New Results on Geodesic and Abstract Voronoi Diagrams

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Given a set $S$ of $n$ point sites on the plane, the Voronoi Diagram $V(S)$ is a planar subdivision:

1. Each region associated with one site $p \in S$ and denoted by $\text{VR}(p, S)$
2. For each point $x \in \text{VR}(p, S)$, $p$ is its nearest site in $S$
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$VR(p, S)$ is the locus of points closer to $p$ than to any other site.
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- $e$ is part of the bisector $B(p, q)$ between $p$ and $q$
Voronoi Edge and Voronoi Vertex

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\( k^{th}\)-order Voronoi Diagram

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\[ V_{n-1}(S) \]
Euclidean $k^{\text{th}}$-order Voronoi Diagram

- $V_k(S)$ has $O(k(n - k))$ regions (Lee 1982)
  - Deterministic Algorithms
    - $O(nk^2 \log n)$ time (Lee 1982)
    - $O(n^2 \log n + k(n - k) \log^2 n)$ time and $O(k(n - k))$ space or $O(n^2 + k(n - k) \log^2 n)$ time and $O(n^2)$ space (Chazelle and Edelsbrunner 1987)
  - Randomized Algorithms
    - $O(kn^{1+\epsilon})$ time (Clarkson 1987)
    - $O(k(n - k) \log n + n \log^3 n)$ time, $O(n \log n + nk \log k)$ time, and $O(n \log n + nk2^{O(\log^* k)})$ time (Agarwal et al. 1998, Chan 1998 and Ramos 1999)
  - On-Line Algorithms
    - $O(n \log n + nk^3)$ time and $O(nk^2)$ space (Boissonnat et al. 1993)
    - $O(nk^2 + nk \log^2 n)$ time and $O(k(n - k))$ space (Aurenhammer and Schwarzkopf 1992)
Various Voronoi Diagrams

How many common properties do Voronoi diagrams of different geometric objects in different distance metrics share?

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Abstract Voronoi diagrams

- Abstract concept of Voronoi diagrams
  - Each site has its own influence on the underlying space
  - A point in the space is assigned to the site with the strongest influence on it.
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- Abstract Voronoi diagrams (Rolf Kein 1989)
  - A bisecting curve system $\mathcal{J}$ of unclosed Jorden curves
    (instead of concrete geometric sites and distance metrics)
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  (A1) For \( \forall S' \subset S \) of size 3, \( \mathbb{R}^2 = \bigcup_{p \in S} \overline{\text{VR}(p, S')} \)

  (A2) For \( \forall S' \subset S \) of size 3, \( \text{VR}(p, S') \) is path-connected
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  - A category of Voronoi diagrams
    - Points in any convex distance function
    - Line segments in Euclidean metric
    - ......
Abstract Voronoi Regions

Bisector $J(p, q)$
Euclidean metric.
Abstract Voronoi Regions

$D(p, q)$

$D(q, p)$

$p \bullet$

$q \bullet$

Bisector $J(p, q)$

A Jorden Curve.
Abstract Voronoi Regions

Bisector $J(p, q)$
A Jorden Curve.

Voronoi Region of $p$:

\[ VR(p, S) := \bigcap_{q \in S \setminus \{p\}} D(p, q) \]
Abstract Voronoi Regions

Voronoi Region of $p$:
$$\text{VR}(p, S) := \bigcap_{q \in S \setminus \{p\}} D(p, q)$$

Voronoi Diagram of $S$:
$$V(S) := \mathbb{R}^2 \setminus \bigcup_{p \in S} \text{VR}(p, S)$$

Bisector $J(p, q)$
A Jordan Curve.
Known Results

- **Abstract Voronoi diagram** can be computed in $O(n \log n)$ steps
  - Divide and Conquer Algorithm (Klein 1989)
  - Randomized Incremental Construction (Klein, Mehlhorn, Meiser, 1993)
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- The bisecting system of geometric sites in a distance metric satisfies necessary axioms
  - The corresponding Voronoi diagram is an abstract Voronoi diagram
  - can be computed in $O(n \log n)$ steps.
Four results

