On the Complexity of Some Simultaneous and Clustered Planarity Problems

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GraDR

Graph Drawings and Representations

AP1-IT (Roma Tre University)

Thanks to: Giuseppe Di Battista, Patrizio Angelini, Giordano Da Lozzo, Fabrizio Frati

Outline

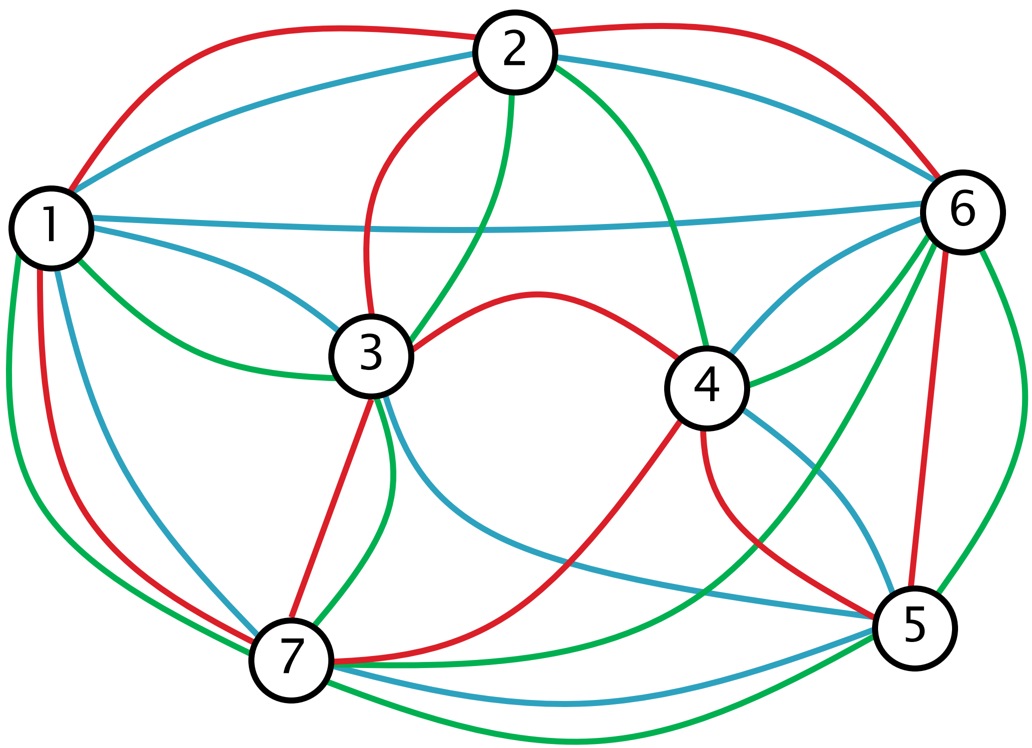
- Two different strains of planarity
  - Simultaneous embedding
  - Clustered planarity
- Some recent results
  - Relationship between the two problems
  - Relaxations
  - Special cases
- Open problems

Simultaneous Embedding

A Brief Survey

Simultaneous Embedding

- Given two or more graphs on the same set of vertices, construct a planar drawing of each graph on the same set of points



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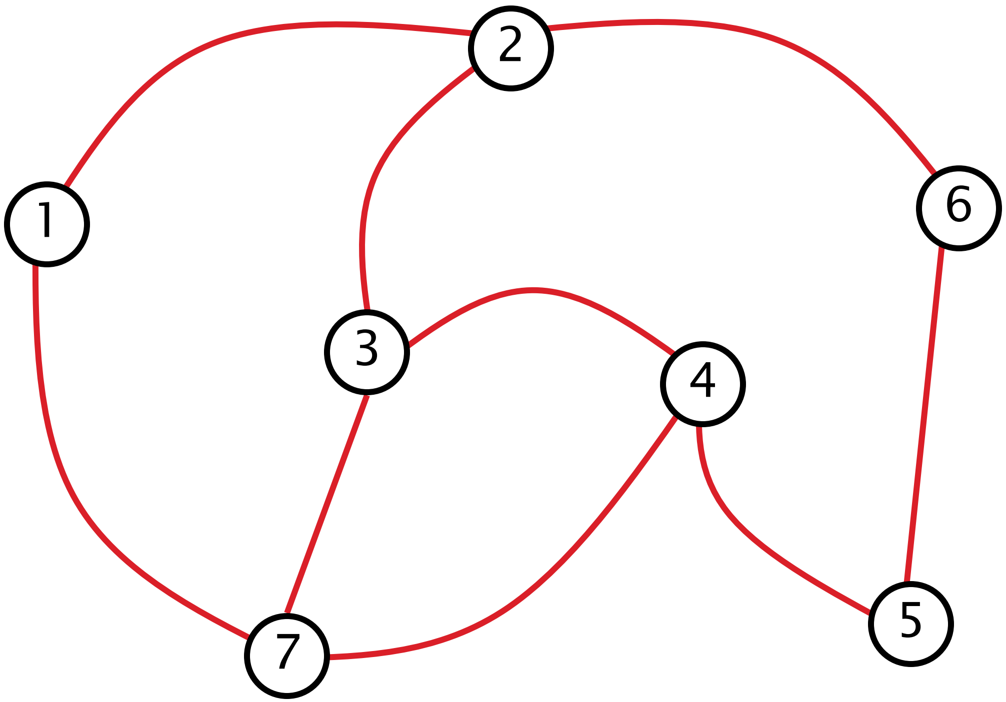

