Location and domination in graphs:

Location-domination in a graph and its complement

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EuroGIGA Final Conference. Berlin, February 17-21, 2014

Detection devices and graphs

Graphs are used to model safeguard problems:

detection devices located at some vertices
to detect/locate an intruder in some vertex,
...of course with a small number of detectors

- Detection: is there any intruder?
- Location: where is the intruder?

Detection devices and graphs

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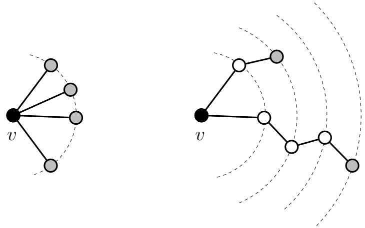
- Detection: is there any intruder?
  Dominating sets
- Location: where is the intruder?
  Locating sets

Restrictions on detection devices

- Detect if there is an intruder in its neighborhood $\rightarrow$ 0, 1
- Detect if there is an intruder at distance $= k$ $\rightarrow$ k



- At most one detection device at a vertex
- Detect an intruder in a vertex not occupied by a detection device

Graphs

$G = (V, E)$ graph,

- $\overline{G}$, *complement* of graph G
- $N(v) = \{u : uv \in E\}$, *open neighborhood*
- $N[v] = \{v\} \cup N(v)$, *closed neighborhood*

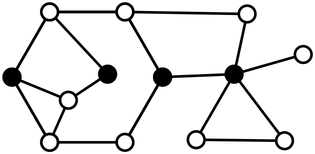
Domination, Location and Location-Domination

- Domination [Berge (1958), Ore (1962)]
- Location [Slater (1975), Harary and Melter (1976)]
- Location and domination [Henning and Oellermann (2004)]
- Location-domination [Slater (1988)]

$G = (V, E)$, $S \subseteq V$:

- *Dominating set:* for all $v \in V \setminus S$, $S \cap N(v) \neq \emptyset$
- *Locating set/Resolving set:*
  every vertex is uniquely determined by its vector of distances to the vertices of S
- *Locating and dominating set (MLD-set):*
  - S is a dominating set
  - S is a locating set
- *Locating-dominating set (LD-set):*
  - S is a dominating set
  - $N_G(u) \cap S \neq N_G(v) \cap S$, if $u, v \in V \setminus S$, $u \neq v$

Dominating set



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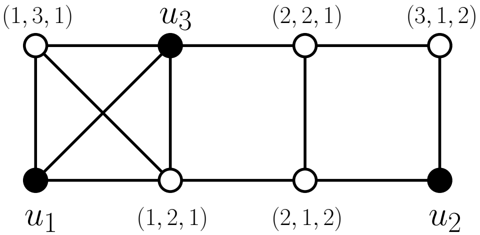
Locating set

$S = \{u_1, u_2, u_3\}$



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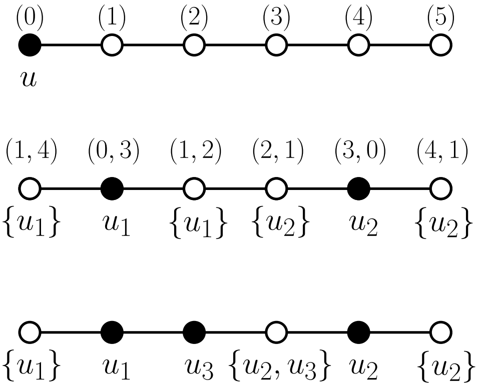
Example



Locating

Locating Dominating

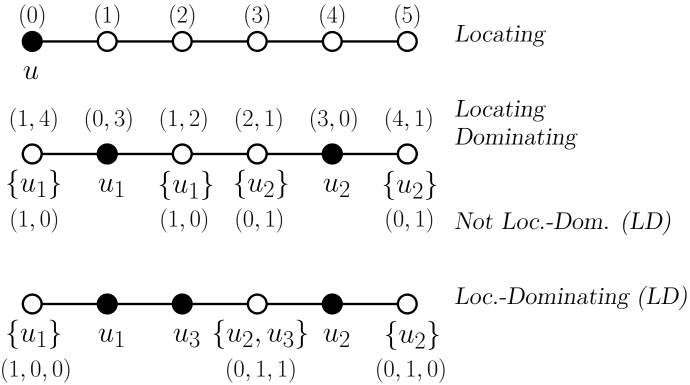
Not Loc.-Dom. (LD)

Loc.-Dominating (LD)

