Two theorems on distances in graphs isometrically embeddable into Cartesian product graphs

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Preliminaries

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- Wiener index of G:

$$W(G) = \frac{1}{2} \sum_{u,v} d_G(u, v) \,.$$

Relation Θ

Edges $e = xy$ and $f = uv$ of G are in Djoković-Winkler relation Θ if

$$d(x,u) + d(y,v) \neq d(x,v) + d(y,u)\,.$$

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- Edges from different blocks are never in relation Θ.

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- Θ^* partitions $E(G)$ into Θ^*-classes.

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- Let $(G/F_i, w)$ be "natural" weighted graphs: the weight of a vertex of G/F_i is the number of vertices in the corresponding connected component of $G - F_i$.

$W(G)$ via quotient graphs

Theorem (K., 2006)

If $F_1, \ldots, F_k$ are the Θ^*-classes of a connected graph G, then

$$W(G) = \sum_{i=1}^{k} W(G/F_i, w)\,.$$

Two fold generalization

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- If all the weights are 1 then $W(G, w) = W(G)$.

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Lemma

Let G be a connected graph and let $\mathcal{E} = \{E_1, \ldots, E_r\}$ be a partition of $E(G)$ coarser than $\mathcal{F}$. Then every connected component of $G \setminus E_j$, $1 \leq j \leq r$, induces a convex subgraph of G.

Coarser partitions cont'd

Proof

- Suppose connected component C of $G \setminus E_j$ not convex.

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- Select $x, y \in C$ and P shortest x, y-path with $P \not\subseteq C$.
- Let $e \in (P \setminus C) \cap E_j$. Let e lie in Θ^*-class F_i.

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- Let $e \in (P \setminus C) \cap E_j$. Let e lie in Θ^*-class F_i.
- Let Q be a x, y-path in C.
- Since P shortest, e is in relation Θ with no edge on P, hence e is in relation Θ with an edge f on Q.
- Contradiction since then $f \notin C$.

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For the proof we essentially need the key lemma of Graham and Winkler from (Trans. Amer. Math. Soc. 288 (1985) 527–536) asserting that if R is a shortest u, v-path in G and Q is an arbitrary u, v-path in G, then

$$|E(R) \cap F| \leq |E(Q) \cap F|$$

holds for any Θ^*-class F.

Main result

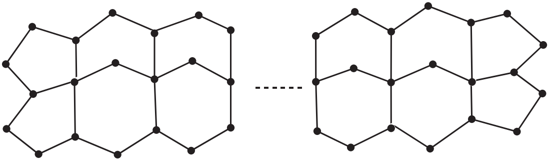
Theorem

Let (G, w) be a connected, weighted graph, and $\{E_1, \ldots, E_r\}$ a partition of $E(G)$ coarser than $\mathcal{F}$. Then

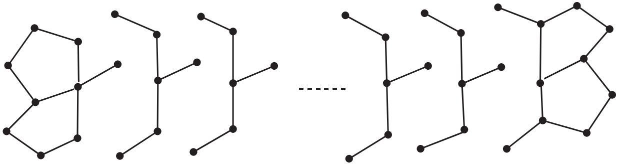
$$W(G, w) = \sum_{j=1}^{r} W(G/E_j, w_j)\,,$$

where $w_j(C) = \sum_{x \in C} w(x)$, for all connected components C of $G \setminus E_j$.

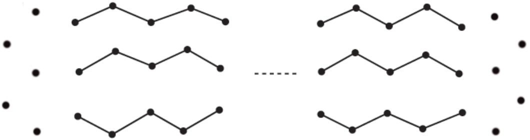
Example: G_n

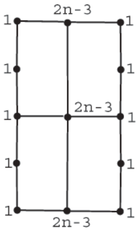


Example cont'd: $G_n \setminus E_1$ and $(G_n/E_1, w)$



Example cont'd: $G_n \setminus E_2$ and $(G_n/E_2, w)$





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- $W(G_n/E_2, w) = 16n^2 + 76n - 28$.

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- Hence by the theorem:

$$\begin{aligned} W(G_n) &= W(G/E_1, w) + W(G/E_1', w) + W(G/E_2, w) \\ &= 12n^3 + 22n^2 - 2n + 8\,. \end{aligned}$$

Edge version

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$$W(G, w_E) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d(u, v),$$

where the distance function is computed in (G, w_E).