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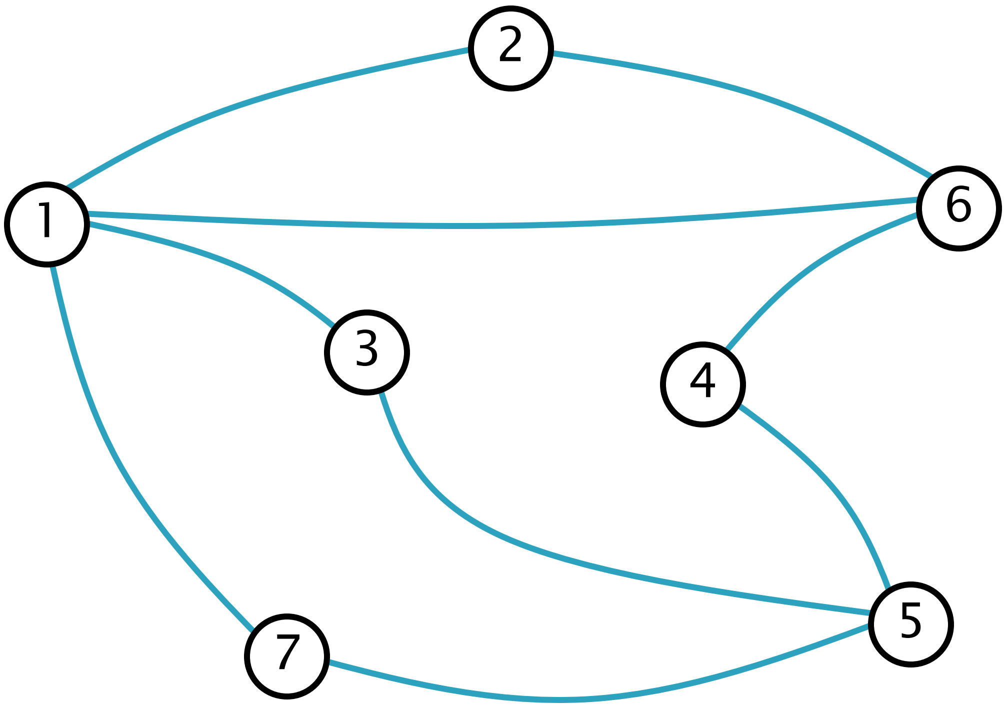

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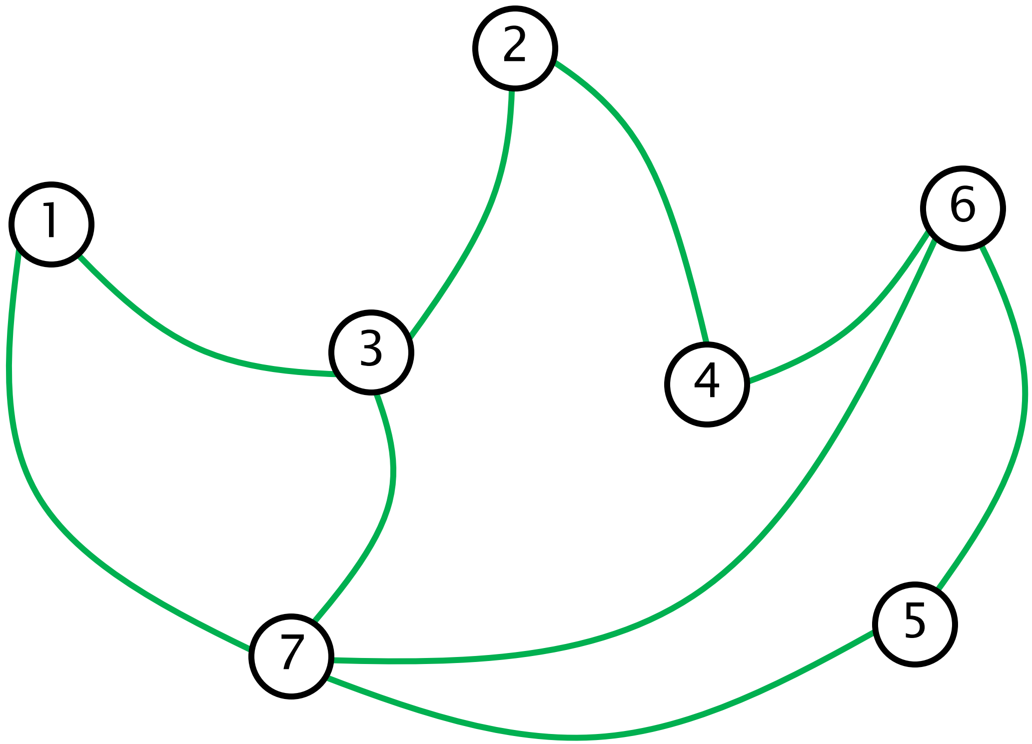

