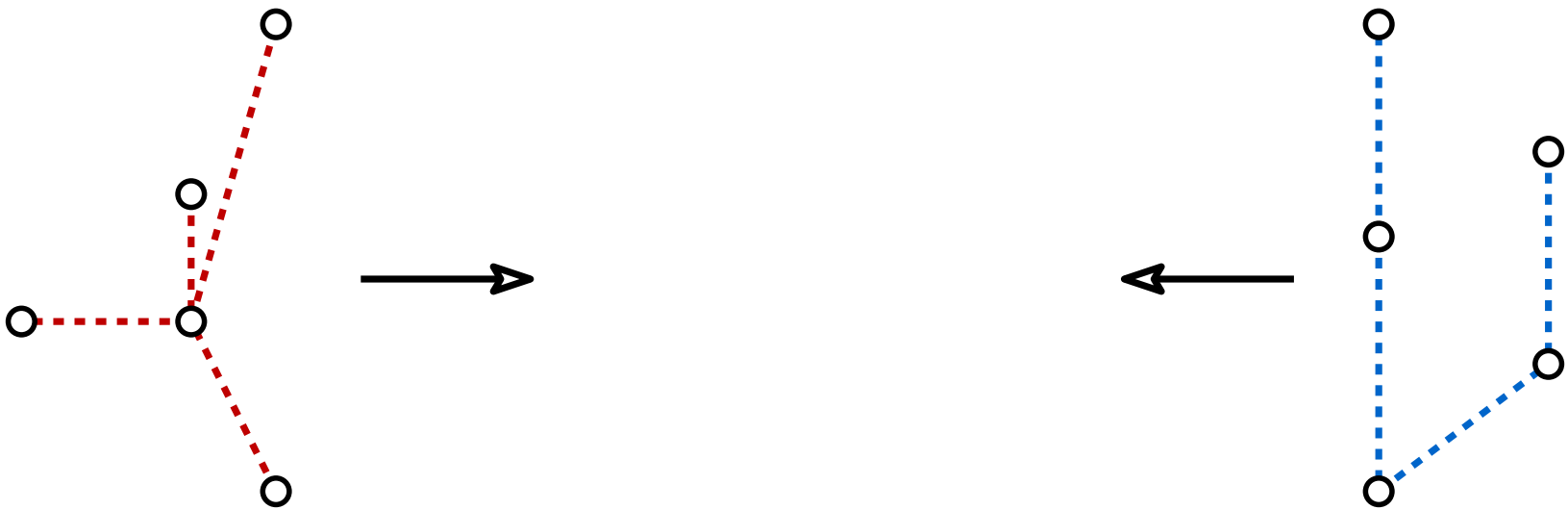

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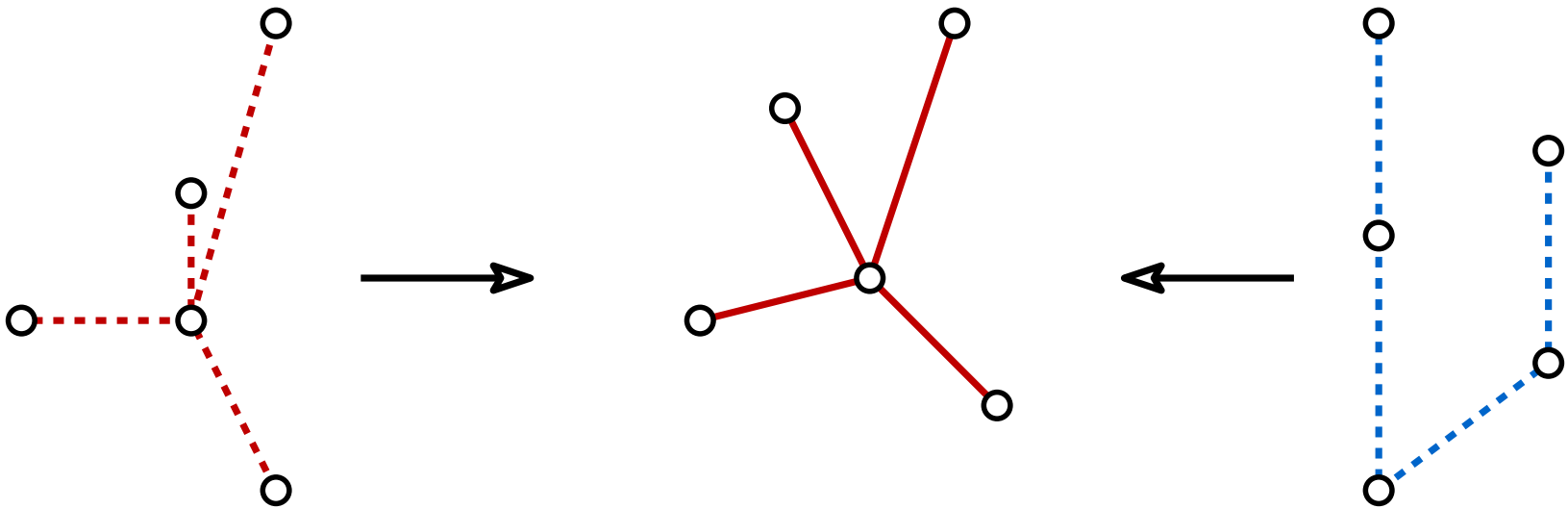

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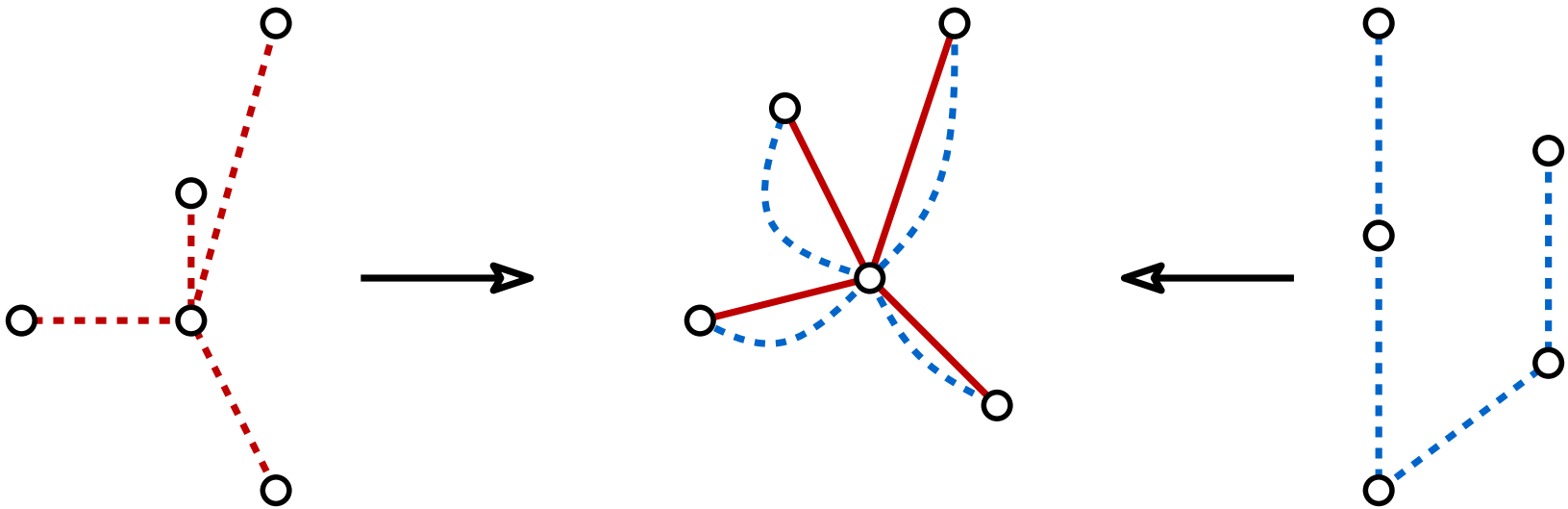

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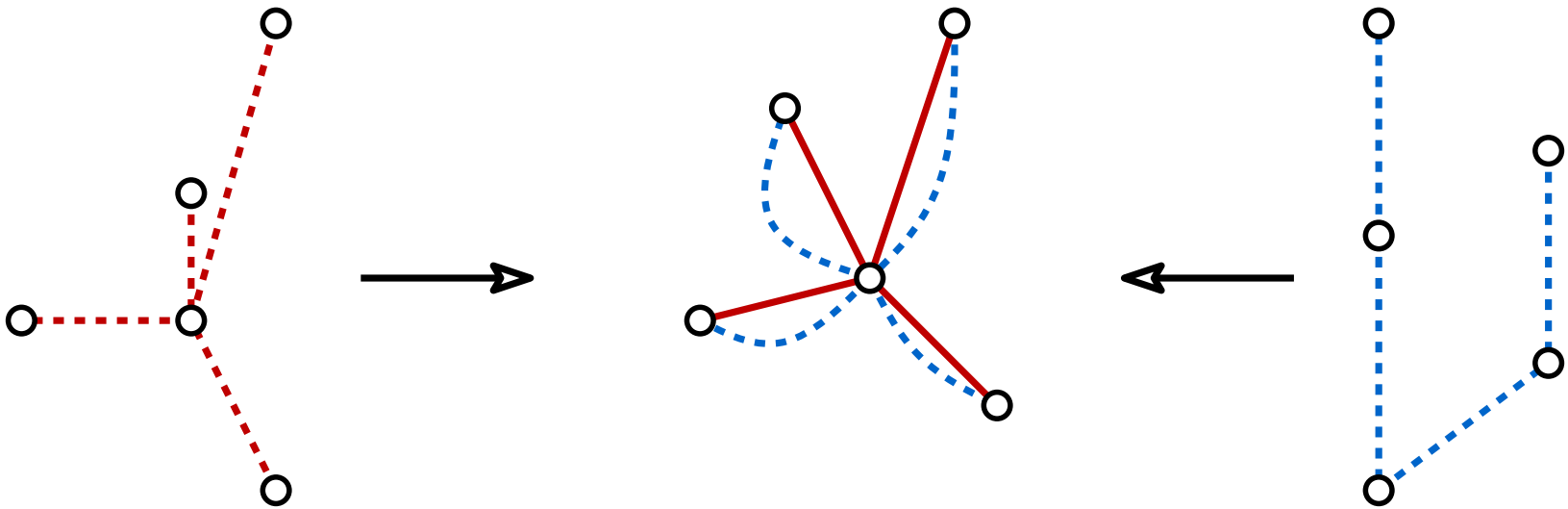

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Every two non-star trees on n vertices admit a planar packing.

Planar tree packing

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Every two non-star trees on n vertices admit a planar packing.

Solved special cases:

- Isomorphic trees (García, Hernando, Hurtado, Noy, and Tejel 2002).
- One tree is a path (García, Hernando, Hurtado, Noy, and Tejel 2002).
- One tree is a caterpillar or a spider of diameter at most four (Oda and Ota 2006).
- One tree is a spider (Frati, Geyer, and Kaufmann 2009).
- Both trees have diameter at most four (Frati 2009).

Theorem

Every two non-star binary trees on n vertices admit a planar packing.

General scheme

1. Place n points on the x-axis.
2. Root the blue tree at one of its leaves.
   Recursively embed the blue tree in the upper halfplane.
3. Root the red tree at one of its leaves.
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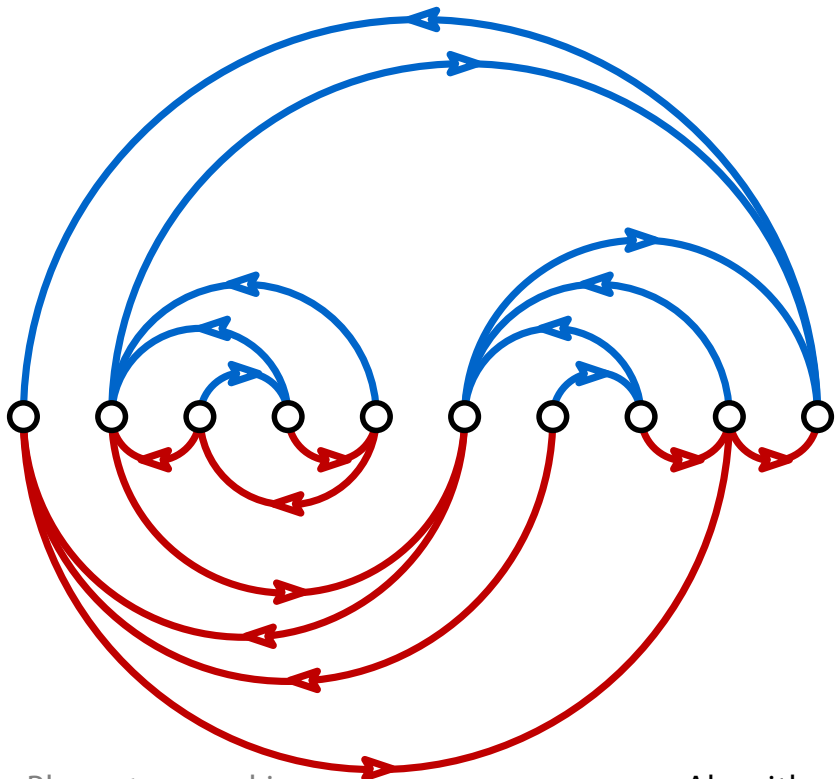
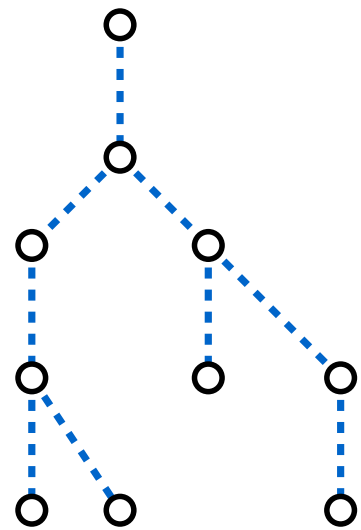


Embedding the blue tree

Embed the blue tree according to two rules:

- **One-side rule**: blue edges incident to each vertex either all go left or all go right.
- **Larger-subtree-first rule**: the larger (if any) subtree is embedded next to its parent.



