Making Octants Colorful

Jean Cardinal (ULB, Belgium), Kolja Knauer (U. Montpellier, France)
Piotr Micek (Jagiellonian U., Poland), Torsten Ueckerdt (KIT, Germany)

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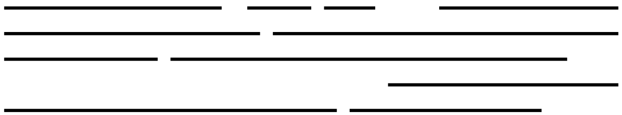
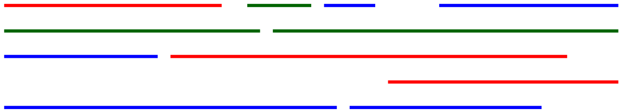
Covering Decomposition



For every collection of intervals such that every point is covered by at least k intervals, one can partition (color) the intervals into k colors so that every point is covered by at least one interval of each color.

Covering Decomposition



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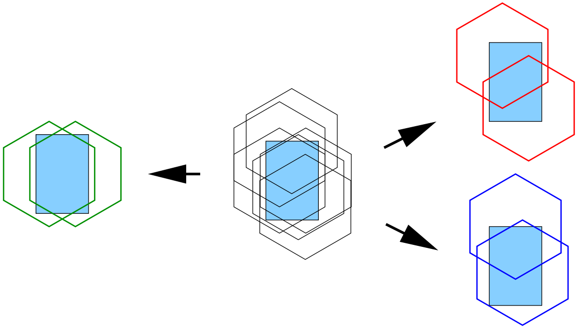


Given a family $\mathcal{F}$ of geometric ranges in $\mathbb{R}^d$, what is the smallest $p(k)$ such that given any (finite) collection of elements of $\mathcal{F}$, there exists a k-coloring of them such that every point covered at least $p(k)$ times is covered by at least one range of each color?

History

- Density-extremal multiple-fold packings and coverings

Fejes Tóth, 197X

- Pach's lecture at the Kolloquium über Diskrete Geometrie, Salzburg, May 1980
- The probabilistic method applies provided we also have an upper bound on the number of times a point is covered.

Mani-Levitska and Pach, 1980

Alon and Spencer, 1991

Recent Results

- $p(k) \leq 3k - 2$ for halfplanes

Smorodinsky and Yuditsky, JCTA 2012.

Aloupis, C., Collette, Langerman, Smorodinsky, DCG 2009.

- $p(k) = O(k)$ for translates of convex polygons

Gibson, Varadarajan, FOCS 2009.

Aloupis, C., Collette, Langerman, Orden, Ramos, SODA 2009.

Pach, Tóth, SoCG 2007.

- $p(k) = \infty$ for translates of a polyhedron in $\mathbb{R}^3$

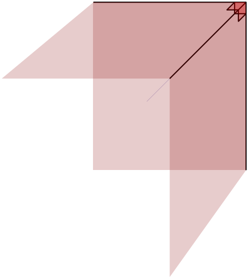
Pálvölgyi, DCG 2010.

- $p(k) = \infty$ for unit disks

Pálvölgyi, 2013 (preprint – cf. previous talk).

Polyhedra and Octants

- What about translates of the negative octant?



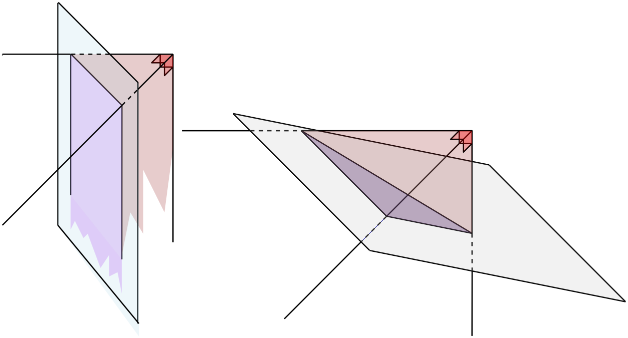
- $p(2) \leq 12$ — Keszegh and Pálvölgyi, DCG 2012.
- $p(k) \leq 12^{2^k}$ — Keszegh and Pálvölgyi, CGTA 2014.