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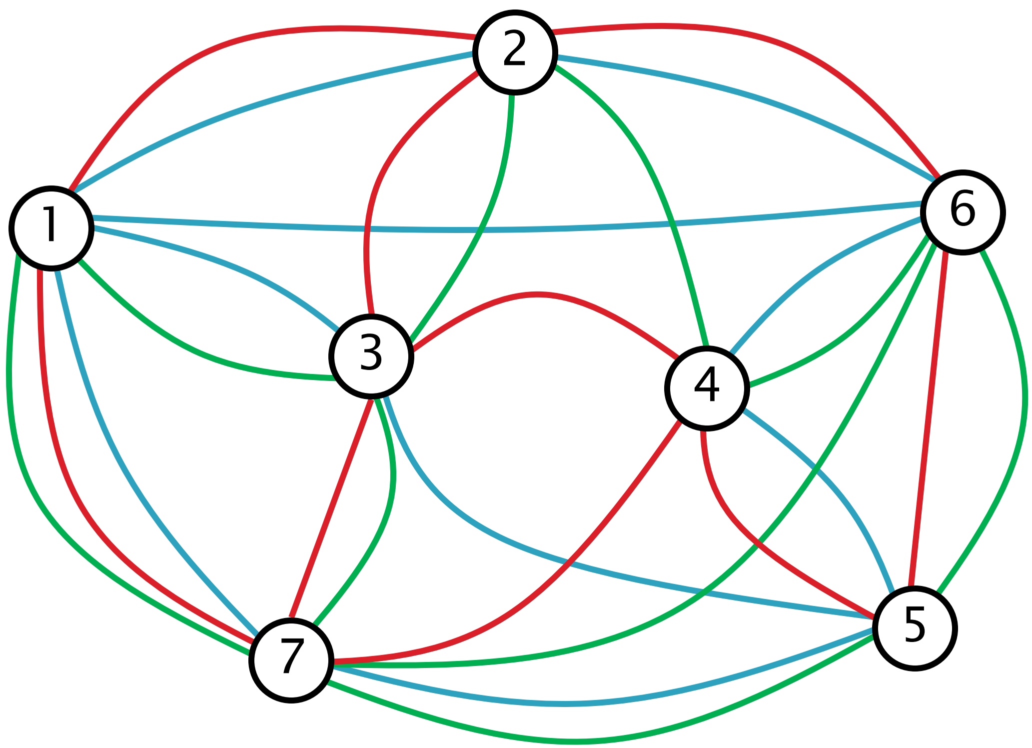
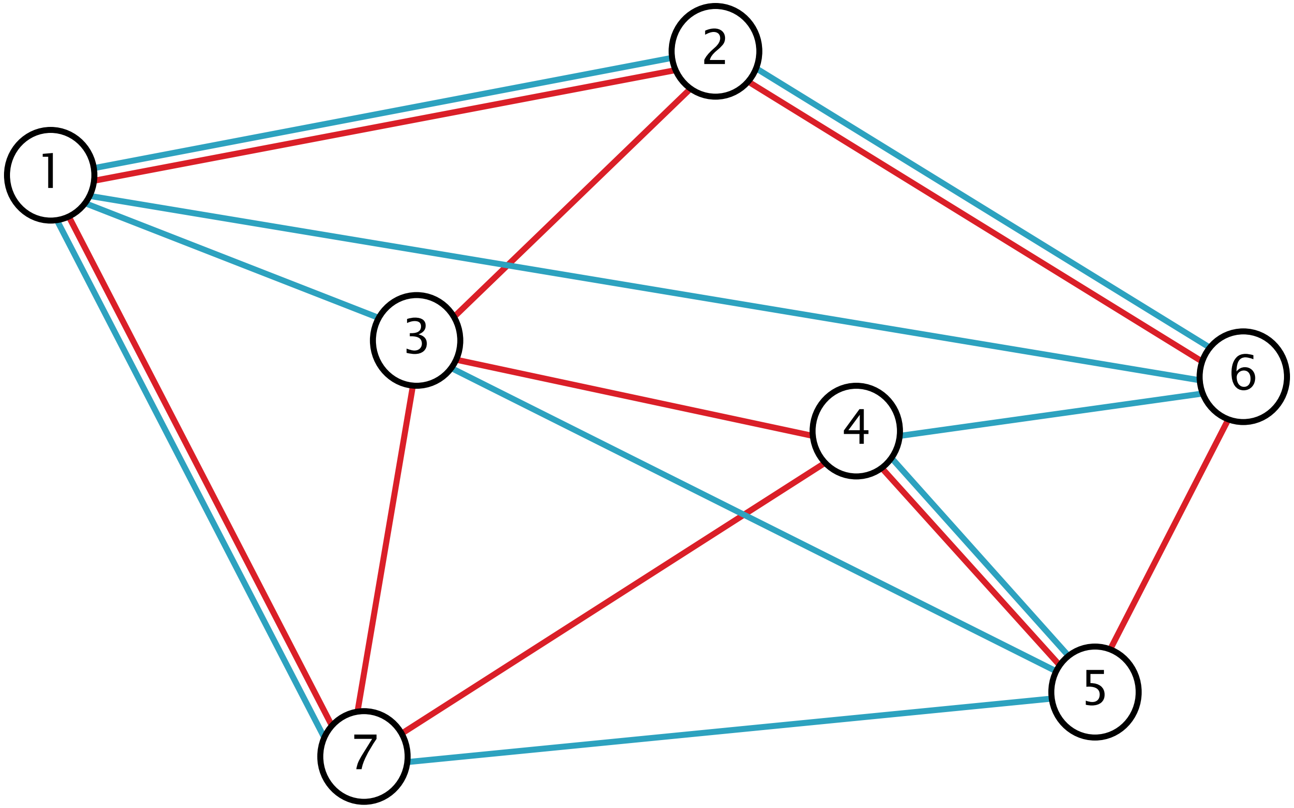