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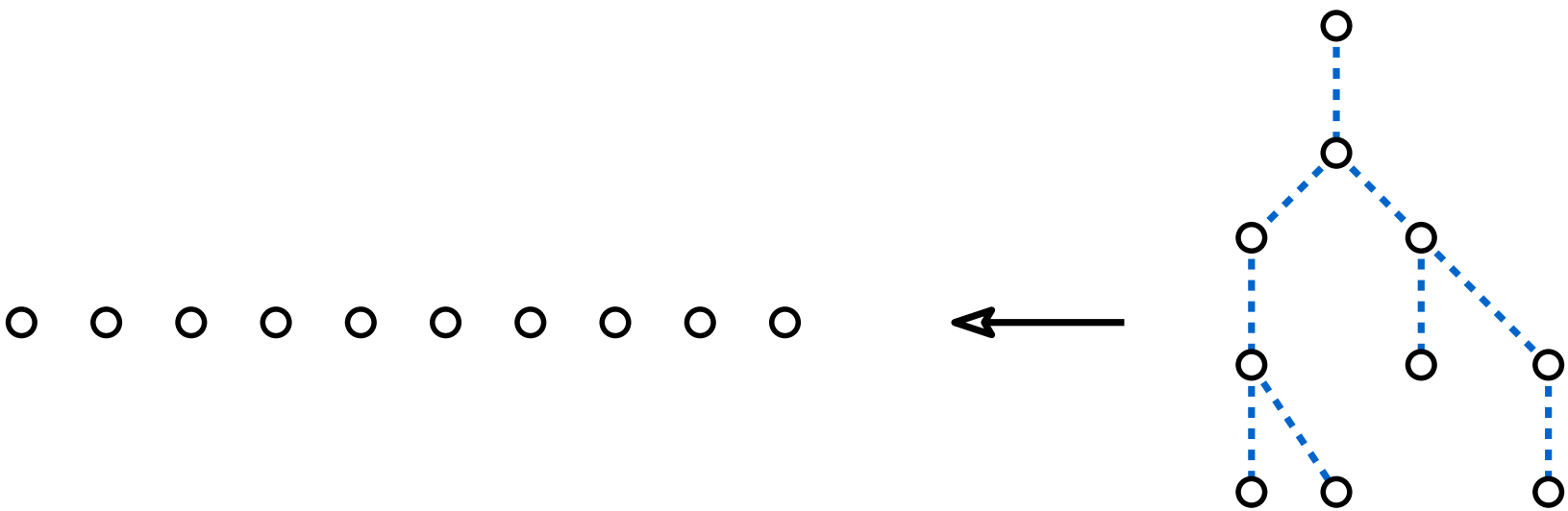

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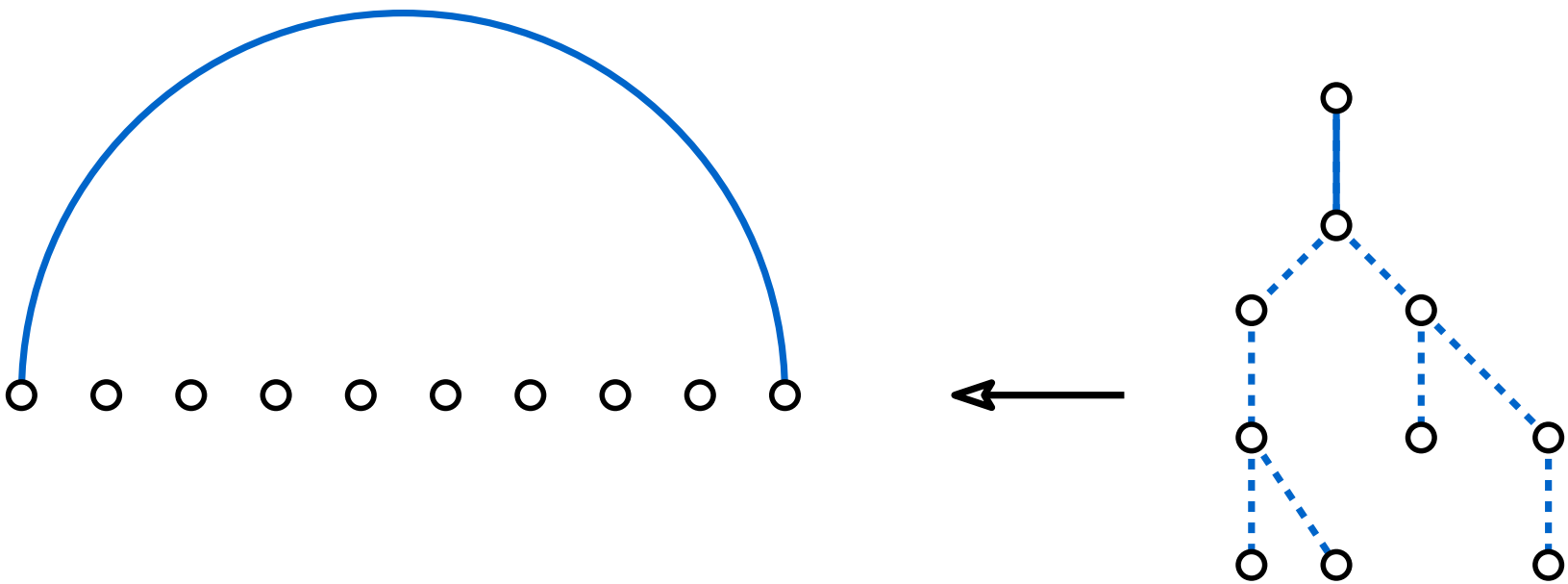

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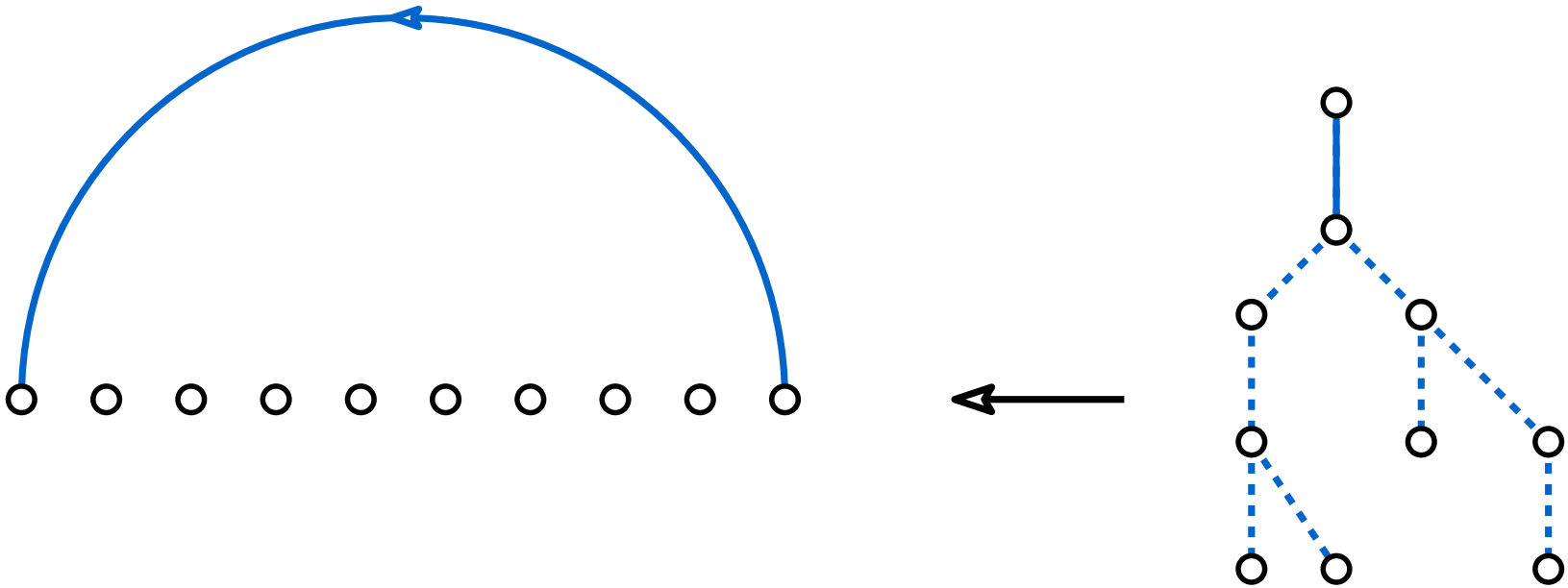

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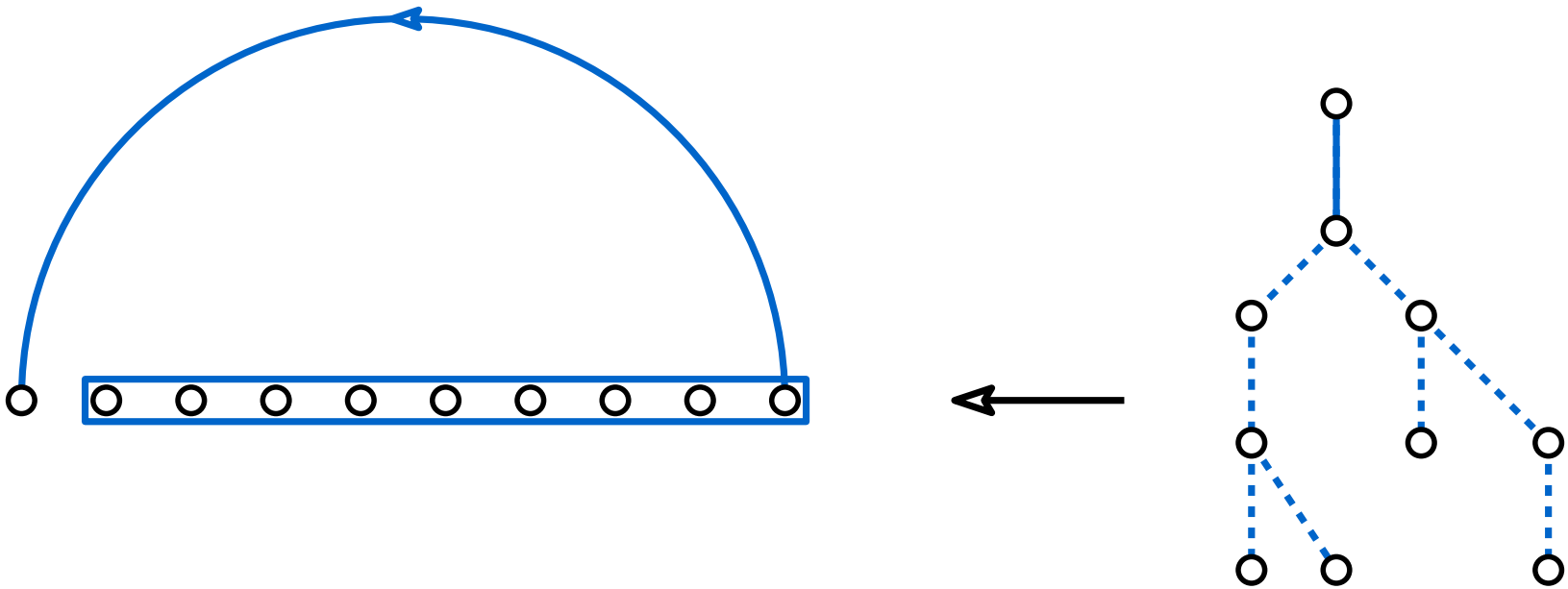

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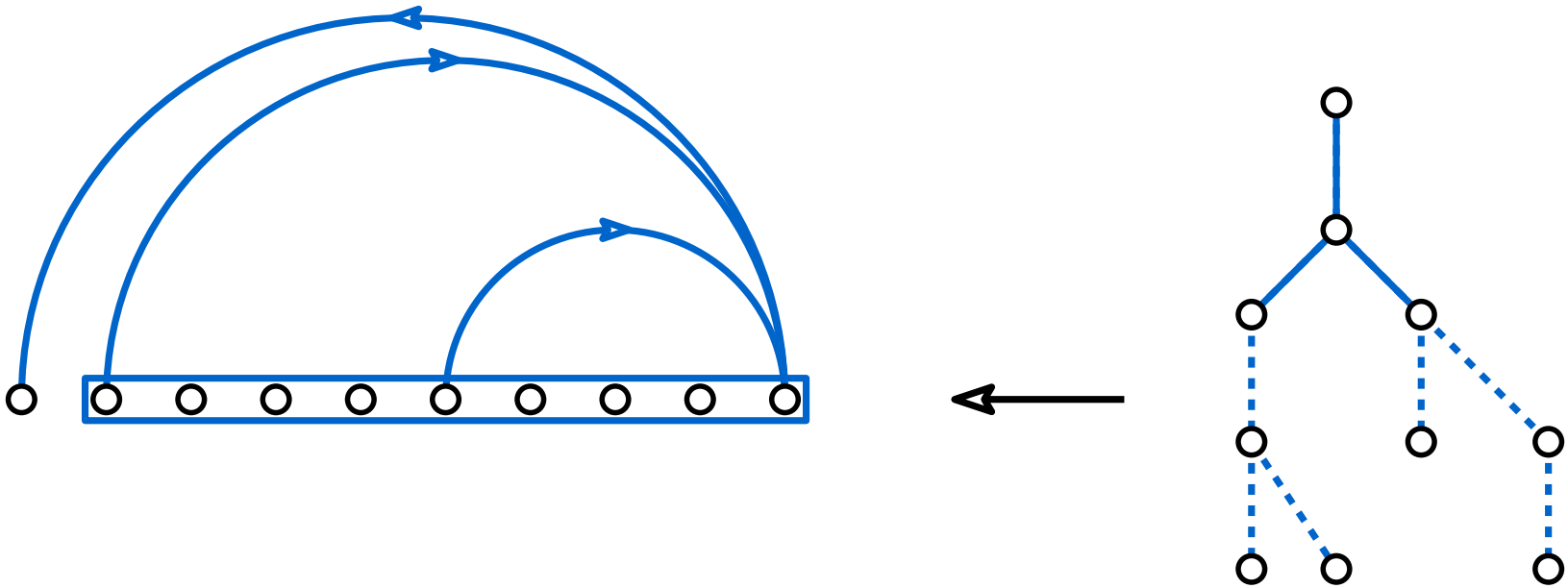

