Counting and Enumerating Crossing-free Perfect Matchings

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ETH Zürich

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Introduction

Crossing-free Perfect Matching: A set of $n/2$ non-intersecting segments with endpoints in a set P of n points.

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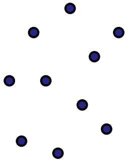

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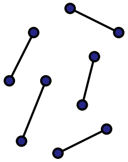

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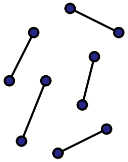

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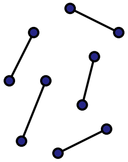

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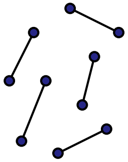

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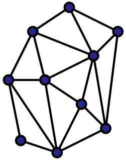

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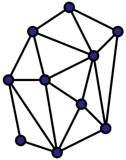

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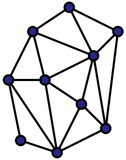

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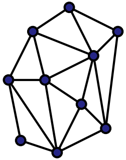

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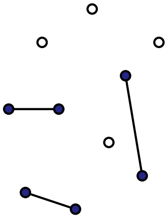

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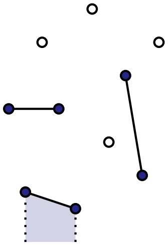

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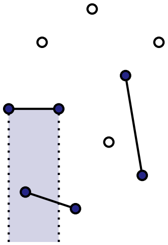

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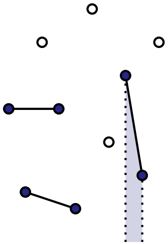

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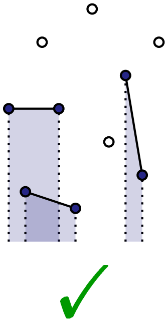

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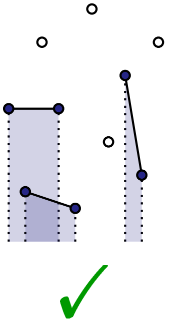

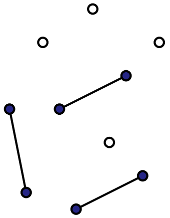

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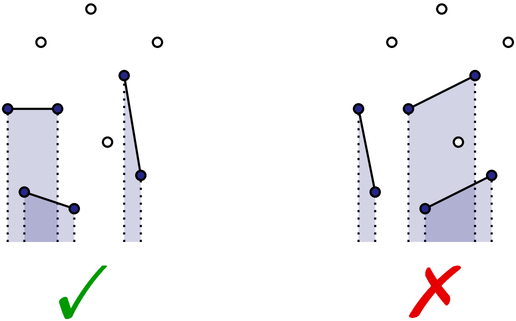

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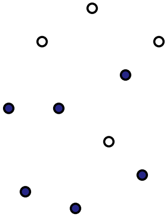

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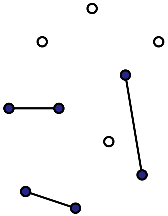

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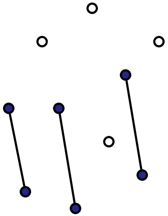

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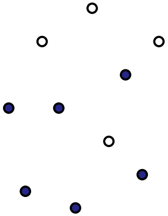

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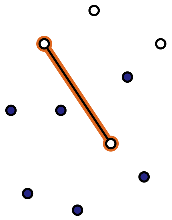

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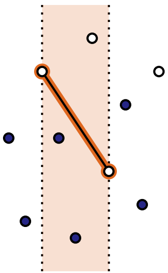

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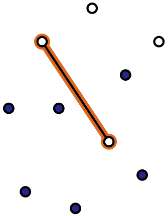

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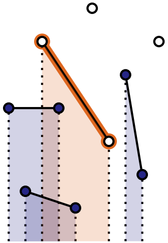

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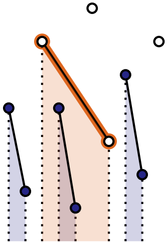

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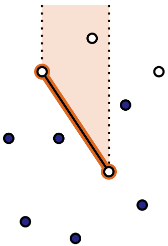

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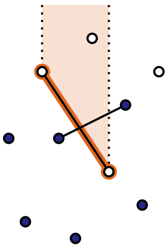

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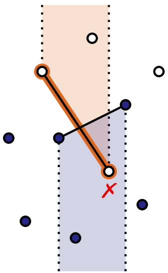

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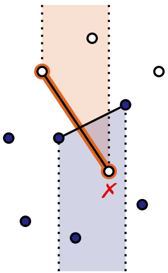

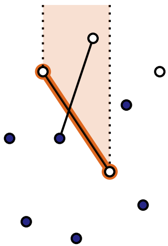

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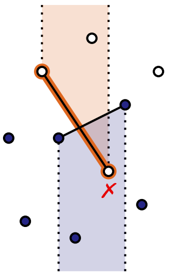