Geometric Simultaneous Embedding

- Edges are straight-line segments



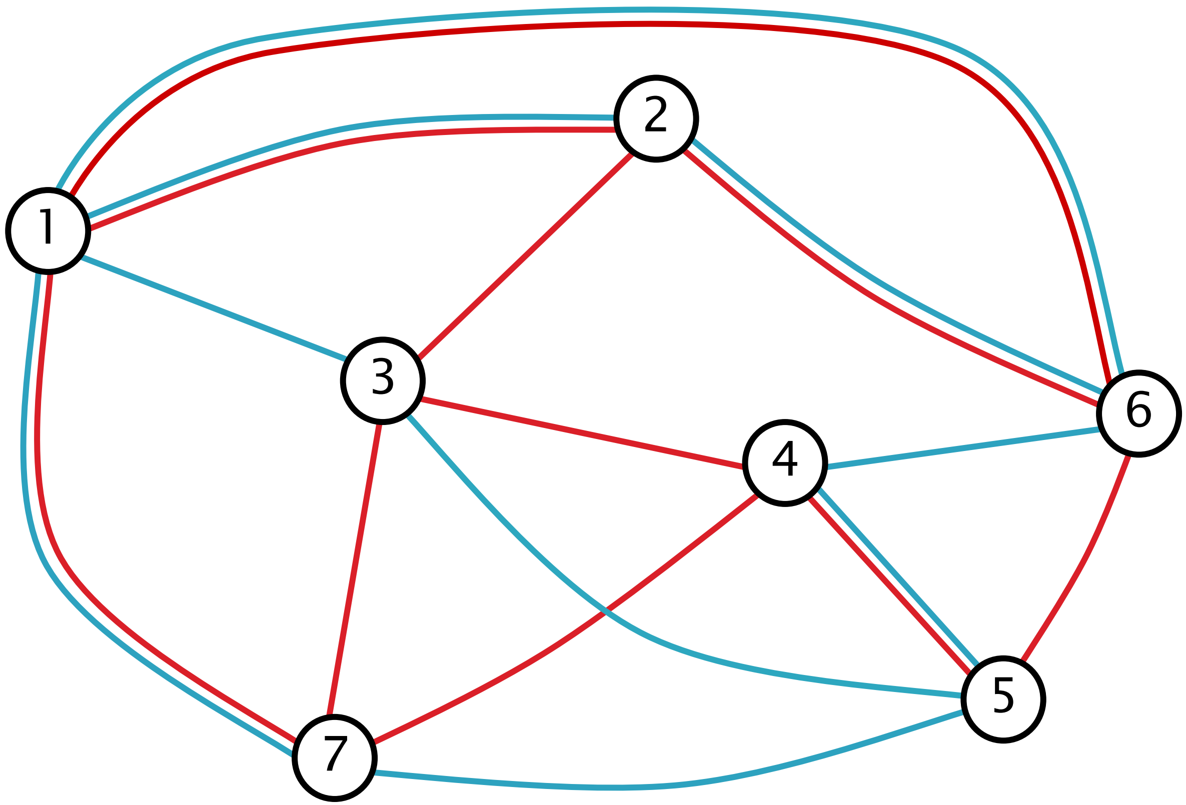
Geometric Simultaneous Embedding

- It is always possible to construct a geometric simultaneous embedding of:
  - two cycles [Brass *et al.*, 2003]
  - two caterpillars [Brass *et al.*, 2003]
- It is not always possible to construct a geometric simultaneous embedding of:
  - a tree and a path [Angelini *et al.*, 2010]
  - three paths [Brass *et al.*, 2003]
- The decision problem for two graphs is NP-Hard
  - [Estrella-Balderrama *et al.*, 2007]

SEFE

Simultaneous Embedding *with Fixed Edges*

Common edges are represented by the *same curve*



SEFE

Simultaneous Embedding *with Fixed Edges*

- It is always possible to construct a SEFE of:
  - a tree and a path (with few bends) [Erten, Kobourov, 2005]
  - an outerplanar graph and a path or a cycle (with few bends) [Di Giacomo, Liotta, 2007]
  - a planar graph and a forest [Frati, 2007], [Fowler *et al.*, 2008]
- It is not always possible to construct a SEFE of:
  - two outerplanar graphs [Frati, 2007]

SEFE

Simultaneous Embedding *with Fixed Edges*

- The decision problem has unknown complexity for two graphs
  - NP-Hard for three graphs [Gassner *et al.*, 2006]
- Characterization of the graphs $G_{1,2}$ such that any two graphs G_1 and G_2 whose intersection is $G_{1,2}$ have a SEFE
  - [Jünger, Schultz, 2009]

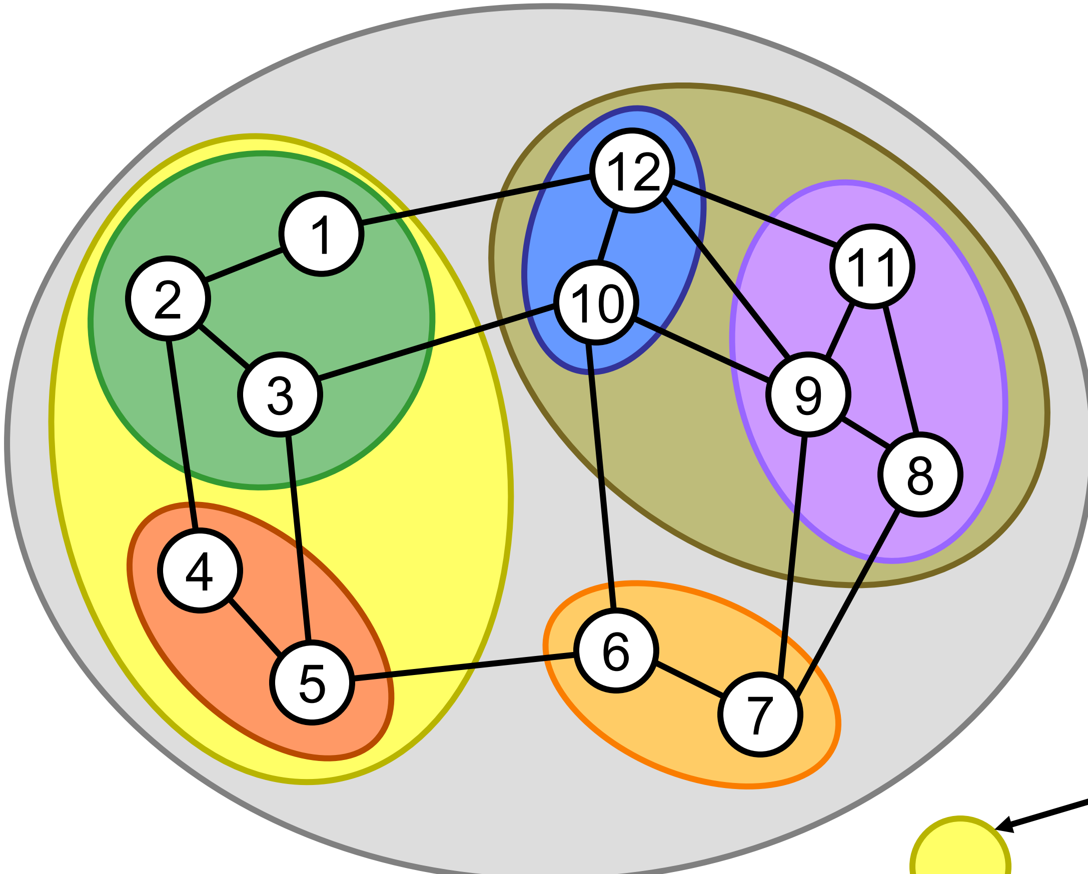
Polynomial Cases of $SEFE_2$

- The $SEFE_2$ decision problem is polynomial when
  - one of G_1 or G_2 has a fixed embedding
    - [Angelini *et al.*, SODA ‘10]
  - each connected component of the intersection graph has a fixed embedding
    - [Bläsius *et al.*, GD ‘13]
  - the intersection graph is one of the following
    - is biconnected [Angelini *et al.*, JDA ‘12][Haeupler *et al*., ISAAC ‘10]
    - is a star graph [Angelini *et al.*, JDA ‘12]
    - is a subcubic graph [Schaefer, JGAA ‘13]
  - G_1 and G_2 are biconnected and $G_{1,2}$ is connected
    - [Bläsius, Rutter, SODA ‘13]

