Coloring geometric intersection graphs via on-line games

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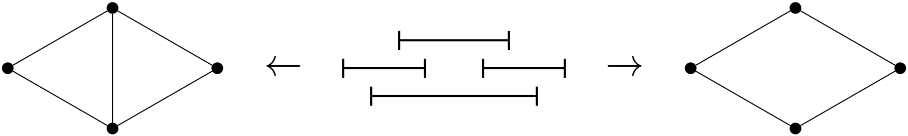
Notation:

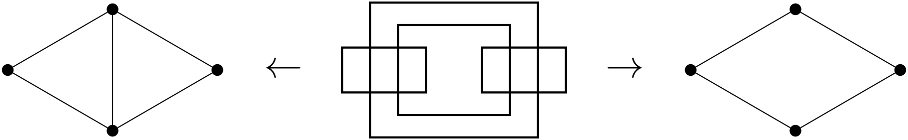
- χ — chromatic number
- ω — maximum clique size
- α — maximum independent set size
- n — number of vertices

Geometric intersection/overlap graphs:

- vertices — geometric objects
- edges — pairs of intersecting/overlapping objects



intersection graph | intervals | overlap graph



intersection graph | axis-parallel rectangles | overlap graph

Folklore result

Interval intersection graphs are perfect; they have $\chi = \omega$.

Asplund, Grünbaum, 1960

Intersection graphs of axis-parallel rectangles in the plane have $\chi = O(\omega^2)$.

Gyárfás, 1985

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Burling, 1965

Triangle-free intersection graphs of boxes in $\mathbb{R}^3$ have unbounded χ.

Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, Walczak, 2013

Triangle-free segment intersection graphs have unbounded χ.

Triangle-free overlap graphs of axis-parallel rectangles have unbounded χ.

For any compact arc-connected set $S \subset \mathbb{R}^2$ (not an axis-parallel rectangle), triangle-free intersection graphs of copies of S scaled independently in horizontal and vertical directions have unbounded χ.

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All the negative results above use essentially the same construction!

It produces graphs with $\chi = \Theta(\log \log n)$ and $\alpha = \Theta(n)$.

McGuinness, 2000; Suk, 2014+

Segment intersection graphs have $\chi = O_\omega(\log n)$.

Fox, Pach, 2014

String graphs (intersection graphs of curves) have $\chi = (\log n)^{O(\log \omega)}$.

Main Problem

Do triangle-free segment intersection graphs have

- $\chi = O(\log \log n)$,
- $\alpha = \Theta(n)$?

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Krawczyk, Pawlik, Walczak, 2013

Triangle-free rectangle overlap graphs have $\chi = O(\log \log n)$.

Krawczyk, Walczak, 2014+

Rectangle overlap graphs have $\chi = O_\omega((\log \log n)^{\omega - 1})$.

Chordal graphs — intersection graphs of subtrees of a tree

Folklore result

Chordal graphs are perfect; they have $\chi = \omega$.

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Subtree overlap graphs — overlap graphs of subtrees of a tree

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There are triangle-free subtree overlap graphs with $\chi = \Theta(\log \log n)$.

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Krawczyk, Walczak, 2014+

Subtree overlap graphs have $\chi = O_\omega((\log \log n)^{\binom{\omega}{2}})$.

There are subtree overlap graphs with $\chi = \Theta_\omega((\log \log n)^{\omega-1})$.

There are string graphs with $\chi = \Theta_\omega((\log \log n)^{\omega-1})$.

On-line graph coloring game:

- Presenter builds a graph presenting vertices one by one with all edges connecting them to previous vertices.
- Algorithm colors each vertex immediately after it is presented.
- Algorithm wants to use as few colors as possible.
- Presenter wants to force Algorithm to use as many colors as possible.

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Possible restrictions of Presenter's play:

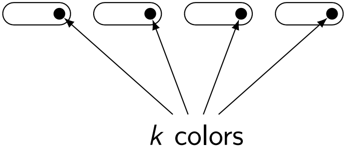
- specific class of graphs,
- representation,
- specific order of presentation of vertices.