Parameters

- *Domination number* of G, $\gamma(G)$:
  minimum size of a dominating set of G
- *Location number/Metric dimension* of G, $\beta(G)$:
  minimum size of a locating set of G
- *Location and domination number*, $\eta(G)$:
  minimum size of a locating and dominating set of G
- *Location-domination number*, $\lambda(G)$:
  minimum size of a locating-dominating set of G

Example


(0) (1) (2) (3) (4) (5)
u
Locating
(1, 4) (0, 3) (1, 2) (2, 1) (3, 0) (4, 1)
Locating Dominating
{u1} u1 {u1} {u2} u2 {u2}
(1, 0) (1, 0) (0, 1) (0, 1)
Not Loc.-Dom. (LD)
Loc.-Dominating (LD)
{u1} u1 u3 {u2, u3} u2 {u2}
(1, 0, 0) (0, 1, 1) (0, 1, 0)

Labels

$S = \{u_1, \ldots, u_r\} \subseteq V$

$v \in V \setminus S \longrightarrow$ label of v, $\ell(v) = (x_1, \ldots, x_r)$

where $x_i = d_i$ (distance) or $x_i = n_i$ (neighborhood):

- $n_i = \begin{cases} 1, & \text{if } x \in N(u_i); \\ 0, & \text{otherwise.} \end{cases}$
- $d_i = d(x, u_i)$

Remark.

$$n_i = \begin{cases} d_i, & \text{if } d_i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Location and Domination/Location-Domination

Parameter	Label, $\ell(v)$	Conditions for $V \setminus S$	
Domination, γ	$(n_1, \ldots, n_r)$	$\exists n_i = 1$	
Location, β	$(d_1, \ldots, d_r)$		$\ell(x) \neq \ell(y)$, if $x \neq y$
Dom. and Loc., η	$(d_1, \ldots, d_r)$	$\exists d_i = 1$	$\ell(x) \neq \ell(y)$, if $x \neq y$
Loc.-Dom., λ	$(n_1, \ldots, n_r)$	$\exists n_i = 1$	$\ell(x) \neq \ell(y)$, if $x \neq y$

Relation between parameters

- S LD-set $\Rightarrow$ S dominating set
- S LD-set $\Rightarrow$ S locating set

$$\boxed{\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \gamma(G) + \beta(G)}$$

$$\boxed{\eta(G) \leq \lambda(G)}$$

Questions

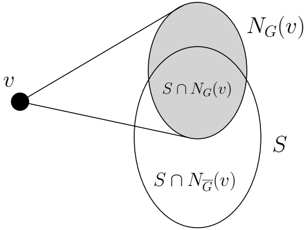
- Bounds relating different parameters
- Bounds relating order, size, diameter, Δ, δ,...
- Values on some families: complete graphs, paths, cycles, wheels, bipartite graphs, trees,...
- Extremal values
- Realization type results
- Graph operations: Cartesian product, strong product, complement,...
- Nordhaus-Gaddum type bounds: $p(G) + p(\overline{G})$

$\lambda(G)$ versus $\lambda(\overline{G})$ [Hernando, Mora and Pelayo (2013)]

S LD-set of $G = (V, E)$:

- dominating set of G
- $N_G(u) \cap S \neq N_G(v) \cap S, \quad \text{if } u, v \in V \setminus S, u \neq v$

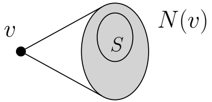
- $N_G(u) \cap S \neq N_G(v) \cap S \Leftrightarrow N_{\overline{G}}(u) \cap S \neq N_{\overline{G}}(v) \cap S$



$\lambda(G)$ versus $\lambda(\overline{G})$

S LD-set of G, then

- LD-set of $\overline{G} \Leftrightarrow$
  $\Leftrightarrow$ S dominating set of $\overline{G}$ $\Leftrightarrow \nexists v \in V \setminus S,\ S \subseteq N_G(v)$



- $\exists v \in V \setminus S,\ S \subseteq N_G(v) \quad \Rightarrow S \cup \{v\}$ is an LD-set of $\overline{G}$

$\lambda(G)$ versus $\lambda(\overline{G})$

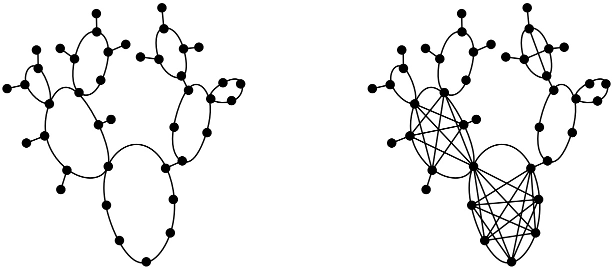
- $|\lambda(G) - \lambda(\overline{G})| \leq 1$

Graphs satisfying $\lambda(\overline{G}) = \lambda(G) + 1$?