Clustered Planarity

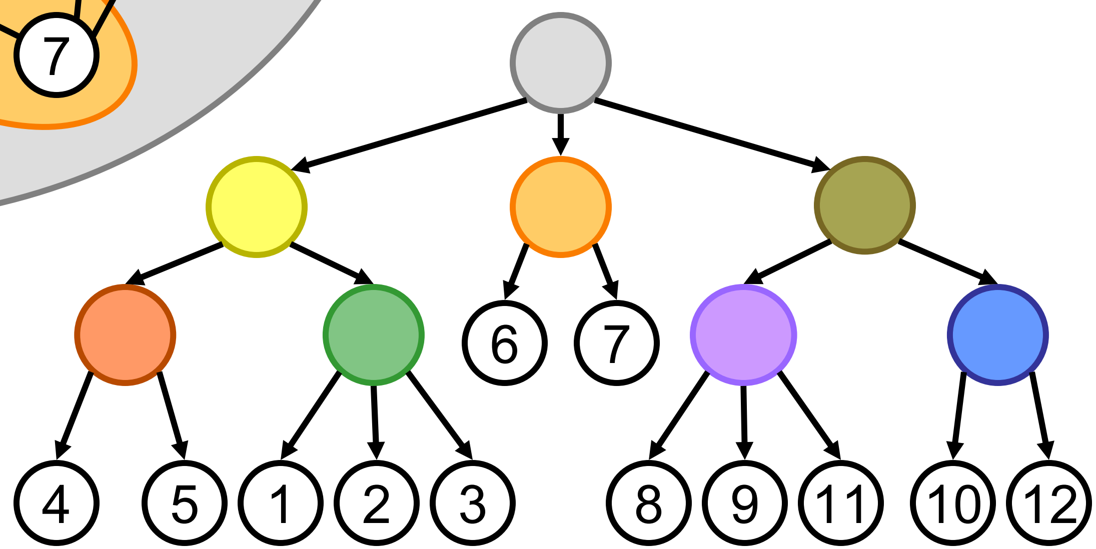
A Brief Survey

Clustered Graph



clustered graph = graph + set of clusters

clusters are described by an *inclusion tree*



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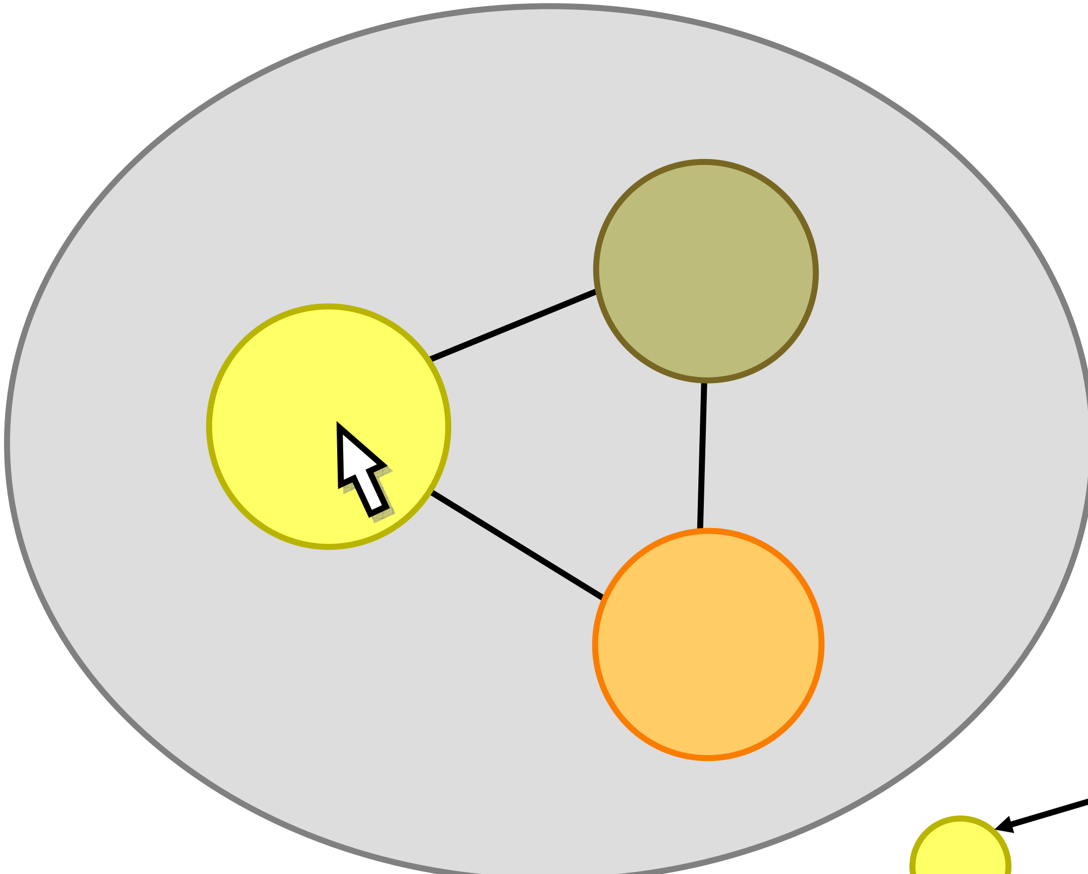
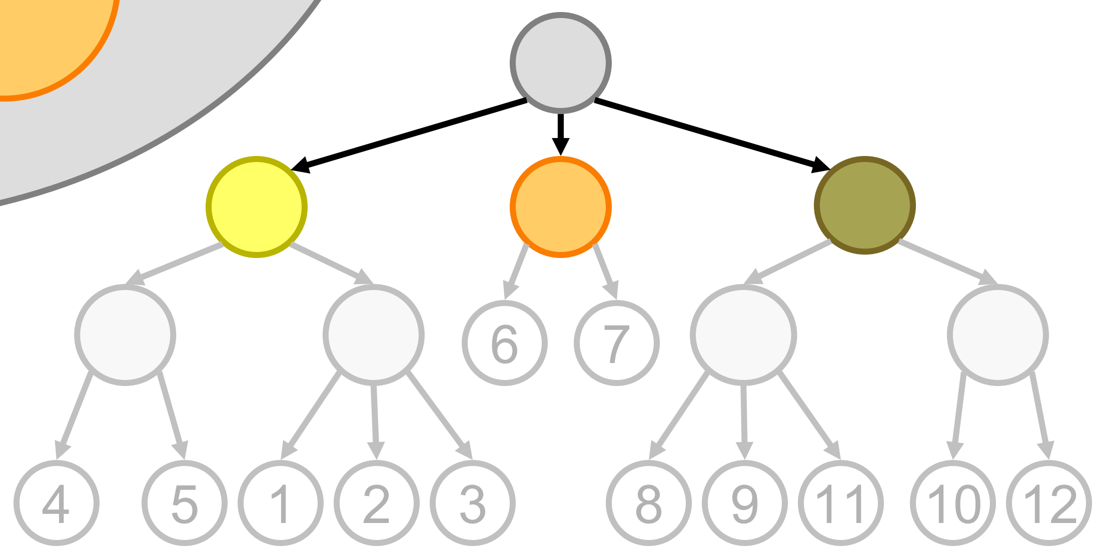