- Higher-Order Geodesic Voronoi Diagrams in a Polygonal Domain with Holes (SODA 2013)
  - $|V_k(S)| = \Theta(k(n - k) + kc)$, # of faces = $\Theta(k(n - k) + kh)$
  - $c =$ # of polygonal vertices, $h =$ # of holes

- On the Complexity of Higher Order Abstract Voronoi Diagrams (ICALP 2013)
  - The number of faces is at most $2k(n - k)$
  - joint work with Lugano group (P. Cheilaris, E. Papadopoulou, M. Zavershynskyi)

- Abstract Voronoi Diagrams with Disconnected Regions (ISAAC 2013)
  - A randomized incremental algorithm.

- Forest-Like Abstract Voronoi Diagrams in Linear Time (EuroCG 2014)
  - A linear time algorithm for special abstract Voronoi diagrams.
Geodesic Distance in Polygonal Domain

- Polygonal Domain with Holes
  - An outer polygon $P$ and a set $\mathcal{H}$ of $h$ polygonal holes inside $P$
  - $V$ is the set of polygonal vertices in $\{P\} \cup \mathcal{H}$, and $c$ is $|V|$
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  - $d(s, t)$: length of shortest path between them.

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- $|V_1(S)| = O(n + c)$, and $V_1(S)$ has $n$ faces.
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- $|V_1(S)| = O(n + c)$, and $V_1(S)$ has $n$ faces
- $|V_{n-1}(S)| = \Theta(nc)$, and $V_{n-1}(S)$ has $\Theta(nh)$ faces
- Bae and Chwa, SoCG, 2009
Our Results

- Euclidean ($L_p$) metric: $|V_k(S)| = O(k(n - k))$
- $|V_1(S)| = O(n)$ and $|V_{n-1}(S)| = O(n)$
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  - Complexity: from $\Theta(n + c)$ to $\Theta(nc)$
  - # of Faces: from $\Theta(n)$ to $\Theta(nh)$
  - How about $k^{th}$-order?
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- Variation between first and $(n - 1)^{st}$ orders.

