New results on the coarseness of bicolored point sets

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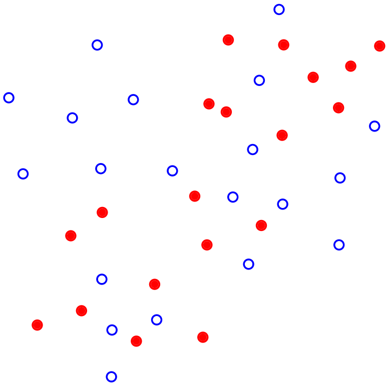
February 16, 2014

Let $S = R \cup B$ be a bicolored set of points.
Is S **"well-separated by color"**, or on the contrary, **"are the colors well-blended"**?



Bereg et al. CGTA, 2013 gave a **formal definition** of **well-blended** point sets!!!

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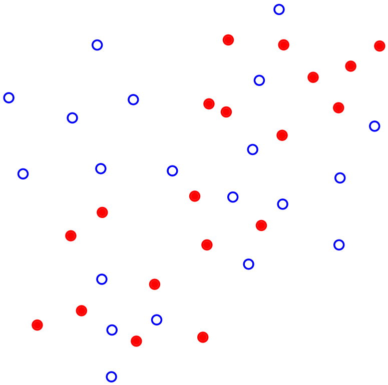

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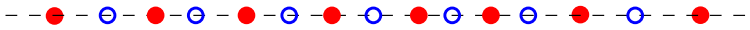
Well-blended point sets

Bereg et al. CGTA, 2013

- On the real line:

  We say that a bicolored point set is **well-blended** if in any interval the discrepancy (difference between the number of red and blue points) is bounded by a constant.



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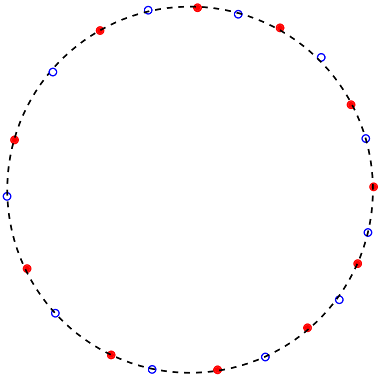

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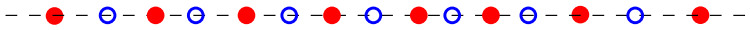

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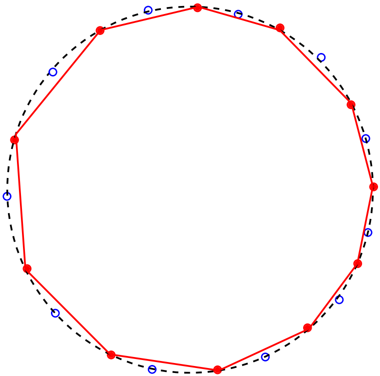

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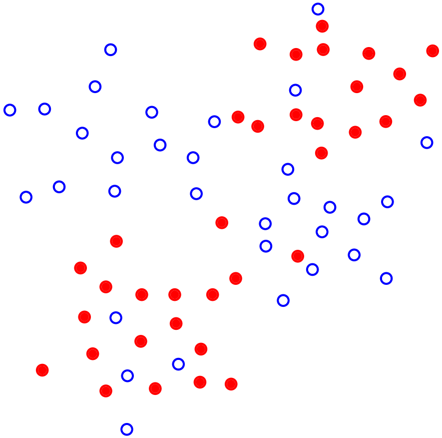

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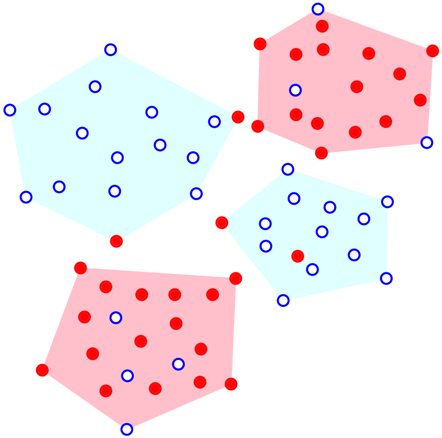
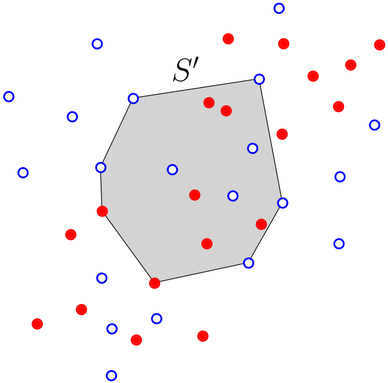


Well-blended point sets

Bereg et al. CGTA, 2013

The formal definition

- [Bautista et al. *Computing maximal islands*. 2011.]: A subset S' of S is an **island** if there is a convex set C on the plane such that $S' = C \cap S$.
- The **discrepancy of an island** is the difference between the number of red and blue points.