to explore complex information at different abstraction levels



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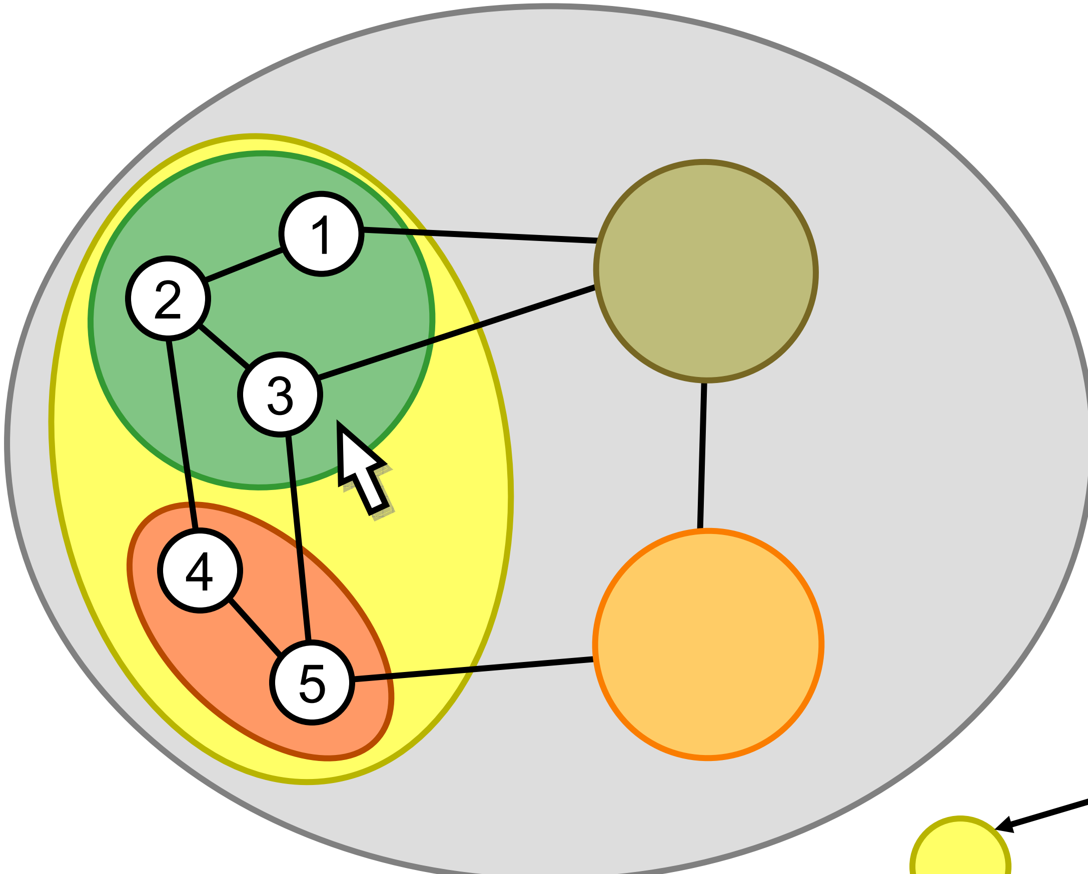

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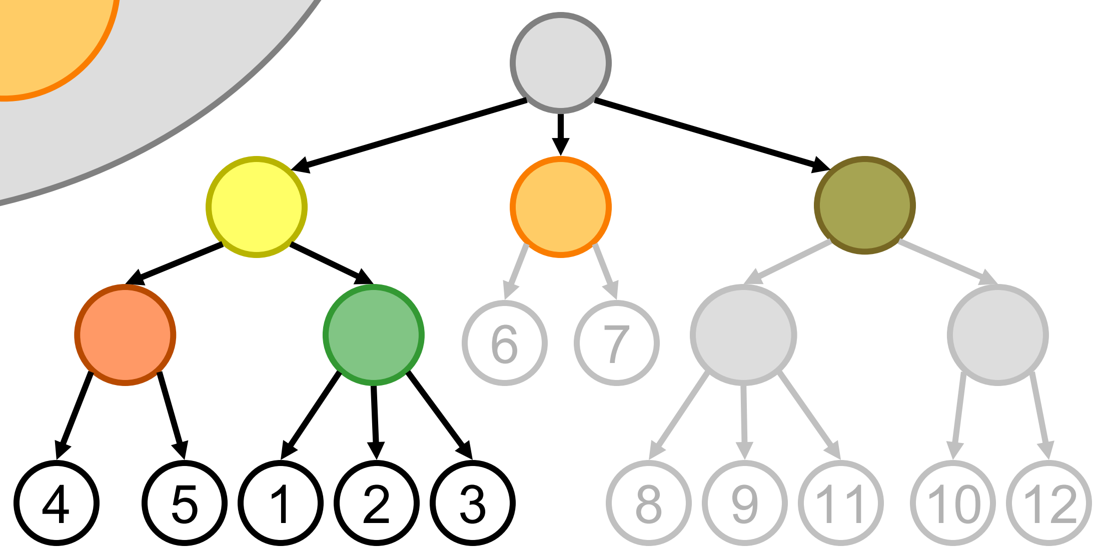