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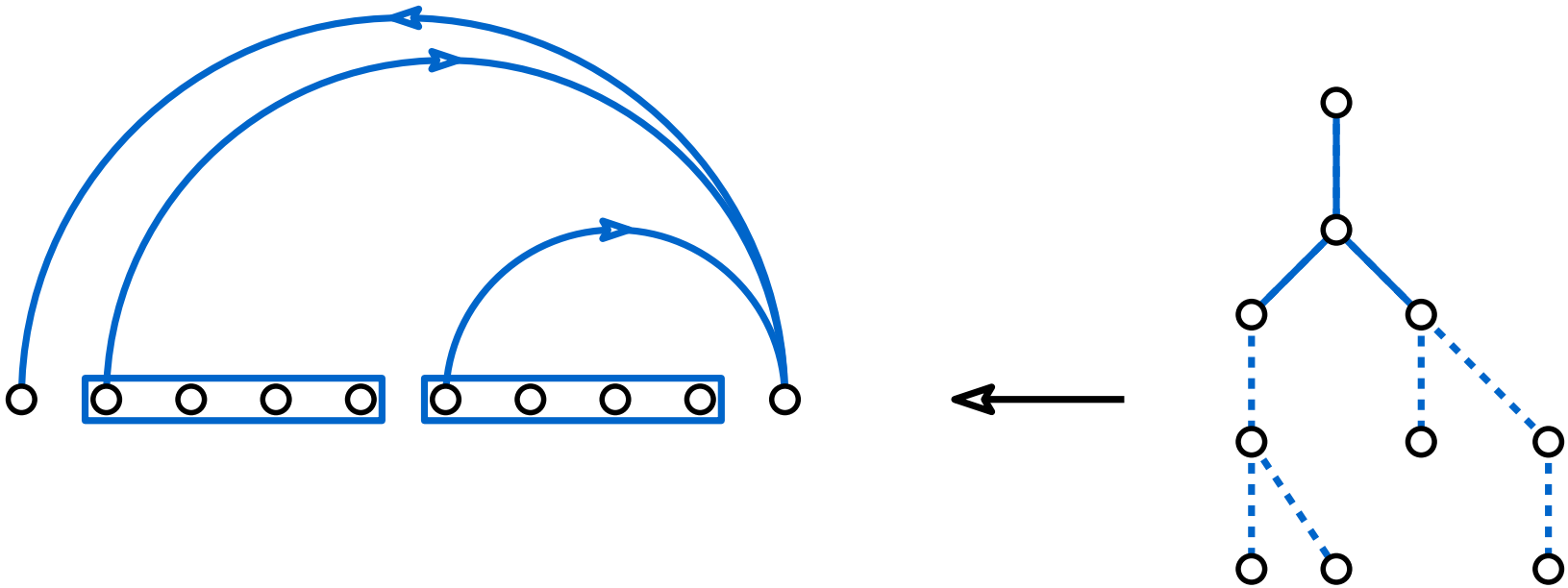

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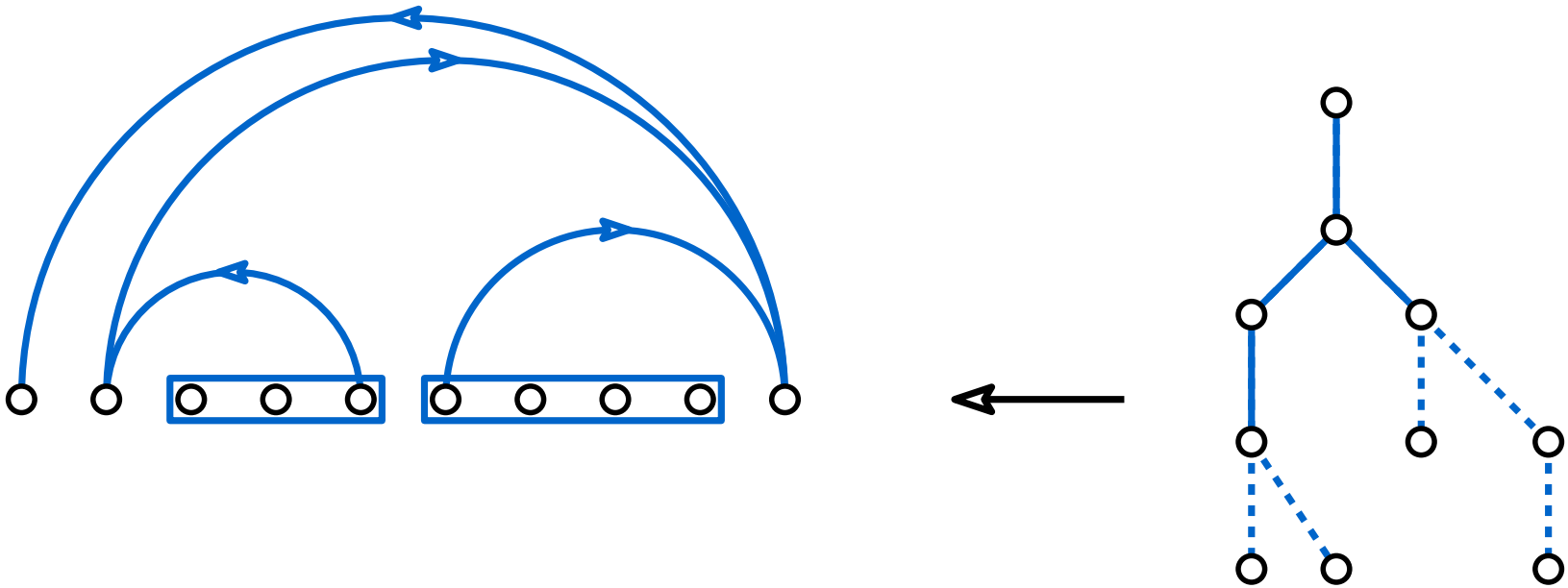

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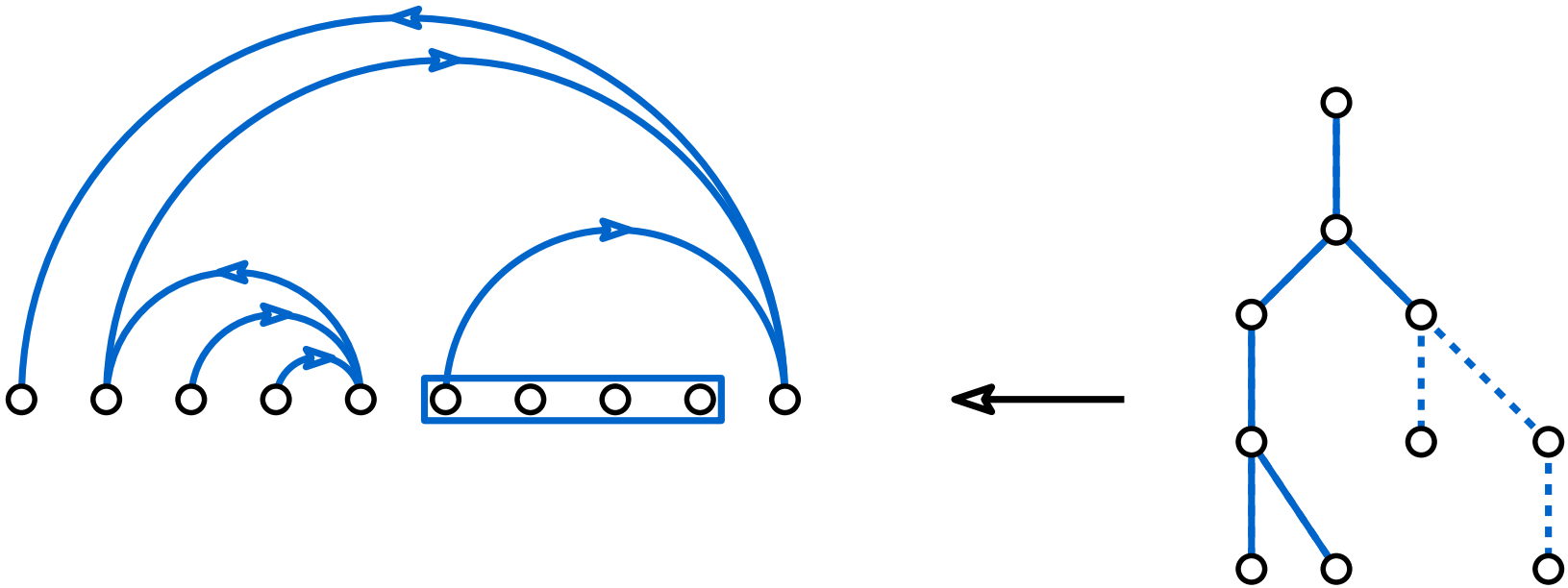

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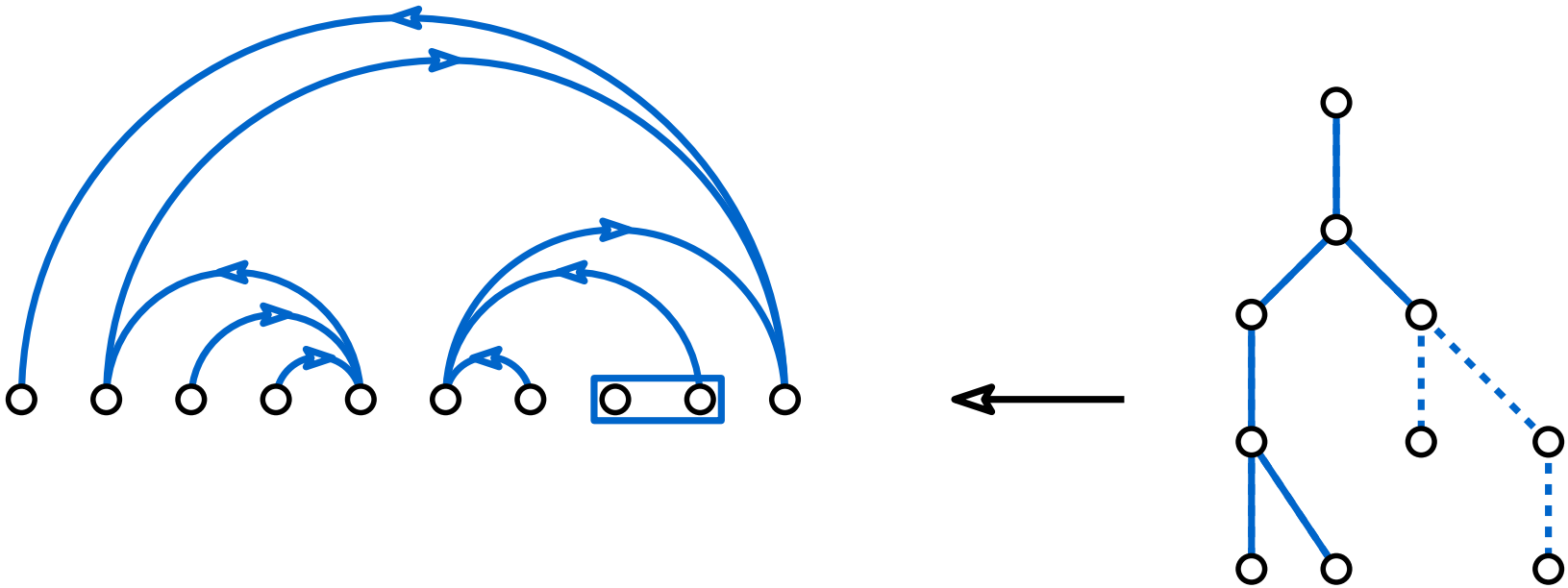