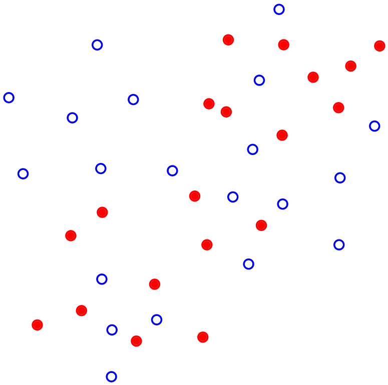


Well-blended point sets

Bereg et al. CGTA, 2013

The formal definition

- A **convex partition** of S is a partition of S into islands, with pairwise disjoint convex hulls.
- The **discrepancy of a convex partition** $\prod = \{S_1, S_2, \ldots S_k\}$ of S, denoted by $disc(\prod)$, is the minimum of $disc(S_i)$ for $i = 1, \ldots, k$.
- The **coarseness** of S is the maximum of $disc(\prod)$ over all the convex partitions of S.



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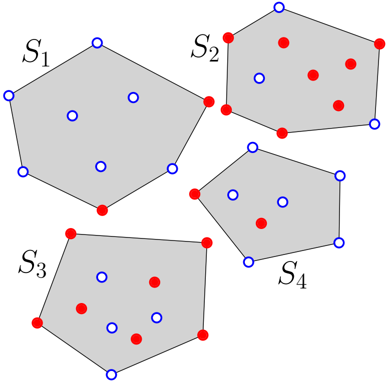
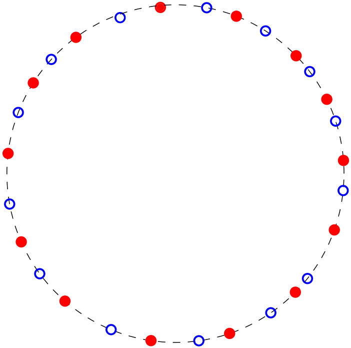


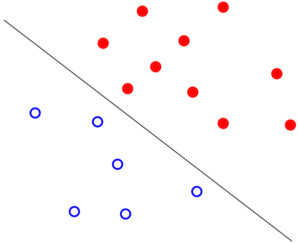
Well-blended point sets

Some examples:

$$C(S) = \max_{\Pi} \min_{S_i \in \Pi} \text{disc}(S_i)$$



$C(S) = 1$



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The more blended set of points has the smaller *coarseness*.

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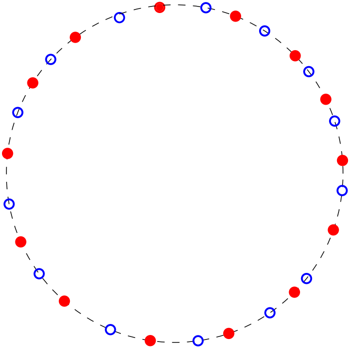

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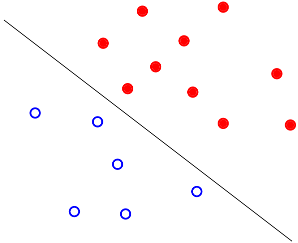

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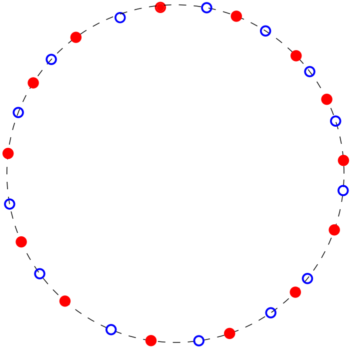

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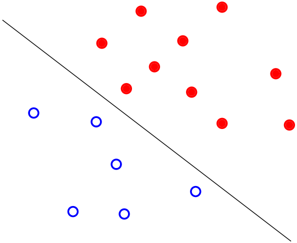

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Well-blended point sets

Bereg et al. CGTA, 2013

The formal definition

- Given r and b, let $\mathcal{C}(S, r, b)$ (**the best coloring**) be the smallest coarseness taken over all the bicolorings $\{R, B\}$ of S such that $|R| = r$, and $|B| = b$.
- A bicoloring $\{R, B\}$ of S is **well blended** if the coarseness of $\{R, B\}$ is within a constant factor of $\mathcal{C}(S, r, b)$.