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How to handle obstacles?

```
1 •
(1, 2)
(1, 2)
(1, 2)

2 •
(1, 3)
(1, 3)
(1, 3)

3 •
(2, 3)
(2, 3)
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```
Continuous Dijkstra Paradigm and Shortest Path Map

- Wavefront $W_p$ from $p$ at time $t$
  - points $x$ with $d(x, p) = t$
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Circular wavelets (Mitchell, 1996)

If a point $u$ is first hit by a wavelet from $v$:

$u$ is associated with $v$

Shortest Path Map (SPM) of $p$:

A point $u \in SPM_p(v)$'s region

shortest path from $u$ to $p$ must pass through $v$
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![Diagram showing wave propagation and shortest path map]
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- **Shortest Path Map** (SPM) of $p$: $\mathcal{SPM}_p$
  - a point $u \in \mathcal{SPM}_p(v)$ (v's region)
    - shortest path from $u$ to $p$ must pass through $v$
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A degree-2 vertex of a bisector $B(p, q)$ results from an edge of $\text{SPM}_p$ or $\text{SPM}_q$. 

**Diagram:**

- $p$
- $q$
- $v_1$
- $v_2$
- $v_3$
- $m_1$
- $m_2$
A degree-2 vertex of a bisector $B(p, q)$ results from an edge of $\text{SPM}_p$ or $\text{SPM}_q$.
A degree-2 vertex of a bisector $B(p, q)$ results from an edge of $SPM_p$ or $SPM_q$.

In different sides of an SPM edge, predecessors are different.

Degree-2 vertices of A Voronoi Edge
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    - $x \in B(p, q) \cap SPM_p(v_1): v_1$

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Degree-2 vertices of A Voronoi Edge

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  - In different sides of an SPM edge, predecessors are different
    - $x \in B(p, q) \cap S\mathcal{PM}_p(v_1): v_1$
    - $x \in B(p, q) \cap S\mathcal{PM}_p(p): p$
Disconnected Voronoi Edge

- Two consecutive disjoint segments of a bisector can be viewed as connected to a **degenerate** degree-2 vertex in a hole.
Disconnected Voronoi Edge

- **Two consecutive disjoint segments** of a bisector can be viewed as **connected** to a *degenerate* degree-2 vertex in a hole.
  - A hole is **NOT** entirely visible from a point.
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  - Boundary of a hole belongs to different SPM regions
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\[ \begin{align*}
\mathcal{SPM}_p(p) \\
\mathcal{SPM}_p(v_1) \\
\mathcal{SPM}_p(v_2)
\end{align*} \]
Disconnected Voronoi Edge

- Two consecutive disjoint segments of a bisector can be viewed as connected to a degenerate degree-2 vertex in a hole.
- A hole is NOT entirely visible from a point.
- Boundary of a hole belongs to different SPM regions.

\[ SP_M_q(v_1) \]
\[ SP_M_q(v_2) \]
\[ SP_M_q(q) \]
Disconnected Voronoi Edge

- Two consecutive disjoint segments of a bisector can be viewed as connected to a degenerate degree-2 vertex in a hole
  - A hole is NOT entirely visible from a point
  - Boundary of a hole belongs to different SPM regions
  - $p$’s or $q$’s SPM region changes
Disconnected Voronoi Edge

- **Two consecutive disjoint segments** of a bisector can be viewed as **connected** to a **degenerate degree-2 vertex** in a hole
  - A hole is **NOT** entirely visible from a point
  - Boundary of a hole belongs to **different SPM regions**
  - p’s or q’s SPM region changes
  - A **degenerate** degree-2 vertex inside a hole
Disconnected Voronoi Edge

- Two consecutive disjoint segments of a bisector can be viewed as connected to a degenerate degree-2 vertex in a hole.
  - A hole is NOT entirely visible from a point.
  - Boundary of a hole belongs to different SPM regions.