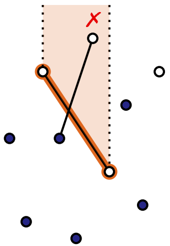

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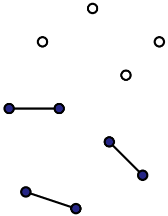

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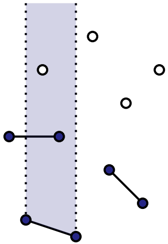

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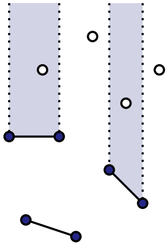

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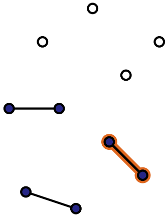

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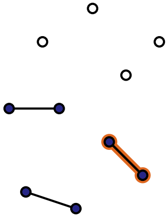

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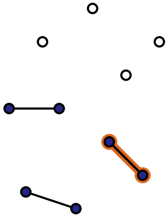

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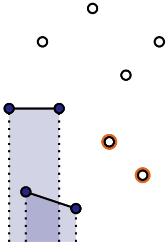

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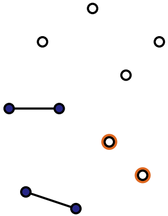

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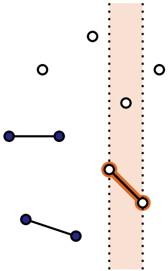

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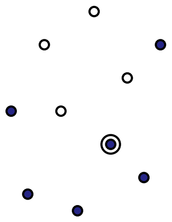

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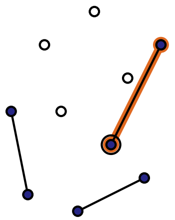

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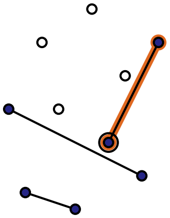

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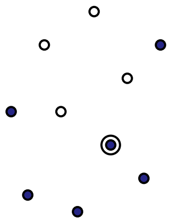

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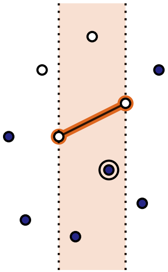

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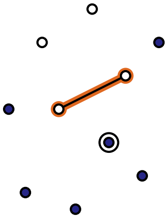

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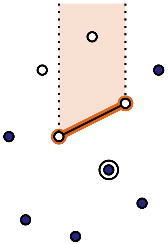

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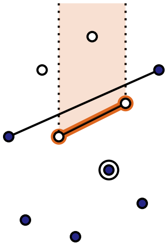

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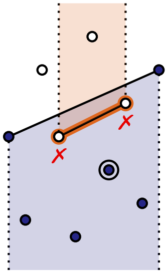

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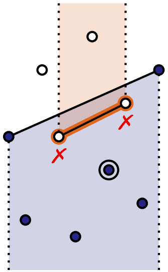

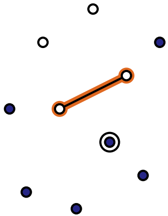

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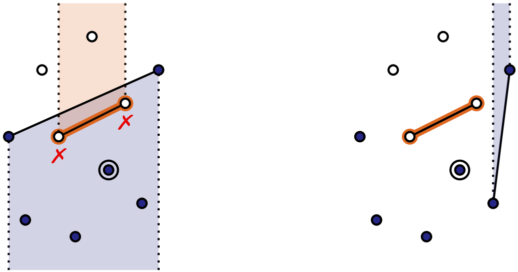

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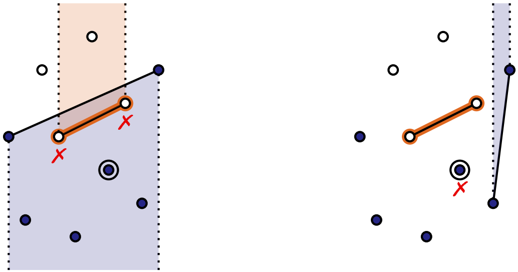

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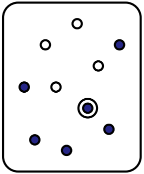

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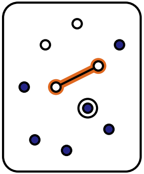

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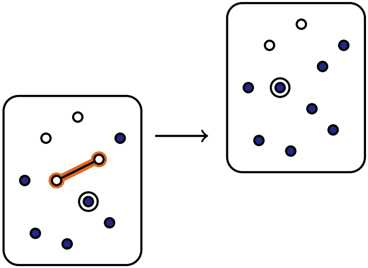

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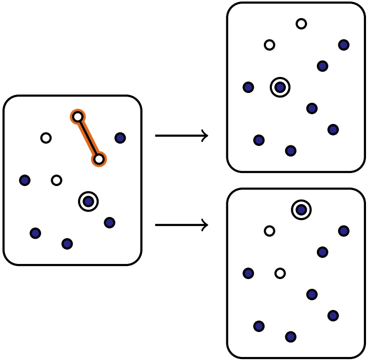