A Polynomial Upper Bound for Octants

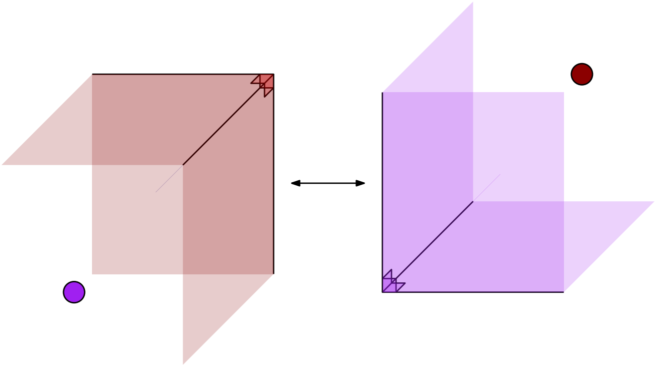
For translates of the negative octant, we have

$$p(k) = O(k^{5.6})$$

$\Rightarrow$ same bound for decomposition of coverings by bottomless rectangles and homothetic triangles

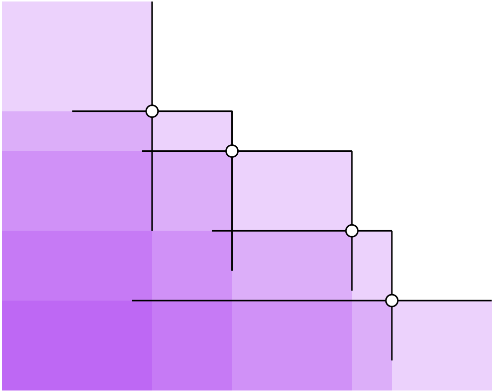


Duality



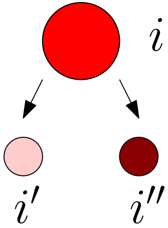
Proof Outline (1)

- We first consider independent point sets.
- Lemma: Given an independent point set P, we can cover all points not above P with $2|P| + 1$ negative octants.



Proof Outline (2)

- Given a valid coloring with k colors, let us call P_i the points of color i
- split each color class P_i simultaneously into two colors, using a proper 2-coloring of P_i for $a = p(2) = 12$.



Proof Outline (3)

- Consider an octant containing a set P' of $(2a-1)p(k)$ points, and suppose one color is missing, say i'
- This means there were not enough points of color i, say at most $a-1$
- From the previous lemma, the points in $P' \setminus P'_i$ can be covered by at most $2(a-1)+1 = 2a-1$ octants
- From the pigeonhole principle, one of these octants must contain at least $p(k)$ points, hence should have contained a point of color i, a contradiction

$$\begin{aligned} p(2k) &\leq (2a-1)p(k) \\ p(k) &\leq O(k^{\log_2(2a-1)}) \\ &\leq O(k^{4.6}) \end{aligned}$$

Proof Outline (4)

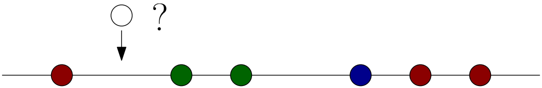
We now have to get rid of the independence assumption.

- Divide the point set greedily into independent layers.
- Use transitivity of the dominance order to apply a pigeonhole argument on the layers.
- Bound raises from $O(k^{4.6})$ to $O(k^{5.6})$.

Dynamic problems

- Turning one geometric dimension into time and perform online coloring
- Related to sweepline algorithms

Coloring Point Sets under Insertions

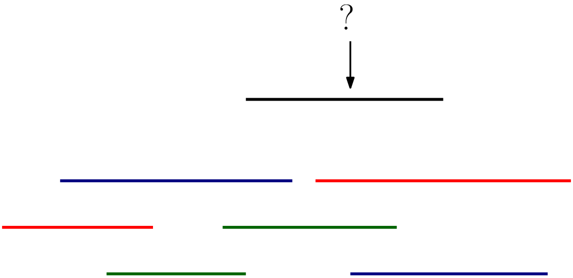


- Points appear one by one on the real line
- At any time, any interval of $p(k)$ contiguous points must contain all k colors
- We can color a point with one of k colors at any time, and cannot change later

$$p(k) \leq 3k - 2$$

ACCCHHKLLMRU, WADS 2013

Coloring Intervals under Insertions



- Intervals appear one by one on the real line
- At any time, any point covered by $p(k)$ intervals must be covered by each of the k colors
- We can color an interval with one of k colors at any time, and cannot change later

Impossible

Closed and Open Questions

- $p(k) = \infty$ for range spaces induced by homothetic copies of an arbitrary polygon S with at least four edges.

  Kovács, 2013 (preprint).

- Only homothetic copies of triangles are cover-decomposable.
- Essentially closes the questions raised by Pach 33 years ago.
- Tight bound for octants?
- Dual point coloring with respect to homothetic copies of a quadrangle?

Thank You!