  - p’s or q’s SPM region changes.
  - A degenerate degree-2 vertex inside a hole.
- \# disjoint segments = \# degenerate degree-2 vertices + 1.
### Abstract Higher-Order Voronoi Diagrams

**Our Result**

\[ V_k(S) \text{ has at most } 2k(n - k) \text{ faces (tight)} \]

- **Two axioms**
  - (A3) At most 3 bisectors intersect at the same point
  - (A4) No first-order Voronoi region is empty
Abstract Higher-Order Voronoi Diagrams

Our Result

\( V_k(S) \) has at most \( 2k(n - k) \) faces (tight)

- Two axioms
  - (A3) At most 3 bisectors intersect at the same point
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- A category of Voronoi diagrams: many concrete cases
Abstract Higher-Order Voronoi Diagrams

Our Result

\[ V_k(S) \] has at most \( 2k(n - k) \) faces (tight)

- Two axioms
  - \((A3)\) At most 3 bisectors intersect at the same point
  - \((A4)\) No first-order Voronoi region is empty
- A category of Voronoi diagrams: many concrete cases
- A shaper bound
  - \( O(k(n - k)) \) for points in \( L_p \) metric (Lee 1982)
  - \( O(k(n - k)) \) for line segments in Euclidean metric (Papadopoulou and Zavershynskyi 2012)
Number of Faces

Lee (1982) in Euclidean

\# faces of \( V_k(S) = 2kn - k^2 - n + 1 - \sum_{i=1}^{k-1} S_i \)

- \( S_i \) is \# of unbounded edges of \( V_i(S) \)
Number of Faces

Lee (1982) in Euclidean

\[ \text{\# faces of } V_k(S) = 2kn - k^2 - n + 1 - \sum_{i=1}^{k-1} S_i \]

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Lee’s formula still holds in abstract version
Number of Faces

Lee (1982) in Euclidean

\[
\# \text{ faces of } V_k(S) = 2kn - k^2 - n + 1 - \sum_{i=1}^{k-1} S_i
\]

- \( S_i \) is \( \# \) of unbounded edges of \( V_i(S) \)

Lee’s formula still holds in abstract version

- If \( \sum_{i=1}^{k-1} S_i \geq k(k-1) \), \( \# \) faces \( \leq 2k(n-k) \)
Number of Unbounded Edges of $V_k(S)$

An **unbounded** Voronoi edge := an **unbounded** piece of a bisector
Number of Unbounded Edges of $V_k(S)$

An unbounded Voronoi edge := an unbounded piece of a bisector $x \in \mathbb{R}^2$ not on any bisecting curve of $J$:

An order of $S$ at $x$: $p <_x q \iff x \in D(p, q)$
Number of Unbounded Edges of $V_k(S)$

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First $k$ elements in this order = $k$ nearest sites of $x$
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An order of $S$ at $x$ : $p <_x q \iff x \in D(p, q)$

First $k$ elements in this order = $k$ nearest sites of $x$

$S_k = \#$ switches between $k^{th}$ and $(k + 1)^{st}$ positions
Number of Switches among the First $k+1$ Elements

Bigger class of permutation sequences:

1. Two consecutive permutations differ by one switch
   - When we cross $B(p, q)$, $p$ and $q$ must be adjacent in the permutations

2. Any two elements switch exactly twice
   - A bisector $B(p, q)$ has exactly two unbounded pieces
Number of Switches among the First $k+1$ Elements

Bigger class of permutation sequences:

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Lemma

If $P$ is such a sequence, then

$$\# \text{switches among first } k+1 \text{ elements} \leq k(2n - k - 1)$$
Total Number of Unbounded Edge

\[
\min \# \text{ unbounded } i^{th}\text{-order edges }, 1 \leq i \leq k - 1 \\
= \min \# \text{ switches at positions } 1, \ldots, k \\
\geq \underbrace{2 \binom{n}{2}}_{= \text{total # switches}} - \max \# \text{ switches at last } n - k + 1 \text{ positions} \\
= \text{first } n-k+1 \text{ positions} \\
\geq n(n-1) - (n - k)(2n - (n - k) - 1) = k(k - 1)
\]
Total Number of Unbounded Edge

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= \min \# \text{ switches at positions } 1, \ldots, k \\
\geq 2 \binom{n}{2} - \max \# \text{ switches at last } n - k + 1 \text{ positions} \\