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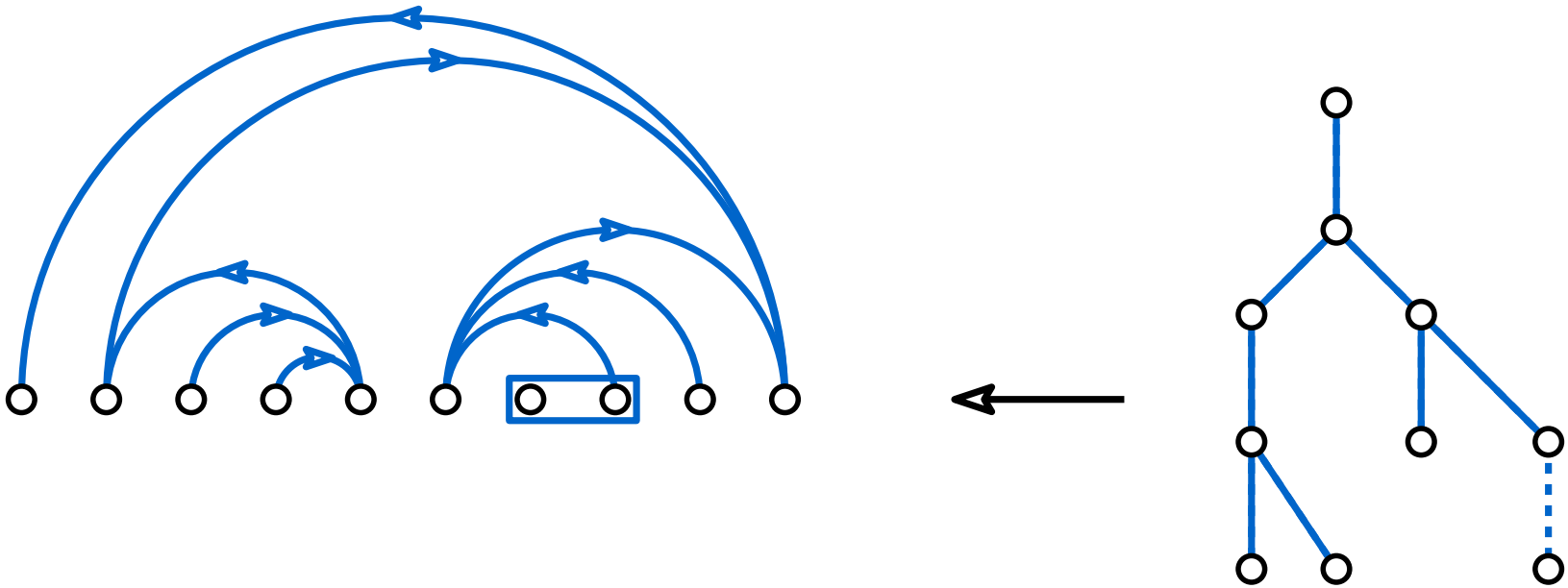

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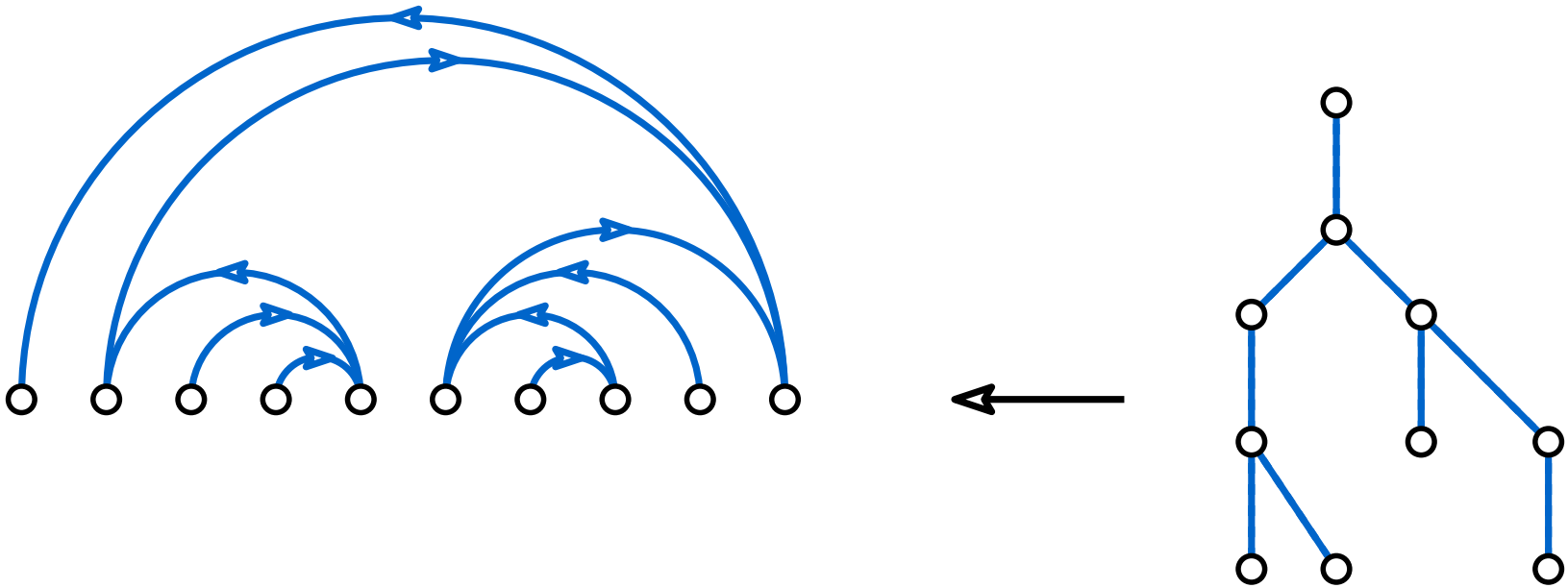

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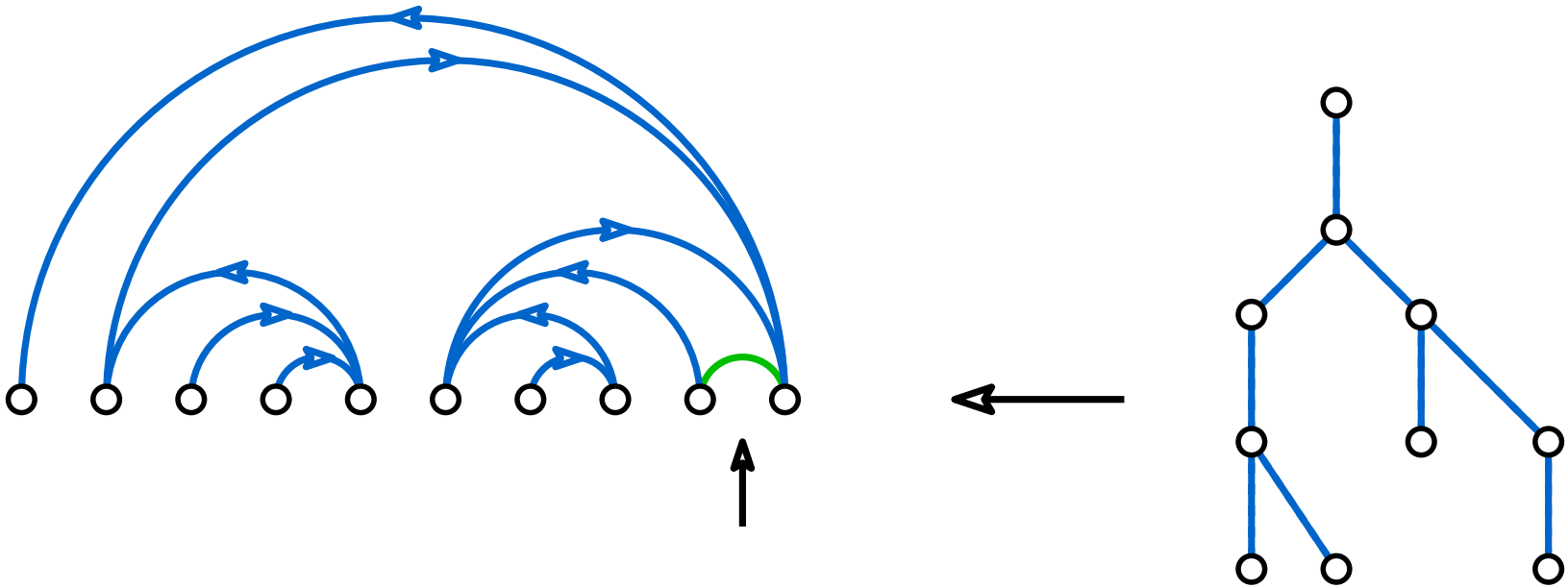

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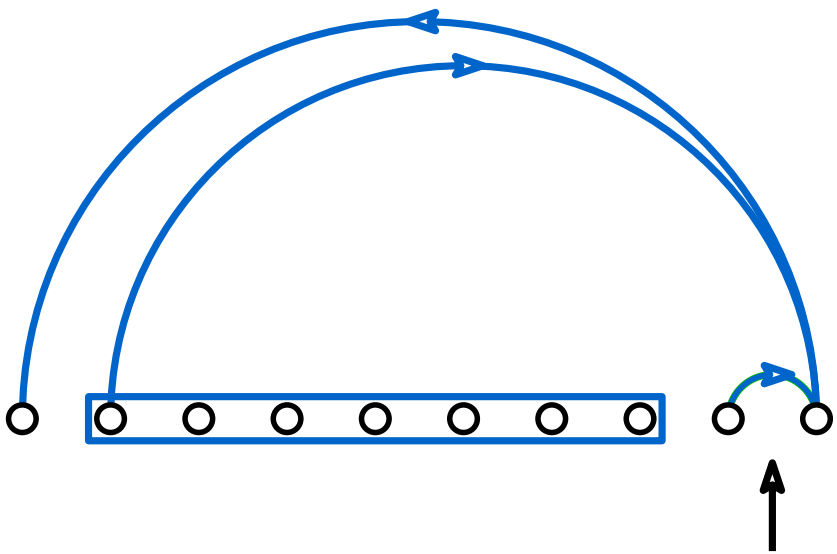

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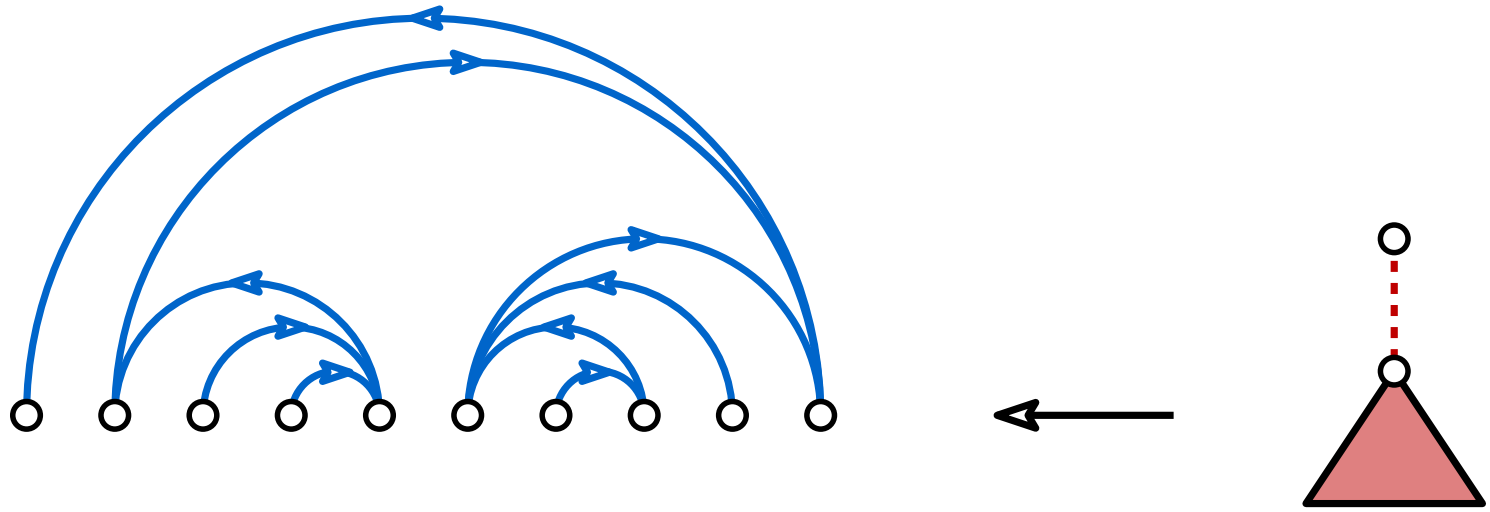

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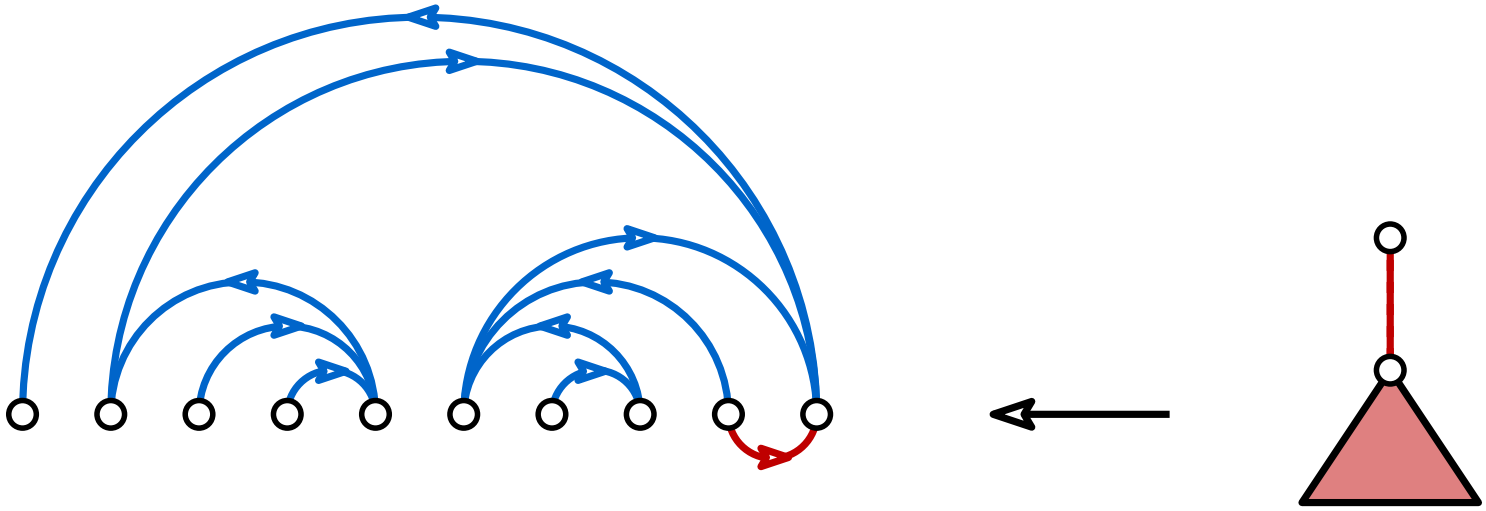