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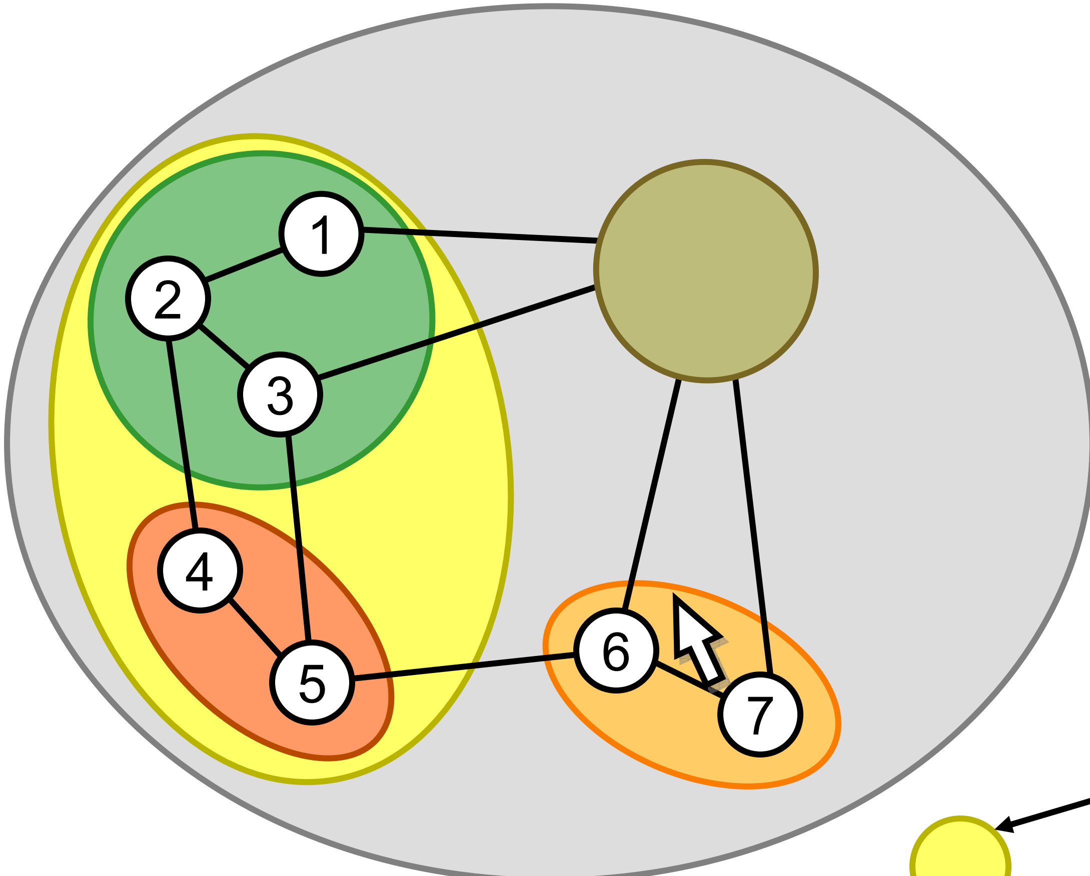

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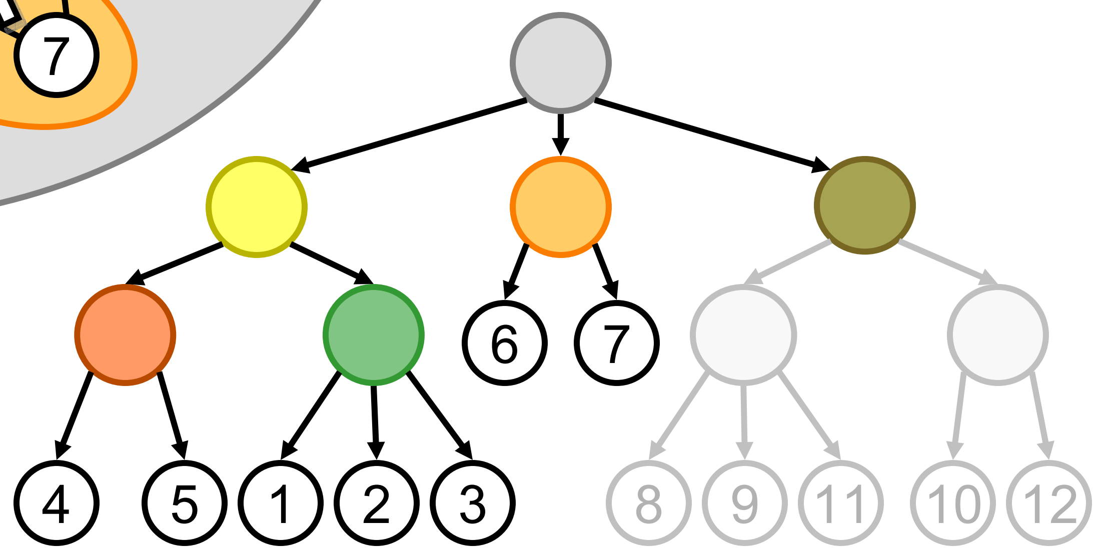

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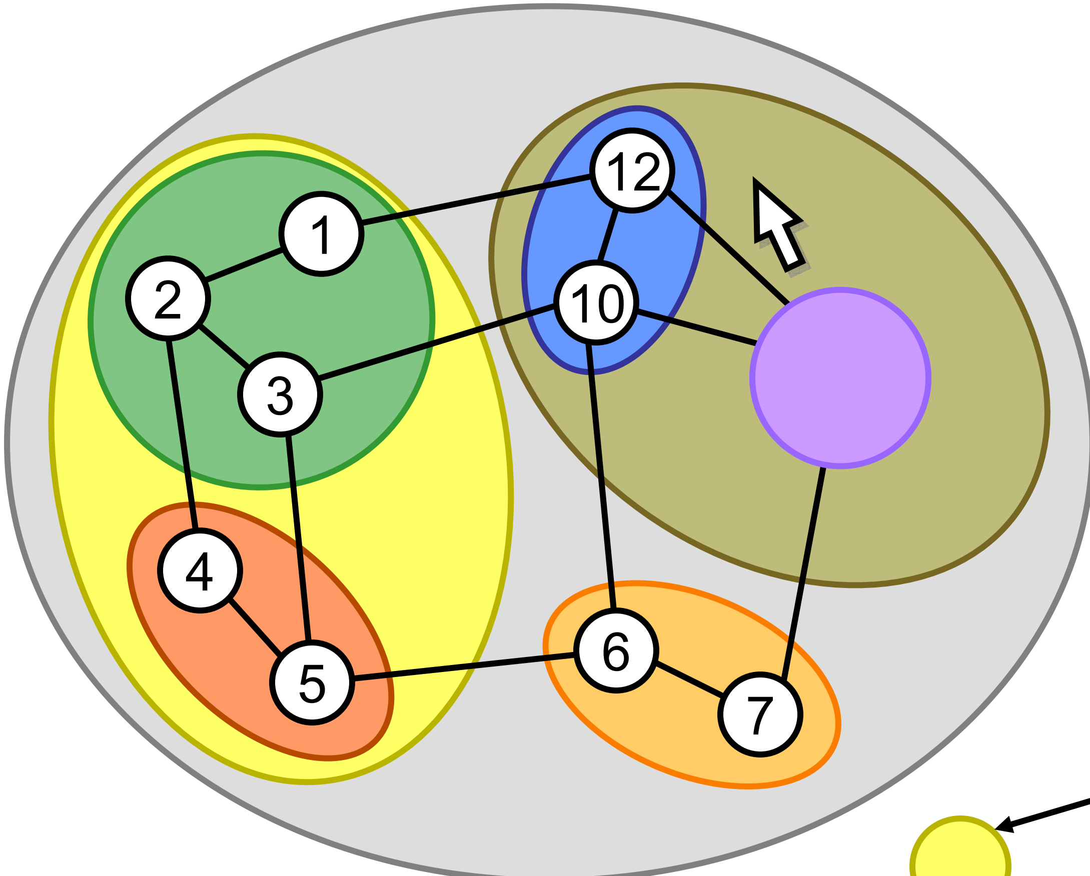