= \text{total } \# \text{ switches} \\
\geq n(n - 1) - (n - k)(2n - (n - k) - 1) = k(k - 1) \\
\]

\# faces of \( V_k(S) \) = \( 2kn - k^2 - n + 1 - \sum_{i=1}^{k-1} S_i \) \\
\geq k(k-1) \\
\leq 2kn - 2k^2 = 2k(n - k)
Abstract Voronoi Diagrams with Disconnected Regions

For all subsets $S' \subseteq S$ of size 3:

\begin{align*}
  R_2 &= \bigcup p_2 \in S' \text{VR}(p_2; S) \\
  \text{Old Axiom:} & \quad \text{VR}(p_2; S) \text{ is path-connected} \\
  \text{New Axiom:} & \quad \text{VR}(p_2; S) \text{ consists of at most} s \text{ connected components}
\end{align*}
Abstract Voronoi Diagrams with Disconnected Regions

For all subsets \( S' \subseteq S \) of size 3:

\[
(A1) \quad \mathbb{R}^2 = \bigcup_{p \in S} \overline{VR(p, S')}
\]
Abstract Voronoi Diagrams with Disconnected Regions

For all subsets $S' \subseteq S$ of size 3:

(A1) $\mathbb{R}^2 = \bigcup_{p \in S} \text{VR}(p, S')$

Old Axiom:

(A2) $\text{VR}(p, S')$ is path-connected
Abstract Voronoi Diagrams with Disconnected Regions

For all subsets $S' \subseteq S$ of size 3:

(A1) $\mathbb{R}^2 = \bigcup_{p \in S} \overline{VR(p, S')}$

Old Axiom:

(A2) $VR(p, S')$ is path-connected

Theorem (R. Klein, K. Mehlhorn, S. Meiser ’93)

If all Voronoi regions are connected, $V(S)$ can be computed in expected time $O(n \log n)$. 
Abstract Voronoi Diagrams with Disconnected Regions

For all subsets $S' \subseteq S$ of size 3:

(A1) $\mathbb{R}^2 = \bigcup_{p \in S} \overline{VR(p, S')}$

Old Axiom:

(A2) $VR(p, S')$ is path-connected

New Axiom:

(A2) $VR(p, S')$ consists of at most $s$ connected components

C. Bohler, R. Klein, and C.-H. Liu

New Results on Geodesic and Abstract Voronoi Diagrams
Main Theorem

Theorem

*If each Voronoi region in a diagram of 3 sites consists of at most $s$ components, $V(S)$ can be computed in expected time $O(s^2 n \sum_{j=2}^{n} \frac{m_j}{j})$.*

$m_j :=$ average number of faces per region over all AVD’s of $j$ sites.
Computing $V(S)$

For $s = 1$

Randomized incremental construction in expected time $O(n \log n)$

K. Mehlhorn, S. Meiser, C. O’Dúnlaing ’91
R. Klein, K. Mehlhorn, S. Meiser ’93
Computing $V(S)$

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$V(R)$ already constructed, history graph $H(R)$ available

- nodes: Voronoi edges ever constructed
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1. pick $t$ randomly in $S \setminus R$
Computing $V(S)$

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1. pick $t$ randomly in $S \setminus R$
2. use $H(R)$ to determine $V(R) \cap VR(t, R \cup \{t\})$
Computing $V(S)$

For $s = 1$

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- nodes: Voronoi edges ever constructed

1. pick $t$ randomly in $S \setminus R$
2. use $H(R)$ to determine $V(R) \cap VR(t, R \cup \{t\})$
3. update $V(R \cup \{t\}) \leftarrow V(R)$ and $H(R \cup \{t\}) \leftarrow H(R)$
Problem with $H(R)$ for disconnected regions

New edge $e$ is made successor of all edges of $P \cup \{e_1, e_2\}$
Problem with $H(R)$ for disconnected regions

New edge $e$ is made successor of all edges of $P \cup \{e_1, e_2\}$

VR($s, R \cup \{s, t\}$) does not intersect the predecessors of $e$
Problem with $H(R)$ for disconnected regions

New edge $e$ is made successor of all edges of $P \cup \{e_1, e_2\}$

$VR(s, R \cup \{s, t\})$ does not intersect the predecessors of $e$

Cannot use $H(R \cup \{t\})$ to find the intersection

C. Bohler, R. Klein, and C.-H. Liu  New Results on Geodesic and Abstract Voronoi Diagrams
New $\mathcal{H}(R)$: Vertical Trapezoidal Decomposition

$s \geq 2$
Assumption: Bisectors are $x$-monotone (at most constantly many points of vertical tangency)