Open problems in Bereg et al. CGTA, 2013

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Problem 1: Coarseness approximation

Let $S = R \cup B$ be a bicolored set of points in the plane, is there any polynomial-time constant approximation algorithm for computing the coarseness of S?

Problem 2: Coarseness bounding

Given a set S of n points in general position in the plane, what is the smallest coarseness of S taken over all the bicolorings $\{R, B\}$ of S such that $|R| = r$, and $|B| = b$?

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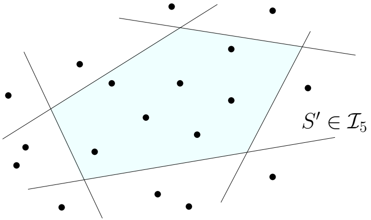
The main tool

An island S' of S is k-**separable** if there exist k halfplanes $H_1, H_2, \ldots, H_k$ such that

$$S' = S \cap (H_1 \cap H_2 \cap \ldots H_k)$$

We denote the family of all the k- separable islands of S with $\mathcal{I}_k$.



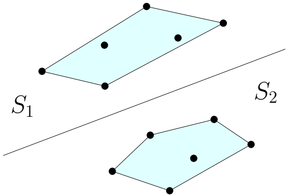
(Edelsbrunner, Robison, and Shen 1990) A collection of n compact, convex, and pairwise disjoint sets in the plane may be covered with n non-overlapping convex polygons with a total of not more than $6n - 9$ sides.

Theorem: Every convex partition Π of S into islands has a 5-separable island.

Problem 1: Coarseness approximation

Lemma 1: If there exists $S_1 \in \mathcal{I}_1$ s.t. $\text{disc}(S_1) \geq t$, then there exists a convex partition Π s.t.

$$\text{disc}(\Pi) \geq \max\{t/2, t - |r - b|\}$$



Proof.

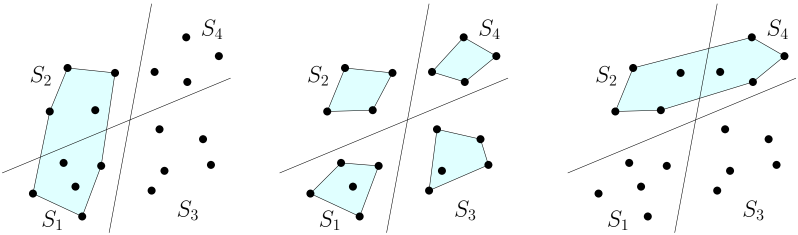
1. $\text{disc}(S_2) \geq t - |r - b|$
2. $t - |r - b| \geq t/2 \Rightarrow$ for $\Pi = \{S_1, S_2\}$, $\text{disc}(\Pi) \geq t - |r - b| = \max\{t/2, t - |r - b|\}$.
3. $t - |r - b| < t/2 \Rightarrow \text{disc}(S) = |r - b| > t/2$
   $\Rightarrow$ for $\Pi = \{S\}$, $\text{disc}(\Pi) > t/2 = \max\{t/2, t - |r - b|\}$.

□

Problem 1: Coarseness approximation

Lemma 2: If there exists $S_1 \in \mathcal{I}_2$ s.t. $\text{disc}(S_1) \geq t$, then there exists a convex partition Π s.t.

$$\text{disc}(\Pi) \geq \max\{t/8, t/4 - |r - b|\}$$



Proof.

1. $\text{disc}(S_2) \leq t/2 \Rightarrow \text{disc}(S_1 \cup S_2) \geq t/2$; $\exists\ \Pi_1 : \text{disc}(\Pi_1) \geq \max\{t/4, t/2 - |r - b|\}$
2. $\text{disc}(S_2) > t/2$ and $\text{disc}(S_3) > t/2$:
   $\text{disc}(S_4) \geq t/4 \Rightarrow \text{disc}(\Pi_2) \geq t/4$ for $\Pi_2 = \{S_1, S_2, S_3, S_4\}$
   $\text{disc}(S_4) < t/4 \Rightarrow \text{disc}(S_2 \cup S_4) > t/4$; $\exists\ \Pi_3 : \text{disc}(\Pi_3) \geq \max\{t/8, t/4 - |r - b|\}$

□

Problem 1: Coarseness approximation

$$D_k := \max_{I \in \mathcal{I}_k} \text{disc}(I)$$

Lemma 3: $D_3 \leq 4D_2$, and $D_{k+1} \leq 2D_k$ for $k \geq 3$.