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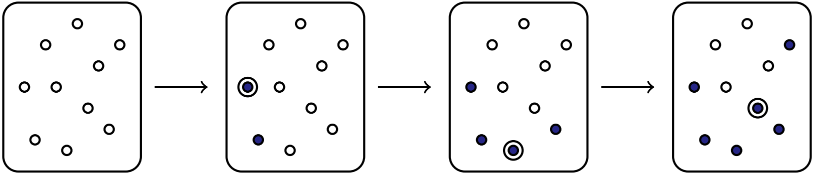

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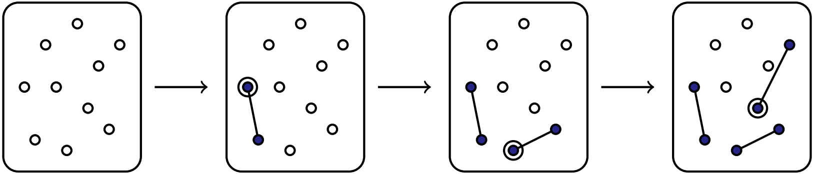

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Theorem

The number $|\mathcal{M}(P)|$ can be computed in time $O(n^3 2^n)$.

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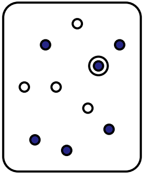

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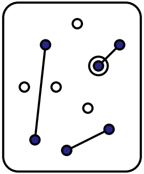

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Theorem

The set $\mathcal{M}(P)$ can be enumerated efficiently, that is, in time $O^*(2^n + |\mathcal{M}(P)|) = O^*(|\mathcal{M}(P)|)$.

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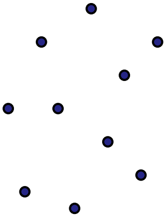

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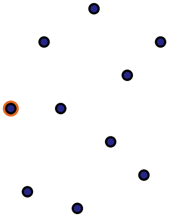

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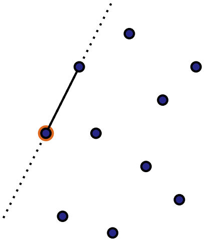

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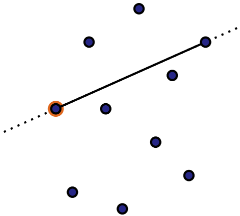

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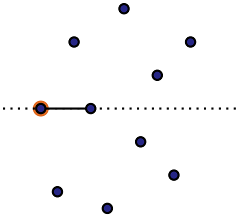

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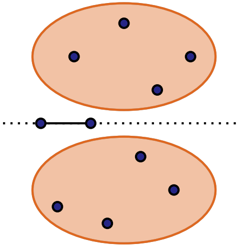

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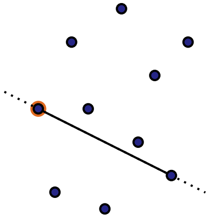

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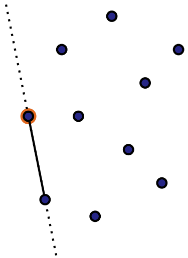

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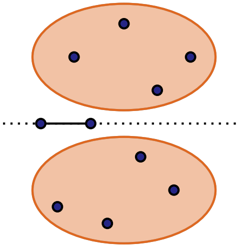

Let $e_m :=$ #easy perfect matchings on $2m = n$ points.

$$e_0 = 1, e_m = \sum_{i=0}^{m-1} e_i \cdot e_{m-i-1} \quad \Rightarrow \quad e_m = \Theta^*(4^m) = \Theta^*(2^n)$$

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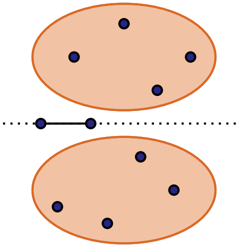

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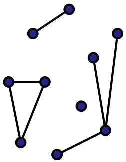
The set $\mathcal{M}(P)$ can be enumerated with polynomial time delay per enumerated object.

All Crossing-free Geometric Graphs



$G \in \mathcal{G}(P)$

- **Lower bound:** $|\mathcal{G}(P)| = \Omega^*(11.65^n)$ for all P. [Flajolet, Noy 99]

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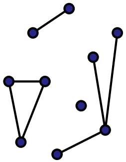

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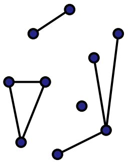

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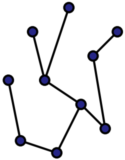
Crossing-free Spanning Trees



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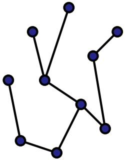

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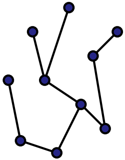

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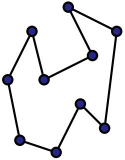

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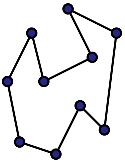

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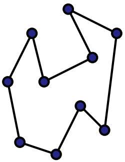

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Thanks for your attention!