New $\mathcal{H}(R)$: Vertical Trapezoidal Decomposition

$s \geq 2$
Assumption: Bisectors are $x$-monotone (at most constantly many points of vertical tangency)
$\rightarrow V^*(R)$: vertical decomposition of $V(R)$ into pseudo-trapezoidal cells

[Seidel, Sharir, Agarwal]
New $\mathcal{H}(R)$: Vertical Trapezoidal Decomposition

$s \geq 2$
Assumption: Bisectors are $x$-monotone (at most constantly many points of vertical tangency)
$\rightarrow V^*(R)$: vertical decomposition of $V(R)$ into pseudo-trapezoidal cells

[Seidel, Sharir, Agarwal]

![Diagram of H(R)](image_url)

$H(R)$
- nodes: all trapezoids ever created
New $\mathcal{H}(R)$: Vertical Trapezoidal Decomposition

$s \geq 2$
Assumption: Bisectors are $x$-monotone (at most constantly many points of vertical tangency)
$\rightarrow V^*(R)$: vertical decomposition of $V(R)$ into pseudo-trapezoidal cells

[Seidel, Sharir, Agarwal]
Constructing $\mathcal{H}(R \cup \{t\})$
Constructing $\mathcal{H}(R \cup \{t\})$

C. Bohler, R. Klein, and C.-H. Liu

New Results on Geodesic and Abstract Voronoi Diagrams
Constructing $\mathcal{H}(R \cup \{t\})$
Constructing $\mathcal{H}(R \cup \{t\})$

**Lemma**

If $t$ is in conflict with a trapezoid $A$ of $V^*(R)$, then $t$ is in conflict with a predecessor of $A$ in $\mathcal{H}(R)$. 

C. Bohler, R. Klein, and C.-H. Liu

New Results on Geodesic and Abstract Voronoi Diagrams
Special Voronoi Diagrams

Tree-Like Voronoi Diagrams
Special Voronoi Diagrams

Tree-Like Voronoi Diagrams

- Points in convex position
Special Voronoi Diagrams

Tree-Like Voronoi Diagrams

- Points in convex position
- Farthest Voronoi diagram
Special Voronoi Diagrams

Tree-Like Voronoi Diagrams

- Points in convex position
- Farthest Voronoi diagram
- Farthest abstract Voronoi diagram

[Mehlhorn, Meiser, Rasch, 2001]
Special Voronoi Diagrams

Tree-Like Voronoi Diagrams

- Points in convex position
- Farthest Voronoi diagram
- Farthest abstract Voronoi diagram

[Mehlhorn, Meiser, Rasch, 2001]

If the ordering of the sites around the tree is known and each site occurs only once. Can we compute the diagram in time $O(n)$?
Theorem (A. Aggarwal, L. Guibas, J. Saxe, P. Shor, ’87)

Given the ordering of $n$ points in convex position, the Voronoi diagram can be computed in time $O(n)$. 
Former Results

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How about abstract VD’s?
Former Results

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Given the ordering of $n$ points in convex position, the Voronoi diagram can be computed in time $O(n)$.

How about abstract VD’s?

Theorem (R. Klein, A. Lingas, ’93)

If $V(S')$ is a tree for all $S' \subseteq S$, given the ordering of the $n$ sites at infinity, the abstract Voronoi diagram can be computed in time $O(n)$.
Basic Idea
Basic Idea

1. Choose a tentative set of **red** sites $R \subseteq S$ such that no two regions of consecutive red sites interfere.
Basic Idea

1. Choose a tentative set of red sites $R \subseteq S$ such that no two regions of consecutive red sites interfere.

2. Compute $V(B)$ recursively, $B = S \setminus R$ ($B$: blue sites).

Diagram:

- $H$ is the horizontal line.
- $x$, $v$, $t$, $s$, and $q$ are points along the line.

Running Time: $O(n)$
Basic Idea

1. Choose a tentative set of red sites $R \subseteq S$ such that no two regions of consecutive red sites interfere.
2. Compute $V(B)$ recursively, $B = S \setminus R$ ($B$: blue sites).
3. Using the structure of $V(B)$, choose a crimson subset $C \subseteq R$, such that no two regions of crimson sites interfere.
Basic Idea

1. Choose a tentative set of red sites $R \subseteq S$ such that no two regions of consecutive red sites interfere.

2. Compute $V(B)$ recursively, $B = S \setminus R$ ($B$: blue sites).