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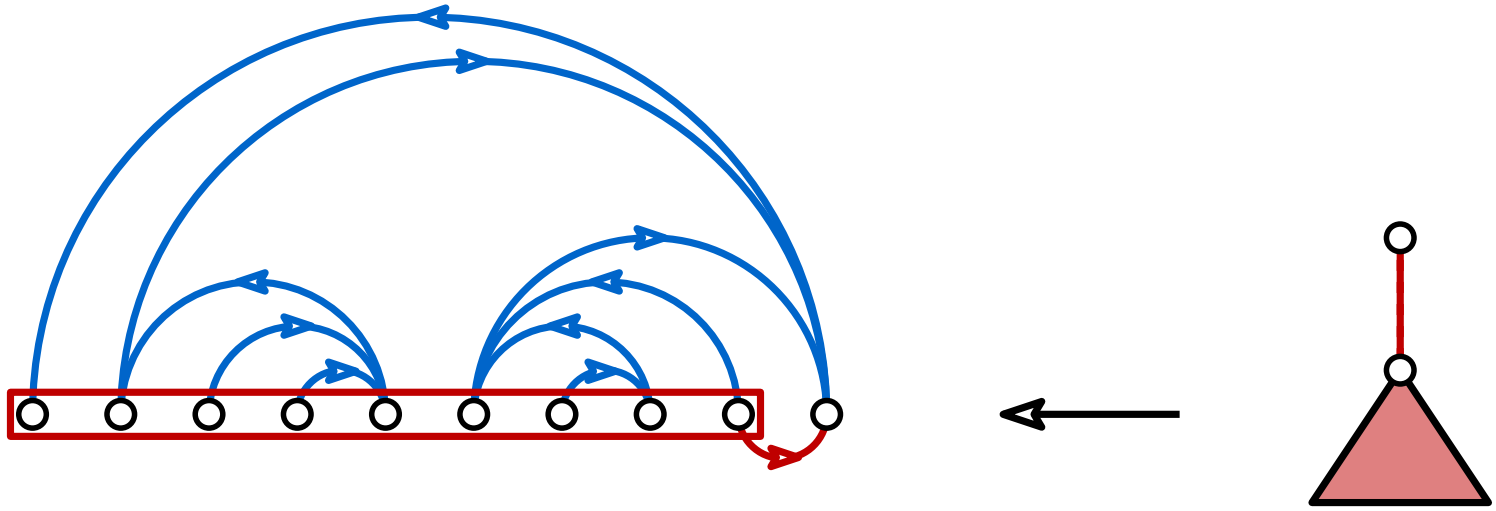

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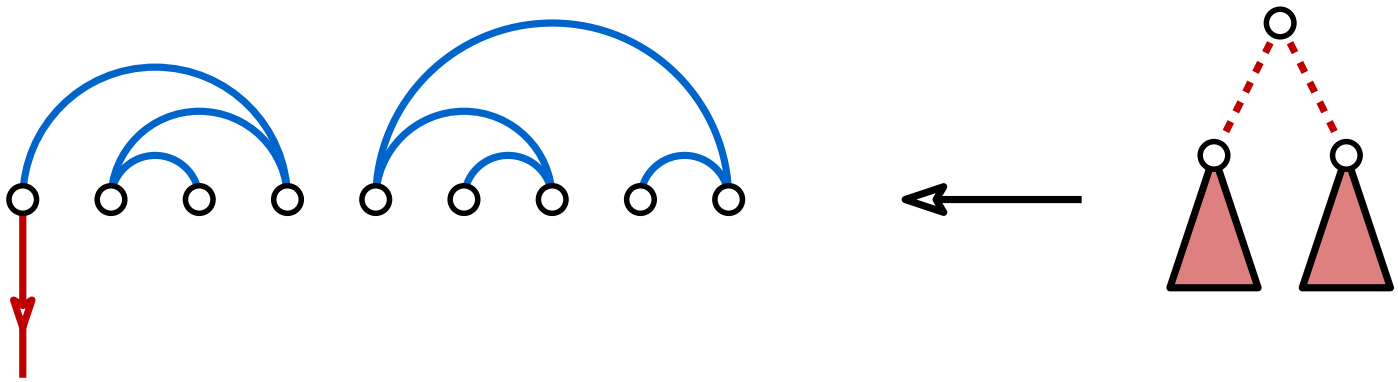

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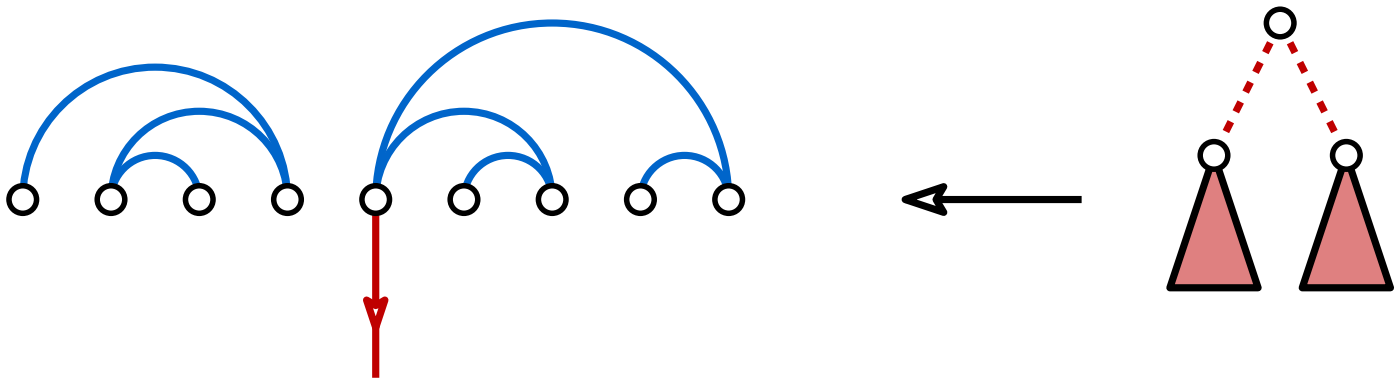

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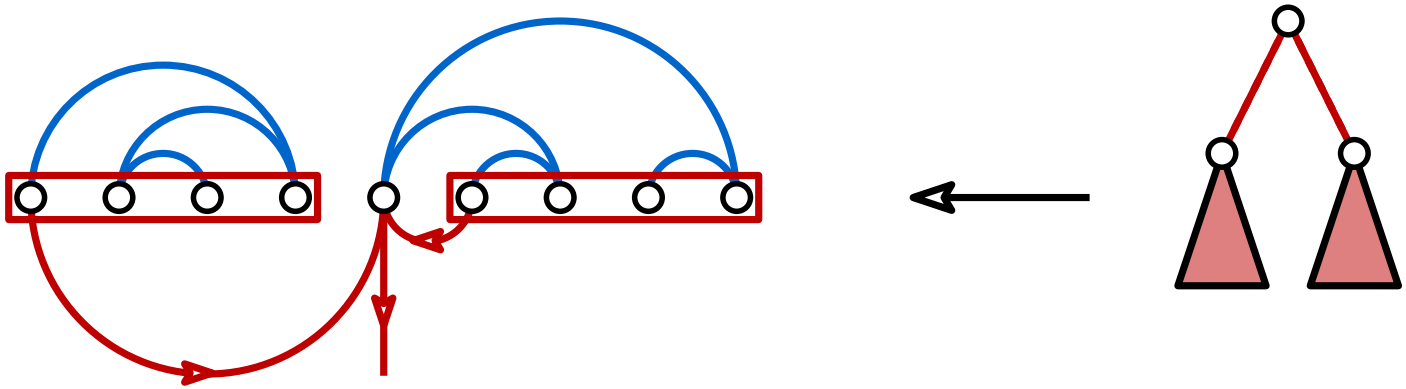

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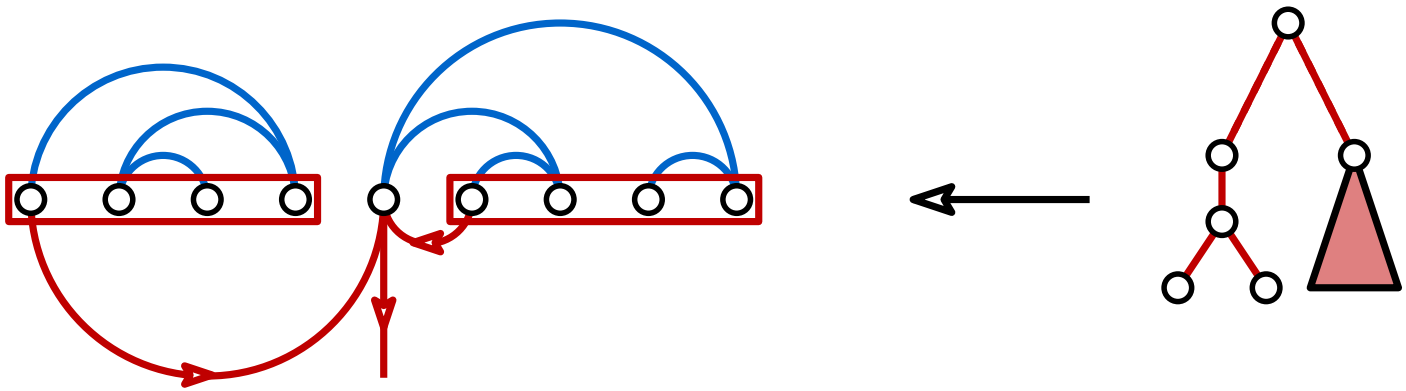

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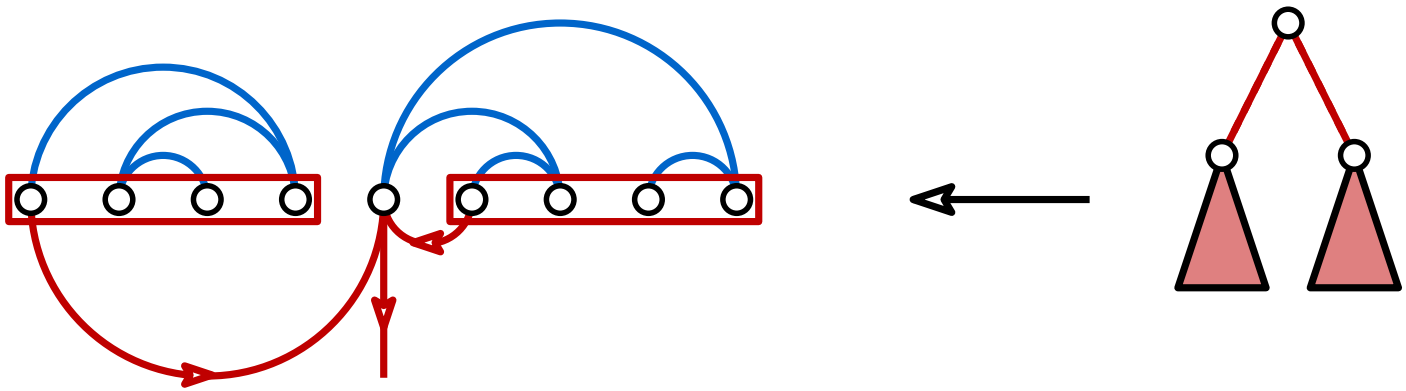

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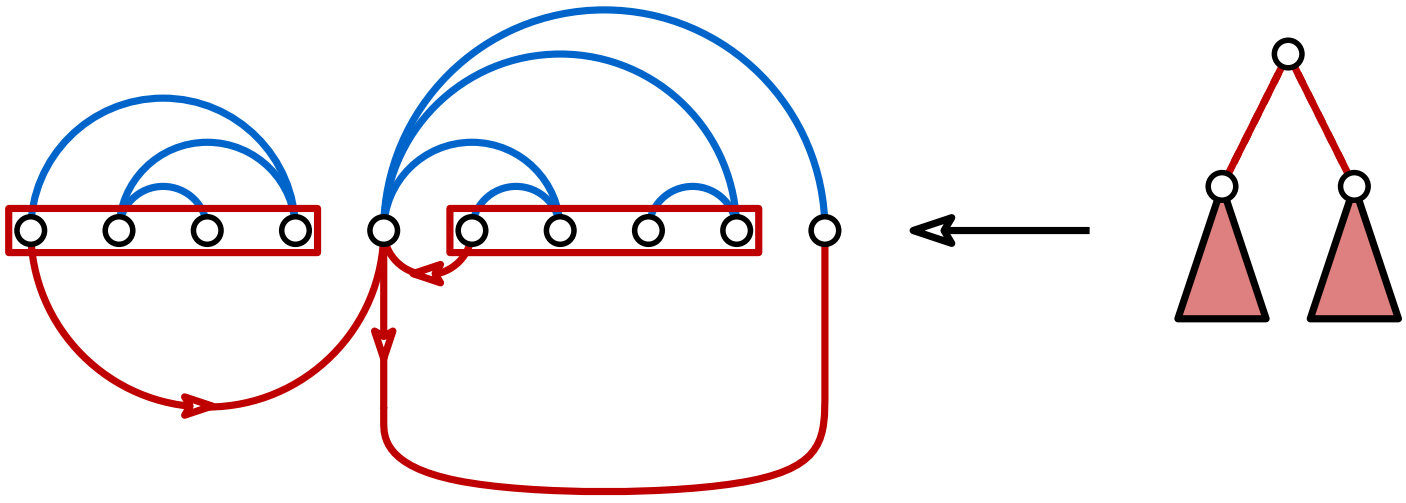