Bean, 1976; Gyárfás, Lehel, 1988

In the on-line graph coloring game played on the class of forests,

- Presenter has a strategy to force the use of $\lfloor \log_2 n \rfloor + 1$ colors,
- First-Fit uses at most $\lfloor \log_2 n \rfloor + 1$ colors.

Presenter has a strategy to force the use of $\lfloor \log_2 n \rfloor + 1$ colors on a forest.



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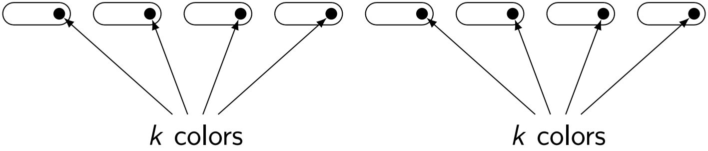

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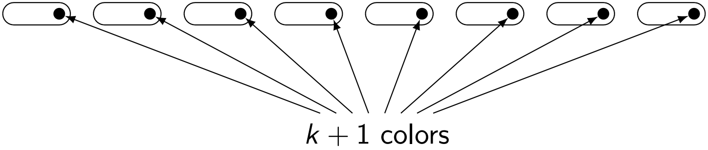

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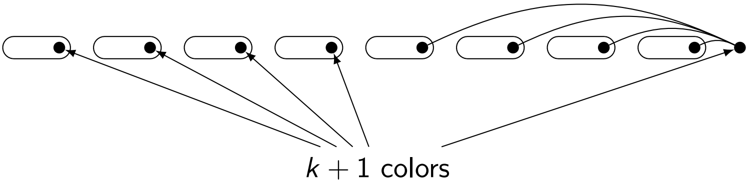

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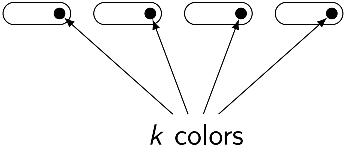
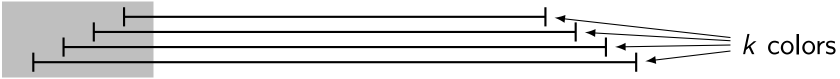


The same strategy works with representation by overlapping intervals.



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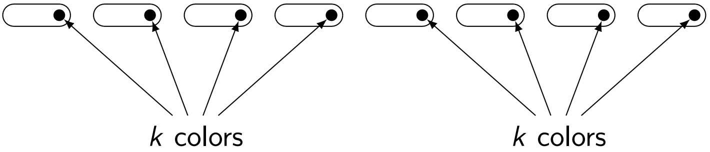

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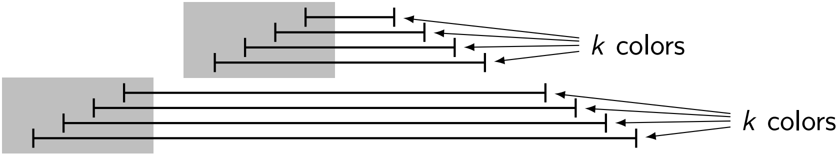

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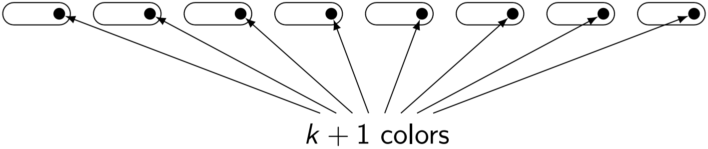

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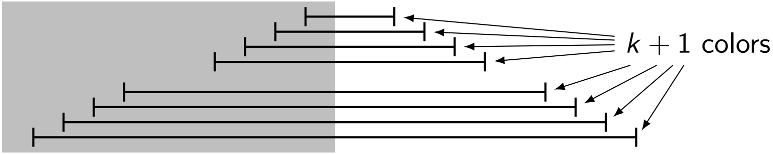

