Making Octants Colorful

Jean Cardinal (ULB, Belgium), Kolja Knauer (U. Montpellier, France)
Piotr Micek (Jagiellonian U., Poland), Torsten Ueckerdt (KIT, Germany)

EuroGIGA project
http://www.eurogiga-compose.eu
For every collection of intervals such that every point is covered by at least $k$ intervals, one can partition (color) the intervals into $k$ colors so that every point is covered by at least one interval of each color.
For every collection of intervals such that every point is covered by at least $k$ intervals, one can partition (color) the intervals into $k$ colors so that every point is covered by at least one interval of each color.
Given a family $\mathcal{F}$ of geometric ranges in $\mathbb{R}^d$, what is the smallest $p(k)$ such that given any (finite) collection of elements of $\mathcal{F}$, there exists a $k$-coloring of them such that every point covered at least $p(k)$ times is covered by at least one range of each color?
• Density-extremal multiple-fold packings and coverings

Fejes Tóth, 197X

• Pach’s lecture at the Kolloquium über Diskrete Geometrie, Salzburg, May 1980

• The probabilistic method applies provided we also have an upper bound on the number of times a point is covered.

Mani-Levitska and Pach, 1980
Alon and Spencer, 1991
Recent Results

- $p(k) \leq 3k - 2$ for halfplanes
  
  Smorodinsky and Yuditsky, JCTA 2012.
  
  Aloupis, C., Collette, Langerman, Smorodinsky, DCG 2009.

- $p(k) = O(k)$ for translates of convex polygons
  
  Gibson, Varadarajan, FOCS 2009.
  
  Aloupis, C., Collette, Langerman, Orden, Ramos, SODA 2009.
  

- $p(k) = \infty$ for translates of a polyhedron in $\mathbb{R}^3$
  
  Pálvölgyi, DCG 2010.

- $p(k) = \infty$ for unit disks
  
  Pálvölgyi, 2013 (preprint – cf. previous talk).
• What about translates of the negative octant?

• $p(2) \leq 12$
• $p(k) \leq 12^k$

Keszegh and Pálvölgyi, DCG 2012.
Keszegh and Pálvölgyi, CGTA 2014.
For translates of the negative octant, we have

\[ p(k) = O(k^{5.6}) \]

⇒ same bound for decomposition of coverings by bottomless rectangles and homothetic triangles
Duality
Proof Outline (1)

- We first consider independent point sets.
- **Lemma**: Given an independent point set $P$, we can cover all points not above $P$ with $2|P| + 1$ negative octants.
Proof Outline (2)

- Given a valid coloring with \( k \) colors, let us call \( P_i \) the points of color \( i \)
- split each color class \( P_i \) simultaneously into two colors, using a proper 2-coloring of \( P_i \) for \( a = p(2) = 12 \).
Proof Outline (3)

• Consider an octant containing a set $P'$ of $(2a - 1)p(k)$ points, and suppose one color is missing, say $i'$

• This means there were not enough points of color $i$, say at most $a - 1$

• From the previous lemma, the points in $P' \setminus P'_i$ can be covered by at most $2(a - 1) + 1 = 2a - 1$ octants

• From the pigeonhole principle, one of these octants must contain at least $p(k)$ points, hence should have contained a point of color $i$, a contradiction

\[
\begin{align*}
p(2k) & \leq (2a - 1)p(k) \\
p(k) & \leq O(k^{\log_2(2a-1)}) \\
& \leq O(k^{4.6})
\end{align*}
\]
Proof Outline (4)

We now have to get rid of the independence assumption.

- Divide the point set greedily into independent layers.
- Use transitivity of the dominance order to apply a pigeonhole argument on the layers.
- Bound raises from $O(k^{4.6})$ to $O(k^{5.6})$. 
Dynamic problems

- Turning one geometric dimension into time and perform online coloring
- Related to sweepline algorithms
Points appear one by one on the real line

At any time, any interval of $p(k)$ contiguous points must contain all $k$ colors

We can color a point with one of $k$ colors at any time, and cannot change later

$$p(k) \leq 3k - 2$$

ACCCHHKLLMRU, WADS 2013
Coloring Intervals under Insertions

- Intervals appear one by one on the real line
- At any time, any point covered by $p(k)$ intervals must be covered by each of the $k$ colors
- We can color an interval with one of $k$ colors at any time, and cannot change later

Impossible
Closed and Open Questions

- \( p(k) = \infty \) for range spaces induced by homothetic copies of an arbitrary polygon \( S \) with at least four edges. Kovács, 2013 (preprint).

- Only homothetic copies of triangles are cover-decomposable.
- Essentially closes the questions raised by Pach 33 years ago.
- Tight bound for octants?
- Dual point coloring with respect to homothetic copies of a quadrangle?
Thank You!