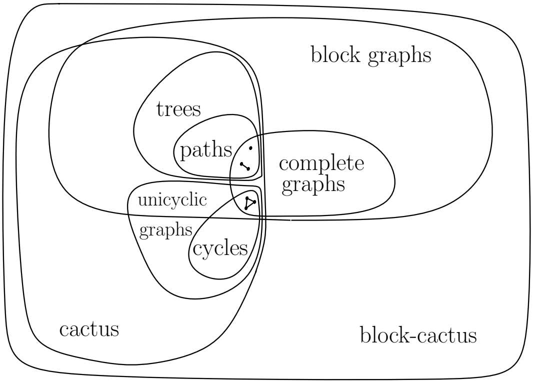
- $\lambda(\overline{G}) = \lambda(G) + 1 \Rightarrow G$ connected and $\mathrm{diam}(G) \leq 4$

Block-cactus

- Block of $G = (V, E)$: maximally connected subgraph with no cut vertices
- Cactus: connected graph s.t. all blocks are cycles or K_2, i.e., there is no edge lying on two different cycles
- Block-cactus: connected graph s.t. all blocks are cycles or complete graphs



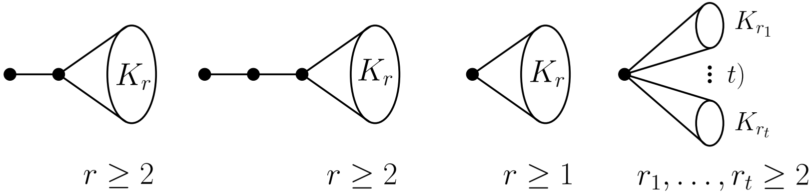
Families of block-cactus



Block-cactus

Block-cactus s.t. $\lambda(\overline{G}) = \lambda(G) + 1$

G is one of the following graphs:



- T tree of order $\geq 3 \Rightarrow \lambda(\overline{T}) \leq \lambda(T)$

Bipartite graphs

Bipartite connected graphs s.t. $\lambda(\overline{G}) = \lambda(G) + 1$

$$G = (V, E),\ V = V_1 \cup V_2,\ |V_1| = r,\ |V_2| = s$$

- S LD-code of G:
  - $S \cap V_1 \neq \emptyset$ and $S \cap V_2 \neq \emptyset \quad \Rightarrow \lambda(\overline{G}) \leq \lambda(G)$.
  - $S = V_2$ and $r < s \Rightarrow \lambda(\overline{G}) \leq \lambda(G)$.
- $r = 1, 2 \Rightarrow \lambda(\overline{G}) \leq \lambda(G)$

Bipartite graphs satisfying $\lambda(\overline{G}) = \lambda(G) + 1$

$$G = (V, E)$$

$$V = V_1 \cup V_2$$

$$|V_1| = r,\ |V_2| = s,\ 3 \leq r \leq s$$

Theorem

$$\lambda(\overline{G}) = \lambda(G) + 1 \Rightarrow \frac{3r}{2} \leq s \leq 2^r - 1$$

Proof of Theorem

$\lambda(\overline{G}) = \lambda(G) + 1 \Rightarrow$

- V_1 LD-code of G
- V_1 is not an LD-set of $\overline{G}$
- there is no LD-code of G with vertices at both stable sets

$s \leq 2^r - 1$

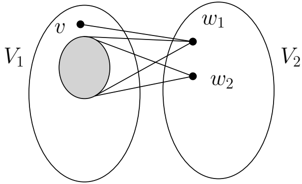
V_1 LD-code $\Rightarrow s \leq 2^r - 1$

Proof of $\frac{3r}{2} \leq s$

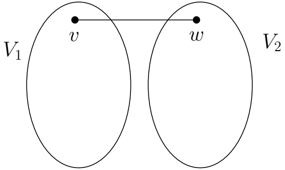
$\lambda(\overline{G}) > \lambda(G) \Rightarrow$
$\Rightarrow$ V_1 LD-code and $\nexists$ LD-code with vertices at both stable sets
$\Rightarrow \forall v \in V_1$, $V_1 \setminus \{v\}$ is not an LD-set of $G - v$
$\Rightarrow \forall v \in V_1$, conditions (a) and/or (b) hold at least twice:

(a) two vertices in V_2 with the same neighborhood in $V_1 \setminus \{v\}$

(b) $V_1 \setminus \{v\}$ is not a dominating set of V_2



$N(w_1) \oplus N(w_2) = \{v\}$



$N(w) \oplus \emptyset = \{v\}$

Proof of $\frac{3r}{2} \leq s$: the graph G^*

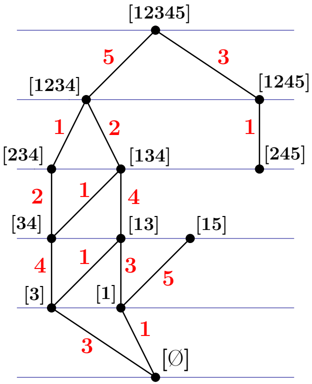
G^* edge-labeled graph associated to G:

- $V(G^*) = V_2 \cup \{w_0\}$, $w_0 \notin V_2$, and define $N(w_0) = \emptyset$
- $w_h w_k \in E(G^*) \Leftrightarrow N(w_h) \oplus N(w_k) = \{v\}$ for some $v \in V_1$
- $\ell(w_h w_k) = N(w_h) \oplus N(w_k) \in V_1$

Example of graph G^*

$V_1 = \{1, 2, 3, 4, 5\}$

$V_2 = \{[12345], [1234], [1245], [134], [234], [245], [13], [15], [34], [1], [3]\}$



Properties of graph G^*

- $|V(G^*)| = |V_2| + 1 = s + 1$
- G^* is bipartite
- incident edges have different labels
- a walk contains an even number of edges labeled v, $\forall v \in V_1$ $\Leftrightarrow$ the walk is closed

Proof of $\frac{3r}{2} \leq s$ using the graph G^* (contd.)

$\lambda(\overline{G}) \geq \lambda(G) \Rightarrow$

- G^* satisfies $|E(G^*)| \geq 2r$
- G^* contains a subgraph H^* of size $2r$ such that all its connected components are cactus
- If all connected components of G are cactus with no C_3, then $|V(G)| \geq \frac{3}{4}|E(G)| + 1$

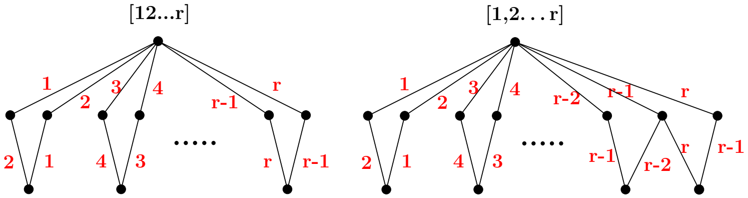
$$\Rightarrow s \geq \frac{3r}{2}$$

Bipartite graphs with $\lambda(\overline{G}) = \lambda(G) + 1$

Theorem

If $r, s \in \mathbb{N}$, $3 \leq r$ and $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$, then $\exists G(r,s)$ bipartite graph such that $\lambda(\overline{G}) = \lambda(G) + 1$.

Proof. $V_1 = \{1, 2, \ldots, r\}$ and $s = \left\lceil \frac{3r}{2} + 1 \right\rceil$:



G^*, r even $\qquad$ G^*, r odd

Bipartite graphs with $\lambda(\overline{G}) = \lambda(G) + 1$

Theorem

If $r, s \in \mathbb{N}$, $3 \leq r$ and $\frac{3r}{2} \leq s < \frac{3r}{2} + 1$, then $\nexists G(r,s)$ bipartite graph such that $\lambda(\overline{G}) = \lambda(G) + 1$.

Proof.

Cases:

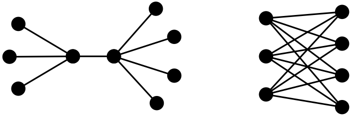
- r even and $s = \frac{3r}{2}$
- r odd and $s = \frac{3r+1}{2}$

Proof based on the structure of the graph G^* and properties of cactus graphs

Bipartite graphs with $\lambda(\overline{G}) - \lambda(G) \in \{-1, 0, 1\}$

Given integers r, s, $3 \leq r \leq s$ there are bipartite graphs G with stable parts V_1, V_2 satisfying $|V_1| = r$, $|V_2| = s$ and:

- $\lambda(\overline{G}) = \lambda(G) - 1$: double star $K_2(r-1, s-1)$
- $\lambda(\overline{G}) = \lambda(G)$: complete bipartite graphs $K(r, s)$



- $\lambda(\overline{G}) = \lambda(G) + 1$, if $\dfrac{3r}{2} + 1 \leq s \leq 2^r - 1$: $G(r, s)$

Thank you for your attention!