3. Using the structure of $V(B)$, choose a crimson subset $C \subseteq R$, such that no two regions of crimson sites interfere.

If $C$ contains a fixed percentage $\delta$ of sites.

Running Time: $O(n)$

C. Bohler, R. Klein, and C.-H. Liu

New Results on Geodesic and Abstract Voronoi Diagrams
Basic Idea

1. Choose a tentative set of red sites \( R \subseteq S \) such that no two regions of consecutive red sites interfere.
2. Compute \( V(B) \) recursively, \( B = S \setminus R \) (\( B \): blue sites).
3. Using the structure of \( V(B) \), choose a crimson subset \( C \subseteq R \), such that no two regions of crimson sites interfere.
4. Construct incrementally \( V(B \cup C) \) from \( V(B) \) by inserting the crimson sites one by one.

---

If \( C \) contains a fixed percentage \( q \).

Running Time: \( O(n) \).

C. Bohler, R. Klein, and C.-H. Liu

New Results on Geodesic and Abstract Voronoi Diagrams
Basic Idea

1. Choose a tentative set of red sites $R \subseteq S$ such that no two regions of consecutive red sites interfere.
2. Compute $V(B)$ recursively, $B = S \setminus R$ ($B$: blue sites).
3. Using the structure of $V(B)$, choose a crimson subset $C \subseteq R$, such that no two regions of crimson sites interfere.
4. Construct incrementally $V(B \cup C)$ from $V(B)$ by inserting the crimson sites one by one.
5. Compute $V(S \setminus (B \cup C))$ recursively.
Basic Idea

1. Choose a tentative set of red sites $R \subseteq S$ such that no two regions of consecutive red sites interfere.
2. Compute $V(B)$ recursively, $B = S \setminus R$ ($B$: blue sites).
3. Using the structure of $V(B)$, choose a crimson subset $C \subseteq R$, such that no two regions of crimson sites interfere.
4. Construct incrementally $V(B \cup C)$ from $V(B)$ by inserting the crimson sites one by one.
5. Compute $V(S \setminus (B \cup C))$ recursively.
6. Merge $V(B \cup C)$ and $V(S \setminus (B \cup C))$. 

If $C$ contains a fixed percentage...
Basic Idea

1. Choose a tentative set of red sites $R \subseteq S$ such that no two regions of consecutive red sites interfere
2. Compute $V(B)$ recursively, $B = S \setminus R$ ($B$: blue sites)
3. Using the structure of $V(B)$, choose a crimson subset $C \subseteq R$, such that no two regions of crimson sites interfere
4. Construct incrementally $V(B \cup C)$ from $V(B)$ by inserting the crimson sites one by one
5. Compute $V(S \setminus (B \cup C))$ recursively
6. Merge $V(B \cup C)$ and $V(S \setminus (B \cup C))$

If $C$ contains a fixed percentage of sites
⇒ Running Time: $O(n)$
Forest-Like Abstract Voronoi Diagrams

Theorem (C. Bohler, R. Klein, C. Liu, 2013)

If $V(S)$ is a tree and $V(S')$ is a forest for all $S' \subseteq S$, given the ordering of the $n$ sites at infinity, the abstract Voronoi diagram can be computed in time $O(n)$. 
Difference

\[ \pi = (p, q, r, s, t, u, v, w, x, y) \]

\[ V(S) \text{ is a tree.} \]
\[ H = (q, r, s, u, v, x) \]

\[ V(S) \text{ is a tree.} \]

\[ V(S') \text{ may be a forest for } S' \subseteq S \]
Problem 1

1 Red-Blue Coloring

\[ \pi = (p, q, r, s, t, u) \]
Problem 1

1. Red-Blue Coloring

\[ \pi = (p, q, r, s, t, u) \]

Consecutive red regions can be adjacent.
Problem 1

1 Red-Blue Coloring

\[ \pi = (p, q, r, s, t, u) \]

Consecutive red regions can be adjacent.

\[ \Rightarrow \text{Can not select Crimson Sites} \]
Problem 2

2. Tree-Lemma

Blue Diagram may not be a tree
Problem 2

2 Tree-Lemma

Blue Diagram may not be a tree
Red regions may not intersect the blue diagram
Problem 2

2 Tree-Lemma

Blue Diagram may not be a tree
Red regions may not intersect the blue diagram
⇒ Can not select crimson site
Thank you for your attention!