Intersection Graphs and Order Dimension

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Outline

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Containment Orders and Dimension
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The Proof
A linear extension of $P = (X, <)$ is a linear order $L$, such that

- $x <_P y \implies x <_L y$
A family $\mathcal{L}$ of linear extensions is a realizer for $P = (X, <)$ provided that

* for every incomparable pair $(x, y)$ there is an $L \in \mathcal{L}$ such that $x < y$ in $L$.

The dimension, $\dim(P)$, of $P$ is the minimum $t$, such that there is a realizer $\mathcal{L} = \{L_1, L_2, \ldots, L_t\}$ for $P$ of size $t$. 

Dimension of Orders I
The dimension of an order \( P = (X, <) \) is the least \( t \), such that \( P \) is isomorphic to a suborder of \( \mathbb{R}^t \) with the product ordering.
Containment Orders and Dimension

- Containment orders of intervals — dimension \( \leq 2 \).
- Containment orders of triangles* — dimension \( \leq 3 \).
- Containment orders of \( n \)-gons* — dimension \( \leq n \).
- Containment orders of \( k \)-boxes — dimension \( \leq 2k \).

*prescribed slopes
Incidence Orders of Planar Maps

The incidence order of vertices, edges, and faces $P_{VEF}(G)$ of a plane graph $G$.

- If $G$ is 3-connected $P_{VEF}(G)$ is the truncated face lattice of the corresponding 3-polytope (Steinitz).
Theorem [Schnyder 1989].
If $G$ is a plane triangulation with a face $F$, then

\begin{itemize}
  \item $\dim(P_{VEF}(G \setminus F)) = 3$
  \item $\dim(P_{VEF}(G)) = 4$
\end{itemize}
Incidence Orders of Planar Maps

The embedded incidence orders $P_{VE}(G)$ and $P_{VEF}(G \setminus F)$ with the graph $G$. 

![Diagram of incidence orders](image-url)
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A bipartite Graph can be viewed as a height 2 order.

- We can talk about $\dim(G)$ when $G$ is bipartite.
Grid Intersection Graphs

A **GIG** is an intersection graphs of horizontal and vertical segments.

- GIGs are bipartite.
In each projection **minimals** are taken early, **maximals** are taken late.

**Theorem.** $G$ a GIG, then $\dim(G) \leq 4$. 

Dimension of GIGs
**Theorem** [generalization]. If a bipartite graph $G = (X, Y; E)$ has a representation as intersection graphs of objects from a $t$-separable class, then $\dim(G) \leq 2t$.

- In each projection, **minimals** are taken early, **maximals** are taken late.

**Theorem.** $G$ a GIG, then $\dim(G) \leq 4$. 
Subclasses of GIGs
Subclasses of GIGs - The Inclusion Order

- GIG
- 3-DORG
- UGIG
- SegRay
- StabGIG
- Stick
- 4-DORG
- Bipartite permutation

- Planar
- Outerplanar
- 3-dimensional
- 4-dimensional
SegRay graphs

For interval dimension we only care of \textit{min-max} pairs.
The interval dimension of a SegRay graph is at most 3.
Outerplanar vertex-face as SegRay

Diagram of a outerplanar graph with vertices labeled from 1 to 9 and edges connecting them.
Outerplanar vertex-face as SegRay
Outerplanar vertex-face as SegRay

- Iterate using leaves of the dual tree
Outerplanar vertex-face as SegRay
Outerplanar vertex-face as SegRay
Dimension of SegRay graphs

- Vertex-face posets of outerplanar maps are SegRay graphs.

**Theorem (F. and Nilsson 2006)**
There are outerplanar maps with a 4-dimensional vertex-face poset.

**Corollary**
There are SegRay graphs of dimension 4.

**Corollary**
The interval dimension of a vertex-face poset of an outerplanar map is 3.
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Theorem [Brightwell+Trotter ’93]. If \( G \) is a 3-connected plane graph with a face \( F \), then

\[
\begin{align*}
\dim(P_{VEF}(G \setminus F)) &= 3 \\
\dim(P_{VEF}(G)) &= 4
\end{align*}
\]
Theorem [Brightwell+Trotter ’97]. If $G$ is a plane multi-graph with loops, then

$$\dim(P_{VEF}(G)) \leq 4.$$
The split of $P = (X, <)$ is $\text{split}(P) = (X', \cup X'', <_s)$ with $x' <_s y''$ iff $x \leq y$.

**Theorem [Kimble 78].**

$\dim(P) \leq \dim(\text{split}(P)) \leq \dim(P) + 1.$
Planar Bipartite Graphs

**Theorem** [Hartman-Newman-Ziv ’91 and de Fraysseix-Ossona de Mendez-Pach ’95].

Every planar bipartite graph $H$ admits a contact representation with interiorly disjoint horizontal and vertical segments.
Angle Graphs of Planar Graphs

- Angle graphs are planar bipartite.
- They are the comparability graphs of vertex-face posets.
The First Step

\[ G \] 2-connected plane multi-graph (no loops).

- The order dimension of \( P_{VF}(G) \), the incidence order of vertices and faces of a planar multigraph \( G \) (no loops) is at most four, moreover \( \text{dim}(\text{split}(P_{VF}(G))) \leq 4 \).
Theorem. If $G$ is a 2-connected and plane multigraph, then $\dim(\text{split}(P_{VEF}(G))) \leq 4$. 
Loops

- Break loops by inserting a new vertex.

\( \text{split}(P_{\text{VEF}}(G)) \) is a suborder of \( \text{split}(P_{\text{VEF}}(G^+)) \)
Cut Vertices

- Use induction: break $G$ into $G_1$ and $G_2$ at a cut vertex:

**Theorem.** If $G$ is a plane multigraph -loops allowed-, then \( \dim(\text{split}(P_{VEF}(G))) \leq 4 \).
The End

Thank You