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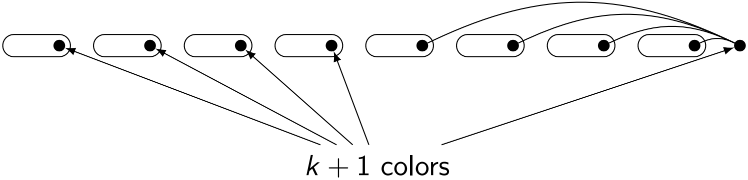

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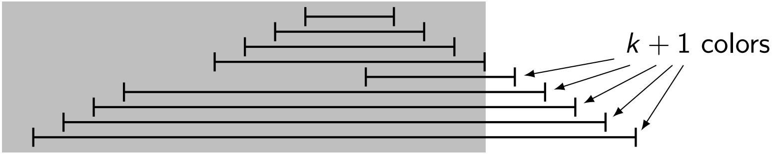

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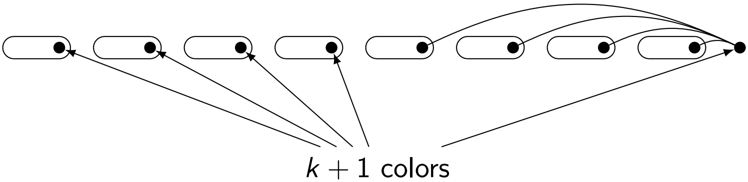

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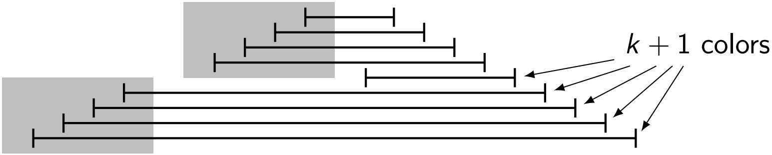
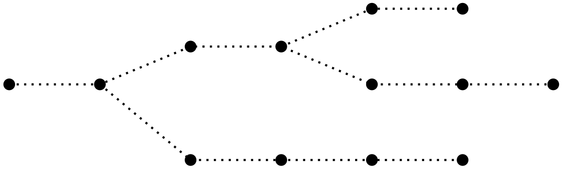


Note: Intervals are presented in the increasing order of left endpoints.

On-line game graphs:

- underlying rooted forest on the vertices,
- representation assigning geometric objects to vertices,



On-line game graphs:

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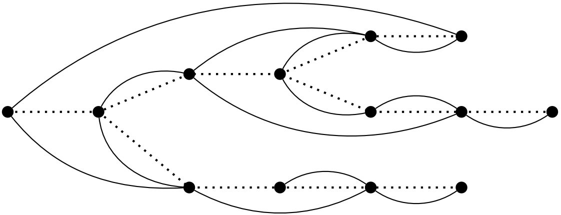

On-line game graphs:

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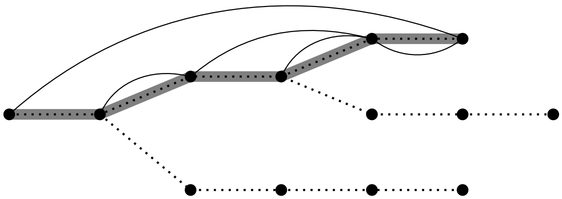
- every root-to-leaf path is a valid presentation scenario in the game.



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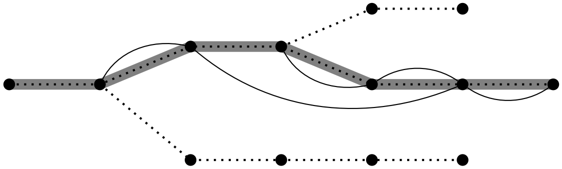

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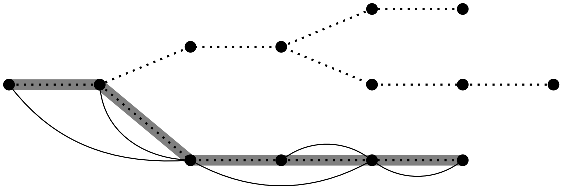