Approximation! Compute $APX := \max\left\{\frac{D_2}{8}, \frac{D_2}{4} - |r - b|\right\}$ which satisfies

$$\max\left\{\frac{\mathcal{C}(S)}{128}, \frac{\mathcal{C}(S)}{64} - |r - b|\right\} \leq APX \leq \mathcal{C}(S)$$

Proof.

- $\max\left\{\frac{D_2}{8}, \frac{D_2}{4} - |r - b|\right\} \leq \mathcal{C}(S)$ from **Lemma 2**.
- $\mathcal{C}(S) \leq D_5 \leq 2D_4 \leq 4D_3 \leq 16D_2$ since $\Pi \cap \mathcal{I}_5 \neq \emptyset$ for all Π and **Lemma 3**
- D_2 can be computed in $O(n^3 \log n)$ time **(Dobkin and Gunopulos 1995)**

□

Open problems in Bereg et al. CGTA, 2013

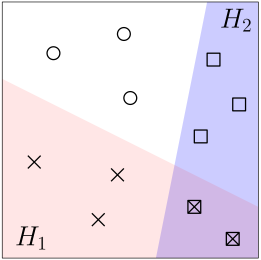
Problem 2: Coarseness bounding

Given a set S of n points in general position in the plane, what is the smallest coarseness of S taken over all the bicolorings $\{R, B\}$ of S such that $|R| = r$, and $|B| = b$?

Problem 2: Coloring with low coarseness

Discrepancy Theory: Chazelle 2004.

- A set system $(S, \mathcal{Y})$, where $|S| = n$ and $\mathcal{Y} \subseteq 2^S$
- **Dual shatter function** $\pi^*_{\mathcal{Y}}(m)$: the maximum number of equivalent classes on S defined by m elements of $\mathcal{Y}$ ($p \equiv q$ iff p and q are covered by the same sets)



$Y_1 = H_1 \cap S$

$Y_2 = H_2 \cap S$

$\mathcal{Y} = \{Y_1, Y_2\}$

$\pi^*_{\mathcal{Y}}(2) = 4$

(Dual shatter function bound) Let $d > 1$ and C be constants such that $\pi^*_{\mathcal{Y}}(m) \leq Cm^d$ for all $m \leq |\mathcal{Y}|$. Then there exists a coloring of S such that:

$$\text{disc}(Y) \text{ is upper bounded by } O\left(n^{1/2-1/2d}\sqrt{\log n}\right) \text{ for every } Y \in \mathcal{Y}.$$

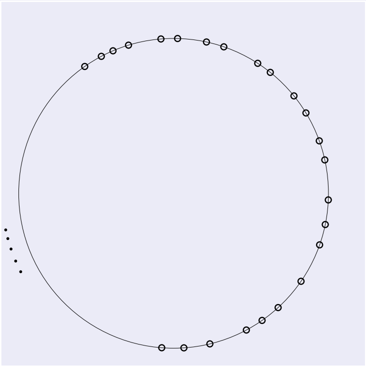
Problem 2: Coloring with low coarseness

The upper bound

Lemma 4: Let S be a set of n points in convex position in the plane then

$$\pi^*_{\mathcal{I}_k}(m) \leq 4km$$

Sketch of the proof: Assume that S is sorted clockwise around its convex hull. A k-separable island must consist of at most k intervals of consecutive points of S.



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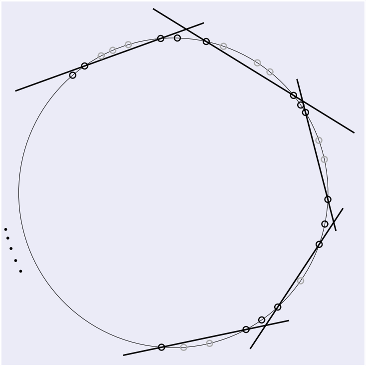

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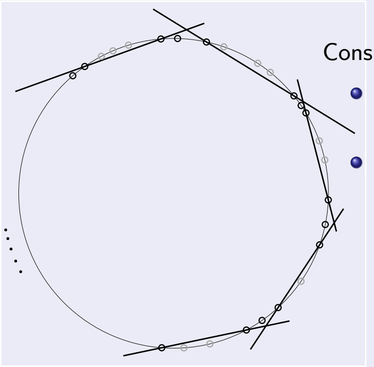

Consider a family of m, k-separable islands.