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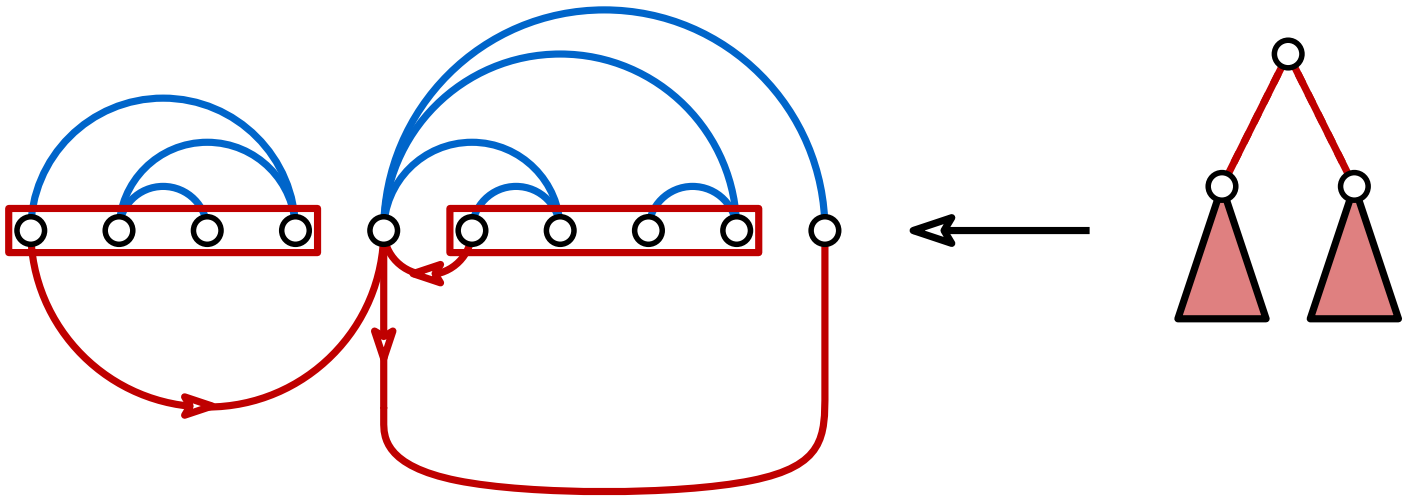

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- A vertex is in conflict if it is connected to the parent of the current root: do not embed the current root there.

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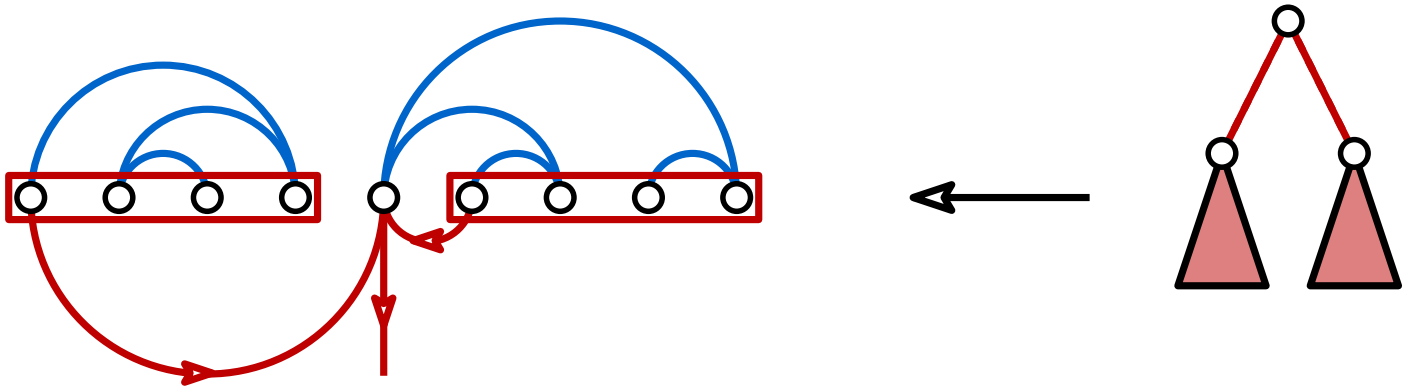

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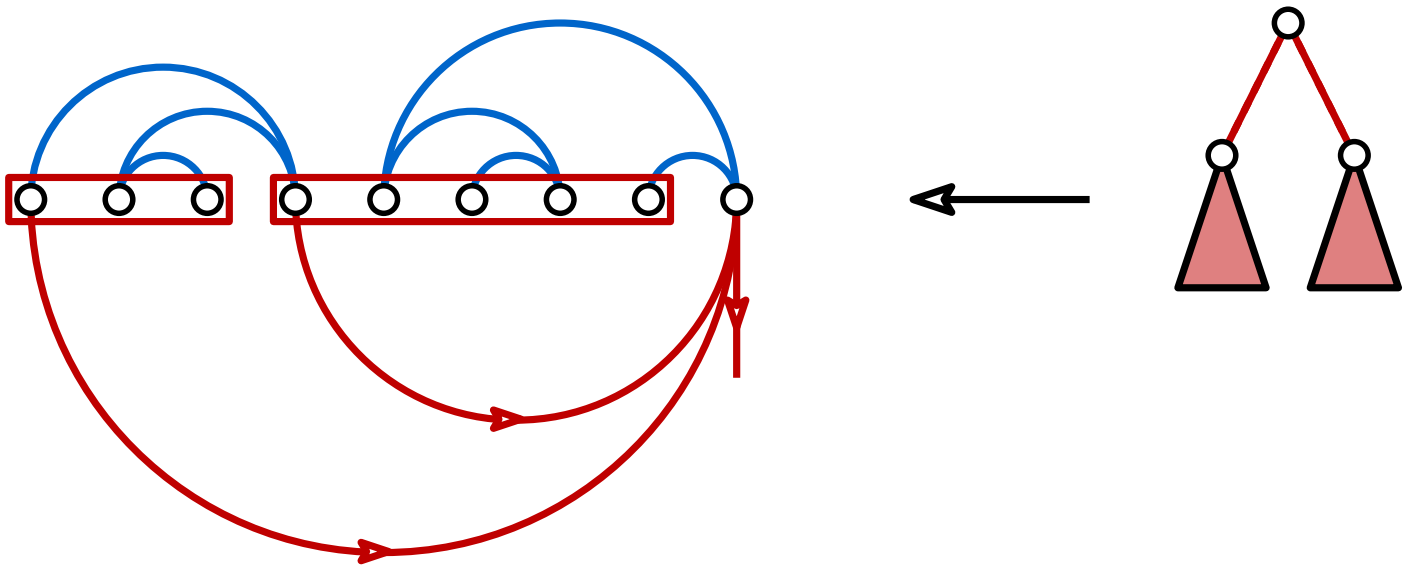

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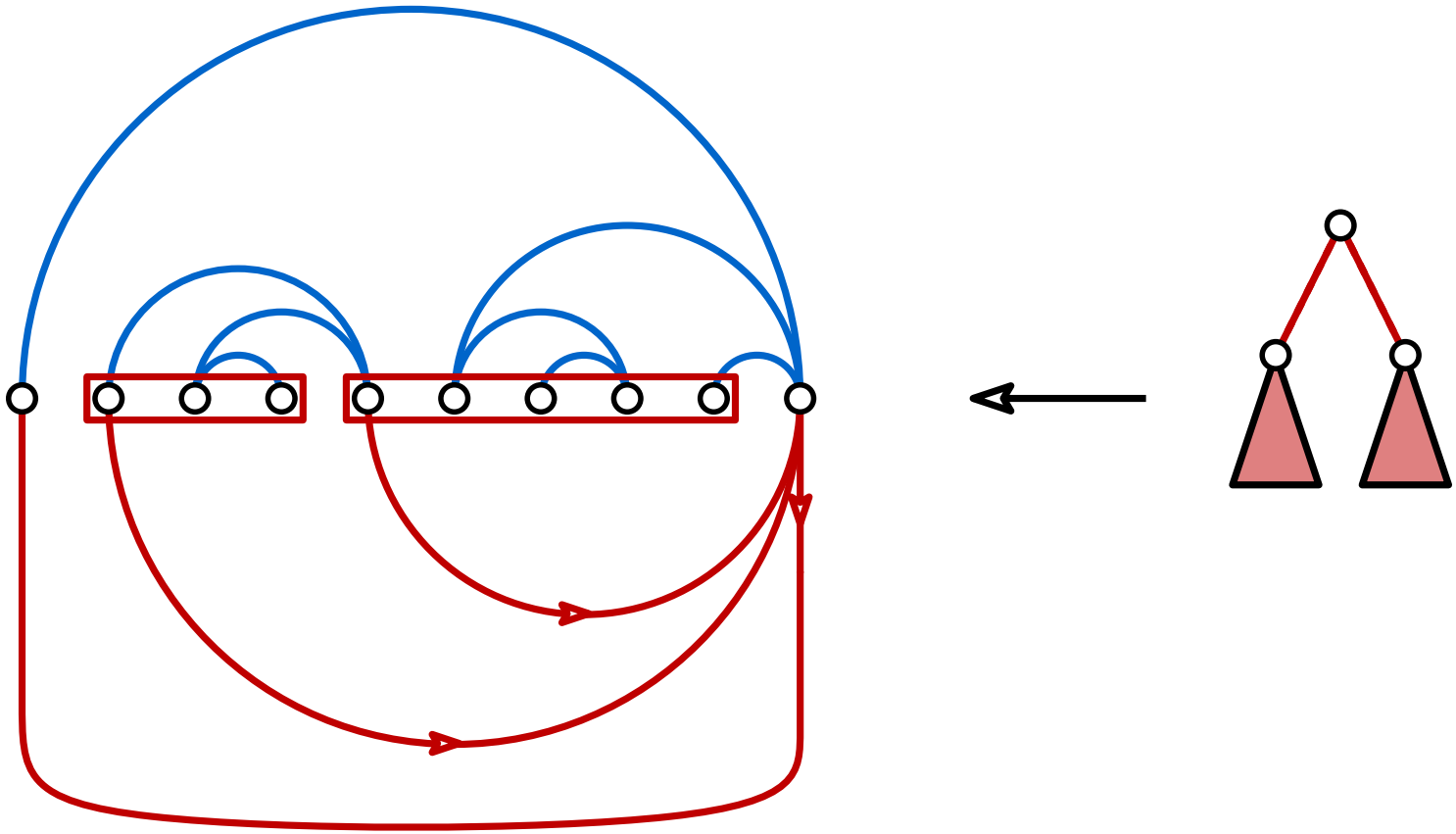

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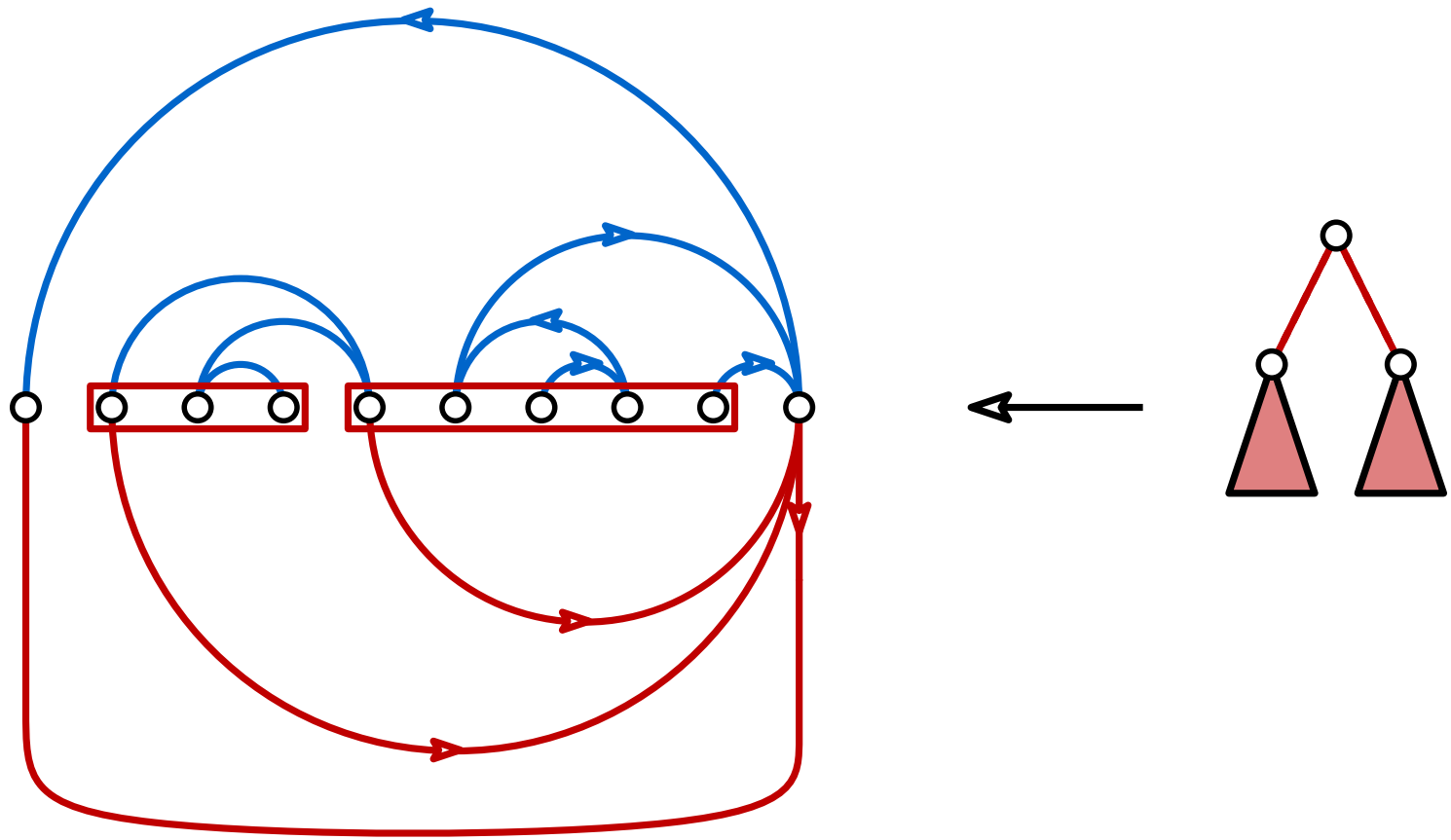