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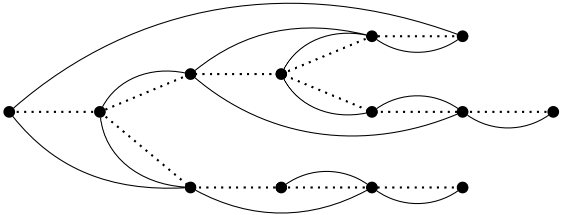
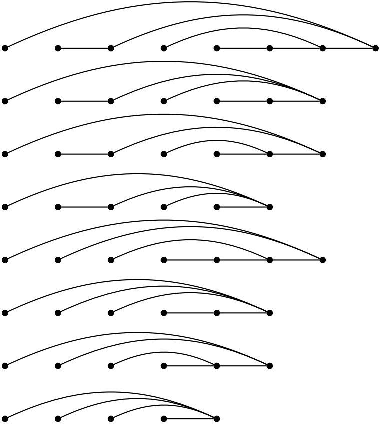
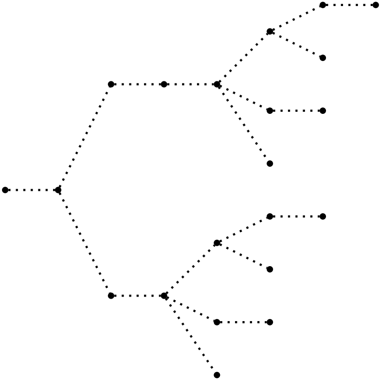


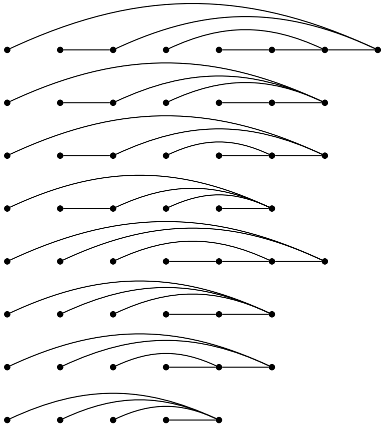
On-line k-coloring algorithm $\rightarrow$ k-coloring of the game graph.



Presenter's strategy



game graph



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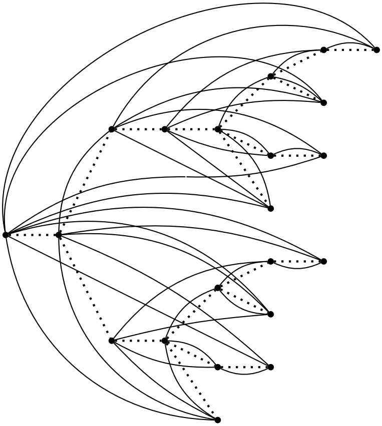

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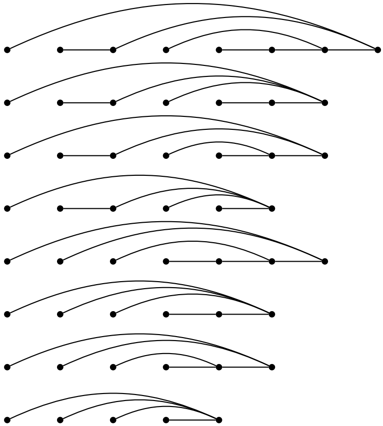

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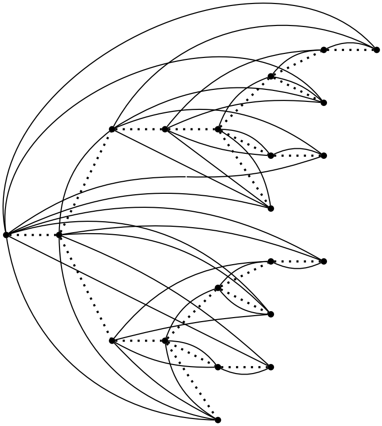
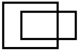
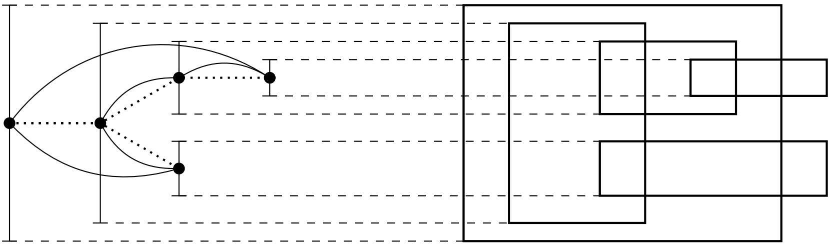
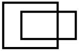