Edge version cont'd

Theorem

Let (G, w_E) be a connected, edge-weighted graph. If $\{E_1, \ldots, E_r\}$ is a partition of $E(G)$ coarser than $\mathcal{F}$ such that for any j, the edges from E_j have the same weight, $w(E_j)$, then

$$W(G, w_E) = \sum_{j=1}^{r} w(E_j) W(G/E_j, w_j)\,.$$

Distance moments on graphs with $\Theta = \Theta^*$

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- $\frac{1}{6}\, W_3 + \frac{1}{2}\, W_2 + \frac{1}{3}\, W_1$ ... Tratch-Stankevich-Zefirov index

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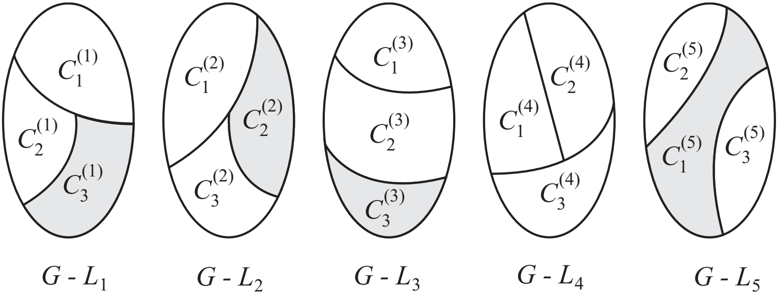
- For $p \geq 1$, pairwise different $i_1, \ldots, i_p \in [k]$, and any $j_1, \ldots, j_p \in [3]$ let

$$n_{j_1,\ldots,j_p}^{i_1,\ldots,i_p} = \left| V(C_{j_1}^{(i_1)}) \cap V(C_{j_2}^{(i_2)}) \cap \cdots \cap V(C_{j_p}^{(i_p)}) \right| .$$

Second theorem cont'd

$n^{1,2,3,5}_{3,2,3,1}$ is the order of the intersection of the gray parts:



Second theorem cont'd

$$N_{i_1,\dots,i_p} = \sum_{\forall r: j_r \neq j'_r} n^{i_1,\dots,i_p}_{j_1,\dots,j_p} \cdot n^{i_1,\dots,i_p}_{j'_1,\dots,j'_p} \,,$$

summation over all admissible indices $j_1, \dots, j_p$ and $j'_1, \dots, j'_p$.

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summation over all admissible indices $j_1, \dots, j_p$ and $j'_1, \dots, j'_p$.

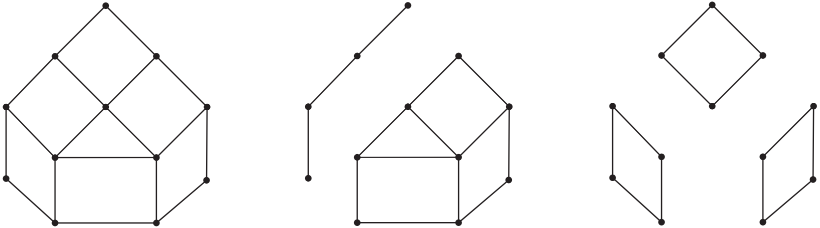
Theorem

If G be a graph with $\Theta = \Theta^*$ and k is a positive integer, then

$$W_k(G) = \sum_{\substack{t_{i_1},\dots,t_{i_p} > 0 \\ t_{i_1}+\dots+t_{i_p}=k}} \binom{k}{t_{i_1}, \dots, t_{i_p}} N_{i_1,\dots,i_p}.$$

Example: G

- G, $\Theta = \Theta^*$, L_1, L_2, L_3, L_4.
- $W_1(G) = W(G) = 3 \cdot (4 \cdot 8) + (4 \cdot 4 + 4 \cdot 4 + 4 \cdot 4) = 144.$
- $W_2(G) = W_1(G) +$
  $2\,[3\,(1 \cdot 5 + 3 \cdot 3) + 3\,(2 \cdot 2 + 2 \cdot 4 + 2 \cdot 2 + 2 \cdot 4 + 0 \cdot 2 + 0 \cdot 2)].$
- $WW(G) = W_1(G)/2 + W_2(G)/2 = 258.$



G $\quad$ G - L_1 $\quad$ G - L_4

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Thank you for your attention!