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Recall:

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When we embed a subtree on an interval $[i, j]$:

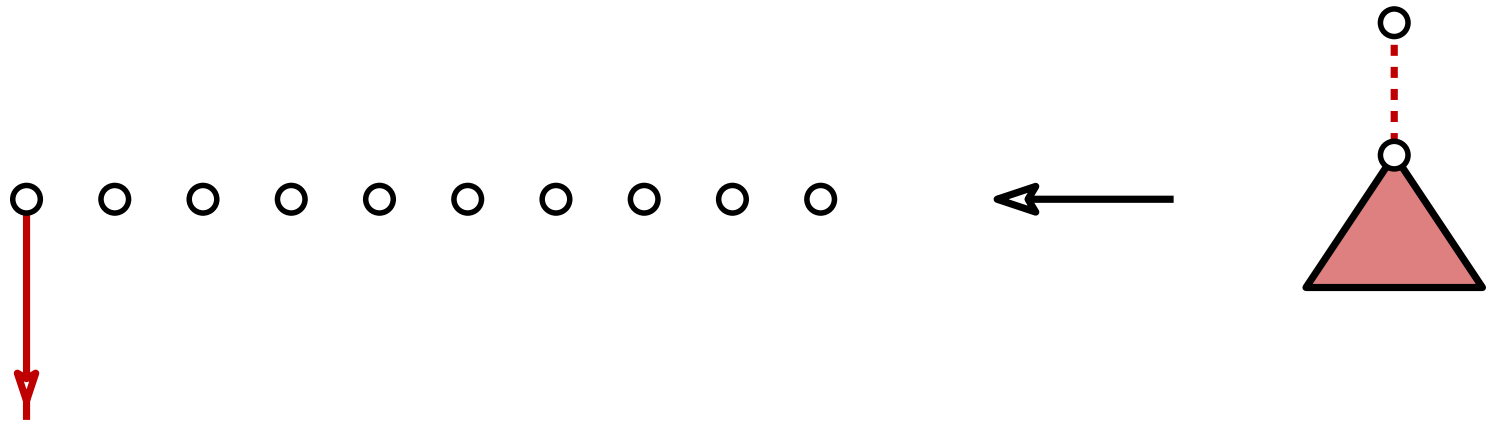
- **Placement invariant**: position i is not in conflict.
- **Local conformance**: the forest on $[i, j]$ satisfies the two rules.
- **No star**: the blue embedding on $[i, j]$ is not a star.

Overview of the cases

We distinguish the following cases in the recursion:

- The red tree is really small ($n \leq 5$).
- The red tree is small ($5 < n \leq 9$).
- The red tree is large and its root has one child.
- The red tree is large and its root has two children.

Embedding a large red tree with one child



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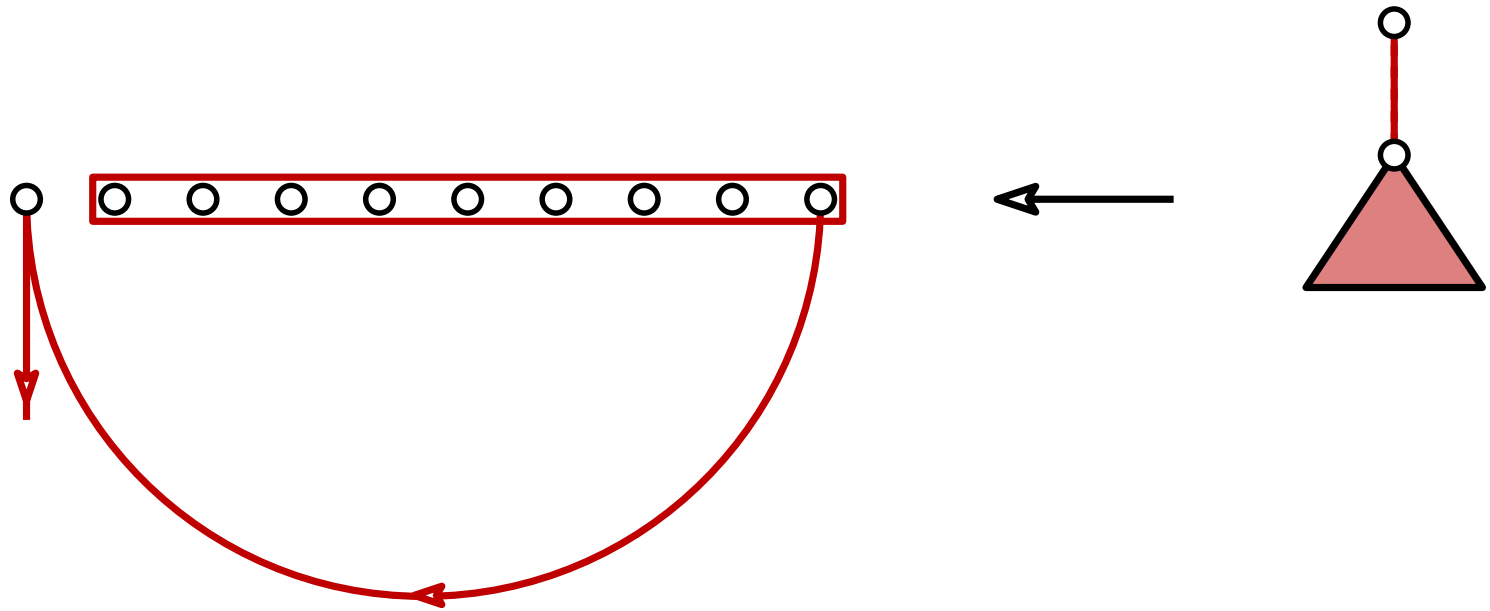
Embedding a large red tree with one child



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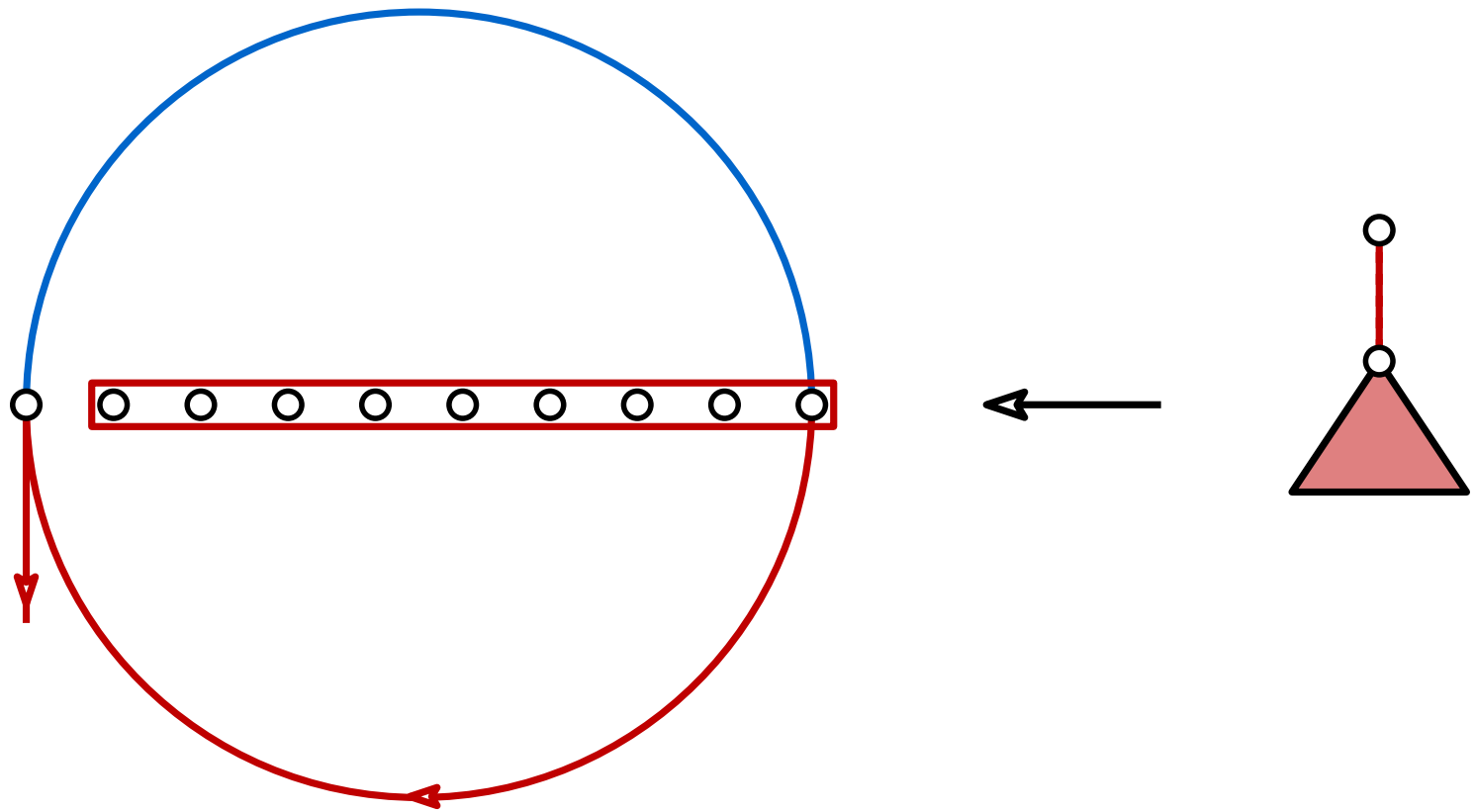

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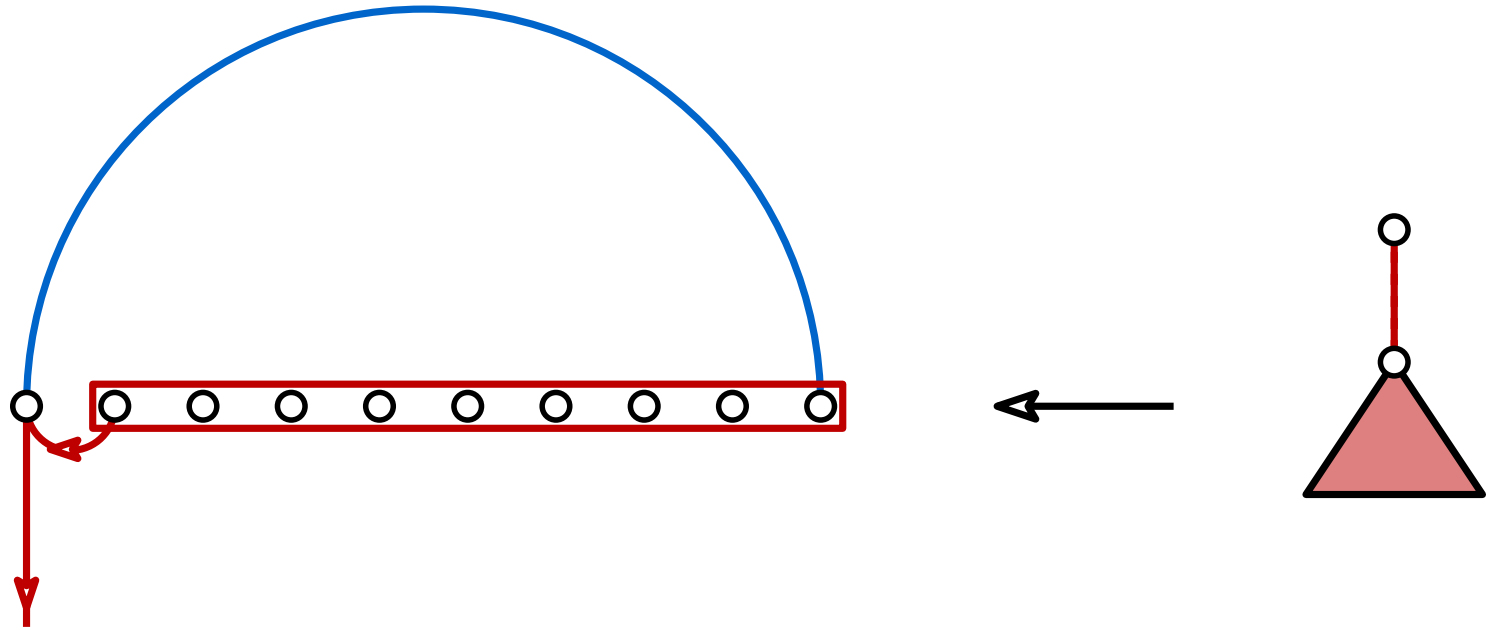