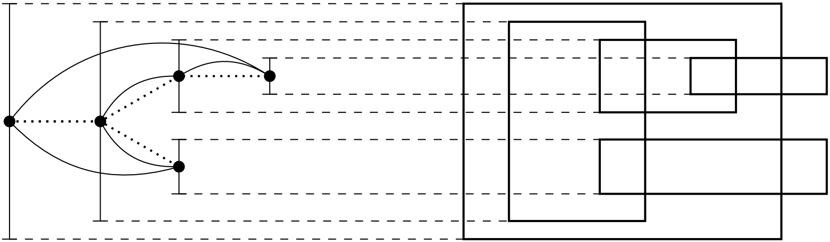
game graph

Presenter's strategy forcing the use of k colors $\rightarrow$ game graph with $\chi \geqslant k$.

game graphs of the on-line coloring game on interval overlap graphs with representation such that intervals are presented in the increasing order of their left endpoints $=$ overlap graphs of axis-parallel rectangles with all overlapping pairs of the form 



game graphs of the on-line coloring game on interval overlap graphs with representation such that intervals are presented in the increasing order of their left endpoints = overlap graphs of axis-parallel rectangles with all overlapping pairs of the form 



These graphs can be also represented as

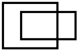
- segment intersection graphs,
- intersection graphs of copies of a fixed compact arc-connected set $S \subset \mathbb{R}^2$ (not an axis-parallel rectangle) scaled independently in horizontal and vertical directions.

Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, Walczak, 2013

In the triangle-free game as before, Presenter has a strategy to force the use of $\lfloor \log_2 n \rfloor + 1$ colors.

$\downarrow$

There are triangle-free

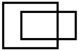
- overlap graphs of axis-parallel rectangles with all overlapping pairs of the form ,
- segment intersection graphs,
- intersection graphs of copies of a fixed compact arc-connected set $S \subset \mathbb{R}^2$ (not an axis-parallel rectangle) scaled independently in horizontal and vertical directions,

with $\chi = \Theta(\log \log n)$.

Krawczyk, Pawlik, Walczak, 2013

In the triangle-free game as before, there is an on-line coloring algorithm using $O(\log n)$ colors.

$\downarrow$ + heavy-light decomposition

Triangle-free overlap graphs of axis-parallel rectangles with all overlapping pairs of the form  have $\chi = O(\log \log n)$.

Triangle-free rectangle overlap graphs have $\chi = O(\log \log n)$.

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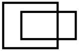
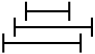
Triangle-free overlap graphs of axis-parallel rectangles with all overlapping pairs of the form  have $\chi = O(\log \log n)$.

Triangle-free rectangle overlap graphs have $\chi = O(\log \log n)$.

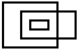
Krawczyk, Walczak, 2014+

In the game as before with clique number ω, there is an on-line coloring algorithm using $O_\omega((\log n)^{\omega-1})$ colors.

In the game as before with clique number ω and excluding , there is an on-line coloring algorithm using $O_\omega(\log n)$ colors.

$\downarrow$

Rectangle overlap graphs have $\chi = O_\omega((\log \log n)^{\omega-1})$.

Rectangle overlap graphs excluding  have $\chi = O_\omega(\log \log n)$.

on-line coloring game on intersection graphs of axis-parallel rectangles with representation $\rightarrow$ intersection graphs of boxes in $\mathbb{R}^3$

Erlebach, Fiala, 2002

In the on-line coloring game on triangle-free intersection graphs of axis-parallel rectangles with representation, Presenter has a strategy to force the use of $\lfloor \log_2 n \rfloor + 1$ colors.