- There are at most $2km$ points of S that are the endpoints of any such intervals.
- There are at most $2km$ regions into which the remaining points (which are not endpoints of any interval) can lie.

Total: at most $4km$ equivalence classes.

□

Problem 2: Coloring with low coarseness

The upper bound

Lemma 5: Let S be a set of n points in general position in the plane then

$$\pi^*_{\mathcal{I}_k}(m) \leq (k^2 + 4k)m^2$$

Sketch of the proof: Let $\mathcal{F}$ be a family of m, k-separable islands on S.

- Points lying in the convex hull of some island.

  There at most $4km^2$ equivalence classes for points in the boundary of some island in $\mathcal{F}$.

- Points not lying in the boundary of any island.

  Each equivalence class is contained in a cell of the arrangement defined by the set of lines that separate each island I from $S \setminus I$.

  This arrangement has at most $k^2 m^2$ cells.

□

Theorem: If k is a constant, the family $\mathcal{I}_k$ of all k-separable islands satisfies:

$$\text{disc}(I) = O\left(n^{1/2-1/2d}\sqrt{\log n}\right) = O\left(n^{1/4}\sqrt{\log n}\right) \text{ for all } I \in \mathcal{I}_k$$

Problem 2: Coloring with low coarseness

Theorem: **The UPPER bound!** For every set S of n points in general position in the plane there exists a coloring such that the coarseness of S is upper bounded by

$$O(n^{1/4}\sqrt{\log n})$$

.

Sketch of the proof:

- There is a coloring such that $\mathrm{disc}(I) = O(n^{1/4}\sqrt{\log n})$ for all $I \in \mathcal{I}_5$.
- $\Pi \cap \mathcal{I}_5 \neq \emptyset$ for all island partitions Π

□

Problem 2: Coloring with low coarseness

(Alexander 1990, Chazelle et al. 1995) For arbitrarily large n, there exist n-point sets S such that for any coloring of S a **halfplane** H exists such that:

$$\mathrm{disc}(S \cap H) = \Omega(n^{1/4})$$

Theorem: **The LOWER bound!** For arbitrarily large values of n, there exist sets of n points in general position in the plane with coarseness $\Omega(n^{1/4})$.

Sketch of the proof: For such point sets S, there exists H s.t. $\mathrm{disc}(S \cap H) \geq Cn^{1/4}$ for any coloring.

Given a convex partition Π, $\mathcal{C}(S) \geq \mathrm{disc}(\Pi)$.

Given a coloring, we have two cases:

1. $\mathrm{disc}(S) \geq \frac{C}{2} n^{1/4} \Rightarrow \Pi = \{S\}$
2. $\mathrm{disc}(S) < \frac{C}{2} n^{1/4} \Rightarrow \mathrm{disc}(\overline{H} \cap S) \geq \mathrm{disc}(S \cap H) - \mathrm{disc}(S) \geq \frac{C}{2} n^{1/4}$
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2. $\mathrm{disc}(S) < \frac{C}{2} n^{1/4} \Rightarrow \mathrm{disc}(\overline{H} \cap S) \geq \mathrm{disc}(S \cap H) - \mathrm{disc}(S) \geq \frac{C}{2} n^{1/4}$
   $\Rightarrow \Pi = \{H \cap S, \overline{H} \cap S\}$

□

Problem 2: Coloring with low coarseness

(Alexander 1990, Chazelle et al. 1995) For arbitrarily large n, there exist n-point sets S such that for any coloring of S a **halfplane** H exists such that:

$$\text{disc}(S \cap H) = \Omega(n^{1/4})$$

Theorem: The LOWER bound! For arbitrarily large values of n, there exist sets of n points in general position in the plane with coarseness $\Omega(n^{1/4})$.

Sketch of the proof: For such point sets S, there exists H s.t. $\text{disc}(S \cap H) \geq Cn^{1/4}$ for any coloring.

Given a convex partition Π, $\mathcal{C}(S) \geq \text{disc}(\Pi)$.

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Open problems

1. Prove or disprove the NP-hardness of computing the coarseness.
2. Improve the constant approximation.
3. Improve the upper bound of the discrepancy of k-separable islands (e.g. the discrepancy of the 1-separable islands is $O(n^{1/4}) \subset o(n^{1/4}\sqrt{\log n})$)
4. Find the exact asymptotic upper bound of the minimum coarseness of point sets $\mathcal{C}$ (interesting and hard).

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