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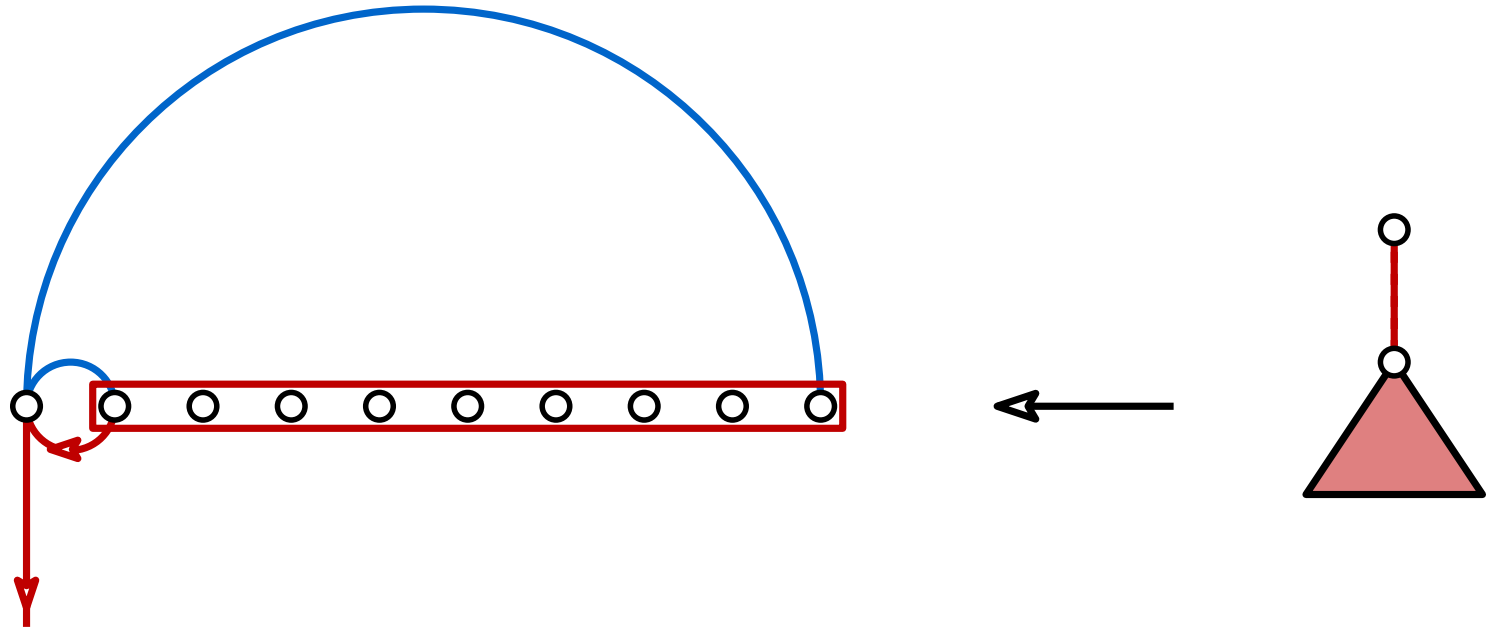

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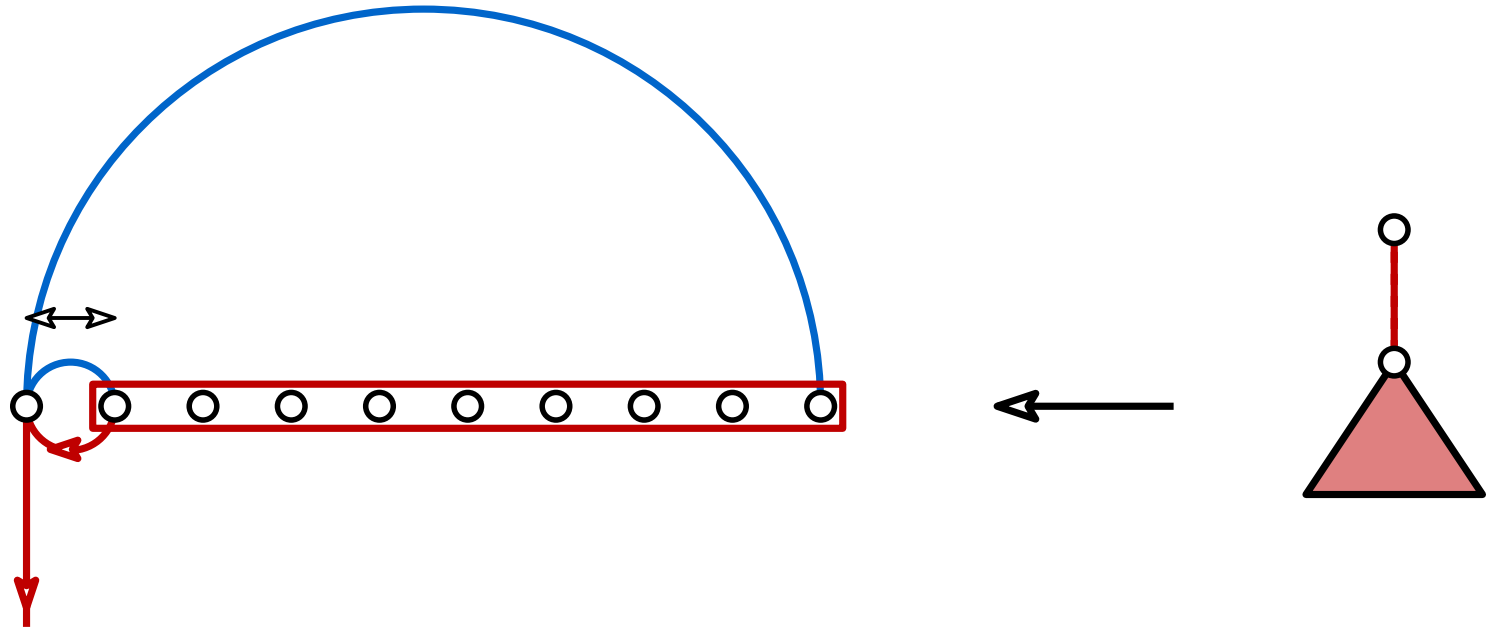

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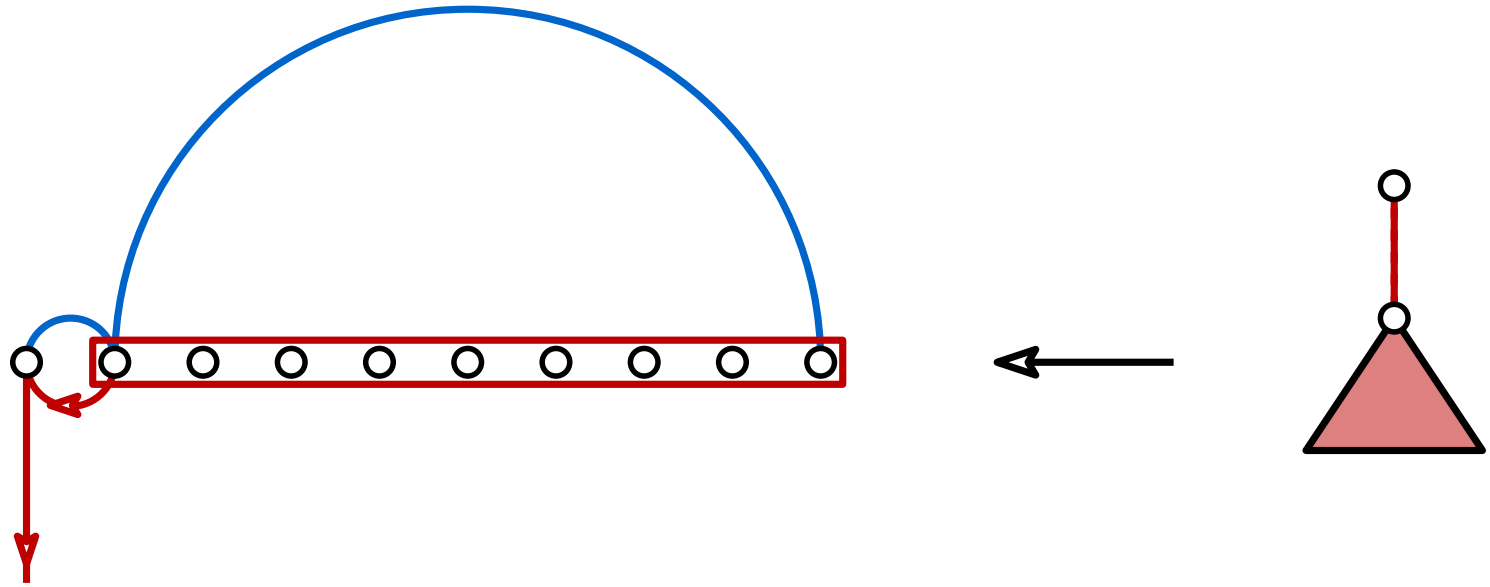

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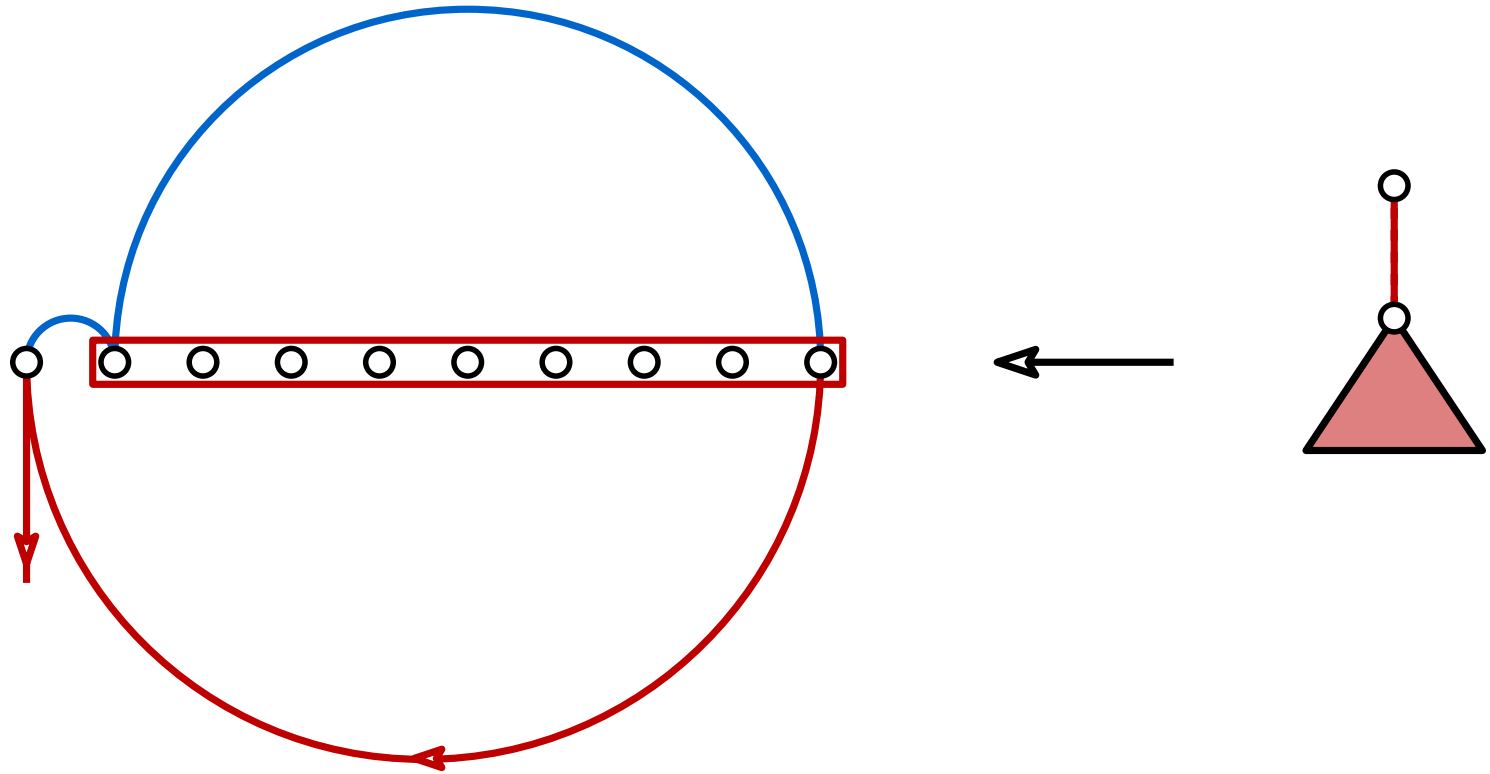

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Conclusion

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Every two non-star binary trees on n vertices admit a planar packing.

Conjecture (García, Hernando, Hurtado, Noy, and Tejel 2002)

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Thanks!

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