$\downarrow$

Burling, 1965

There are triangle-free intersection graphs of boxes in $\mathbb{R}^3$ with $\chi = \Theta(\log \log n)$.

on-line coloring game on interval graphs with representation	$\rightarrow$	intersection graphs of axis-parallel rectangles

Kierstead, Trotter, 1981

The value of the on-line coloring game on interval graphs with representation is $3\omega - 2$.

$\downarrow$

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on-line coloring game on interval graphs with representation $\rightarrow$ intersection graphs of axis-parallel rectangles

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There are intersection graphs of axis-parallel rectangles with $\chi = 3\omega - 2$.

Kostochka (unpublished)

There are intersection graphs of axis-parallel rectangles with $\chi = 3\omega$.

on-line coloring game on cocomparability graphs with "up-growing" representation	$\leftrightarrow$	interval filament graphs with non-overlapping domains

Felsner, 1997

The value of the on-line coloring game on cocomparability graphs with "up-growing" representation is $\binom{\omega+1}{2}$.

$\downarrow$

Krawczyk, Walczak, 2014+

There are interval filament graphs with non-overlapping domains and with $\chi = \binom{\omega+1}{2}$.

All interval filament graphs with non-overlapping domains have $\chi \leqslant \binom{\omega+1}{2}$.

All interval filament graphs have $\chi = O_\omega(1)$.

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All interval filament graphs have $\chi = O_\omega(1)$.

Rok, Walczak, 2014+

All outerstring graphs have $\chi = O_\omega(1)$.

a specific on-line coloring game on interval filament graphs	$\leftrightarrow$	subtree overlap graphs with no subtree contained in two overlapping subtrees

Krawczyk, Walczak, 2014+

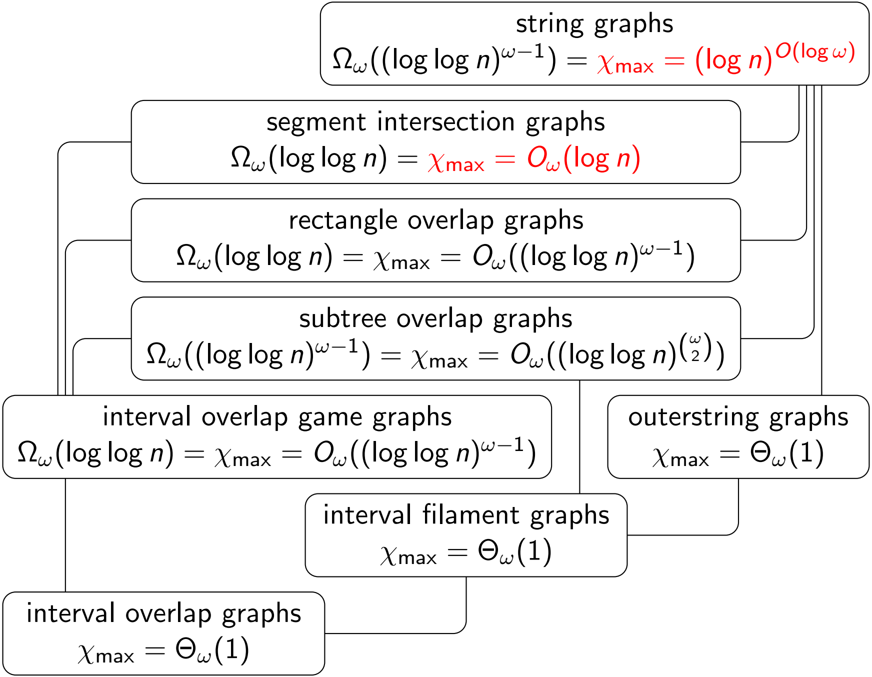
The value of the specific on-line coloring game on interval filament graphs is $\Theta((\log n)^{\omega-1})$.

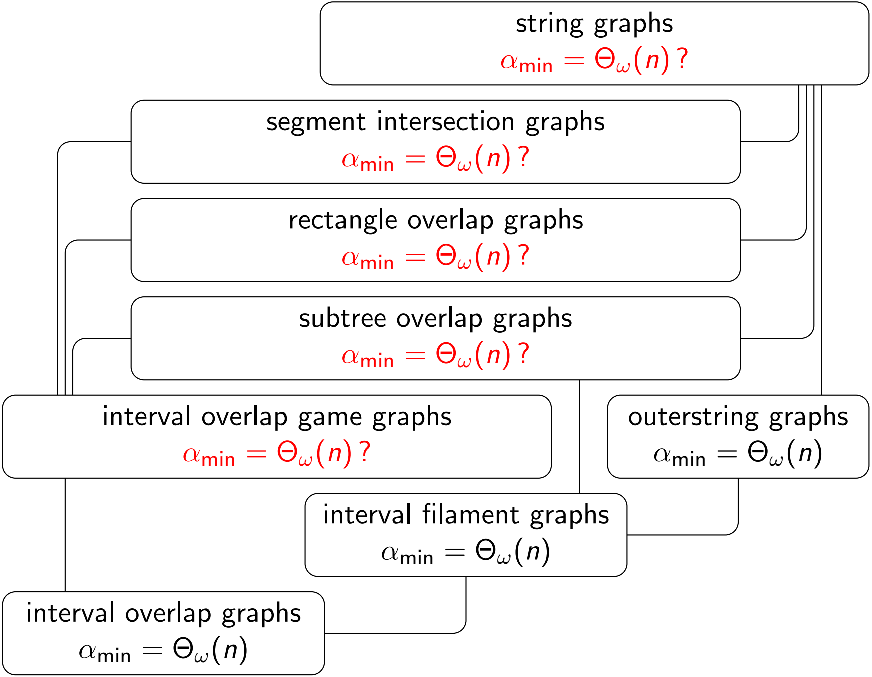
$\downarrow$

There are subtree overlap graphs with no subtree contained in two overlapping subtrees and with $\chi = \Theta_\omega((\log\log n)^{\omega-1})$.

All subtree overlap graphs with no subtree contained in two overlapping subtrees have $\chi = O_\omega((\log\log n)^{\omega-1})$.

All subtree overlap graphs have $\chi = O_\omega((\log\log n)^{\binom{\omega}{2}})$.


string graphs
$\Omega_\omega((\log\log n)^{\omega-1}) = \chi_{\max} = (\log n)^{O(\log\omega)}$
segment intersection graphs
$\Omega_\omega(\log\log n) = \chi_{\max} = O_\omega(\log n)$
rectangle overlap graphs
$\Omega_\omega(\log\log n) = \chi_{\max} = O_\omega((\log\log n)^{\omega-1})$
subtree overlap graphs
$\Omega_\omega((\log\log n)^{\omega-1}) = \chi_{\max} = O_\omega((\log\log n)^{\binom{\omega}{2}})$
interval overlap game graphs
$\Omega_\omega(\log\log n) = \chi_{\max} = O_\omega((\log\log n)^{\omega-1})$
outerstring graphs
$\chi_{\max} = \Theta_\omega(1)$
interval filament graphs
$\chi_{\max} = \Theta_\omega(1)$
interval overlap graphs
$\chi_{\max} = \Theta_\omega(1)$


string graphs
$\alpha_{\min} = \Theta_\omega(n)$?
segment intersection graphs
$\alpha_{\min} = \Theta_\omega(n)$?
rectangle overlap graphs
$\alpha_{\min} = \Theta_\omega(n)$?
subtree overlap graphs
$\alpha_{\min} = \Theta_\omega(n)$?
interval overlap game graphs
$\alpha_{\min} = \Theta_\omega(n)$?
outerstring graphs
$\alpha_{\min} = \Theta_\omega(n)$
interval filament graphs
$\alpha_{\min} = \Theta_\omega(n)$
interval overlap graphs
$\alpha_{\min} = \Theta_\omega(n)$