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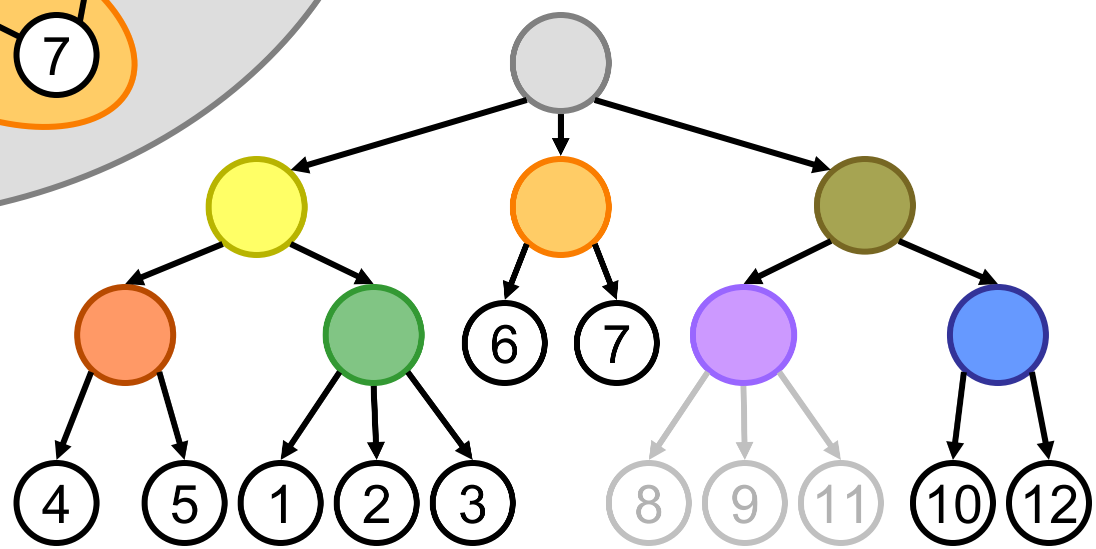

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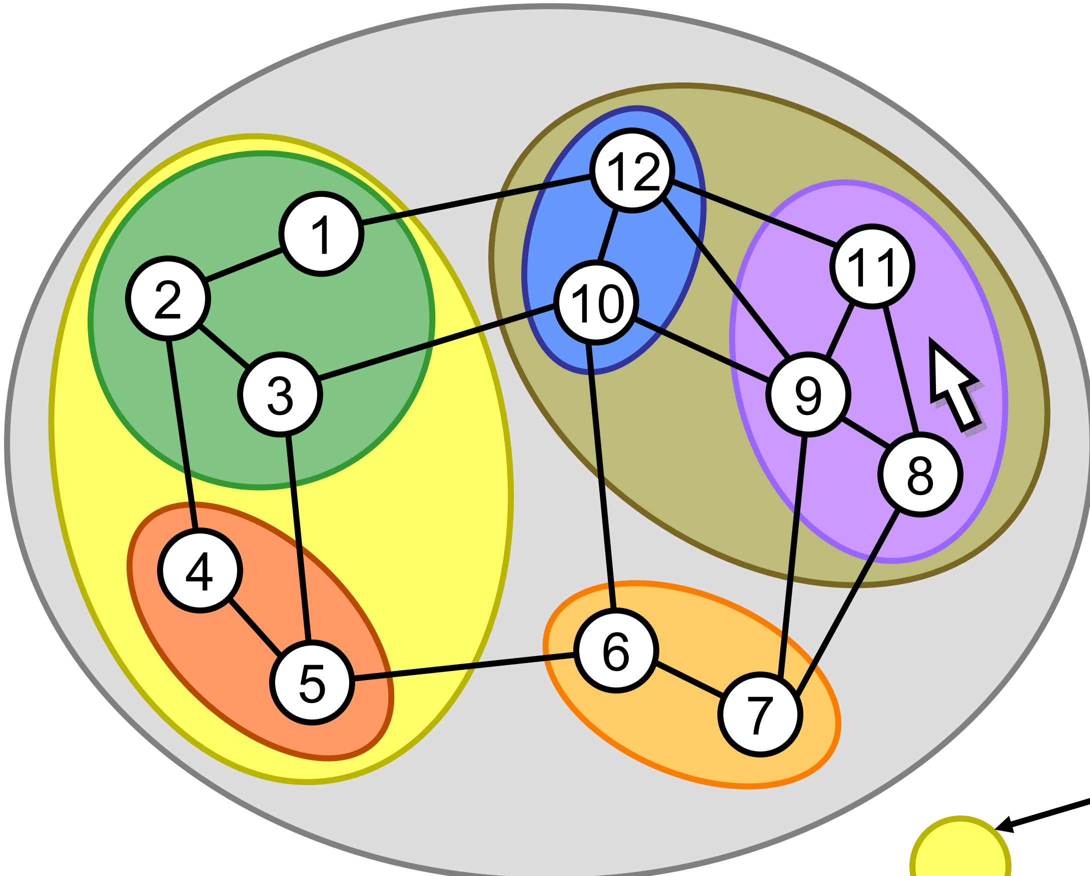

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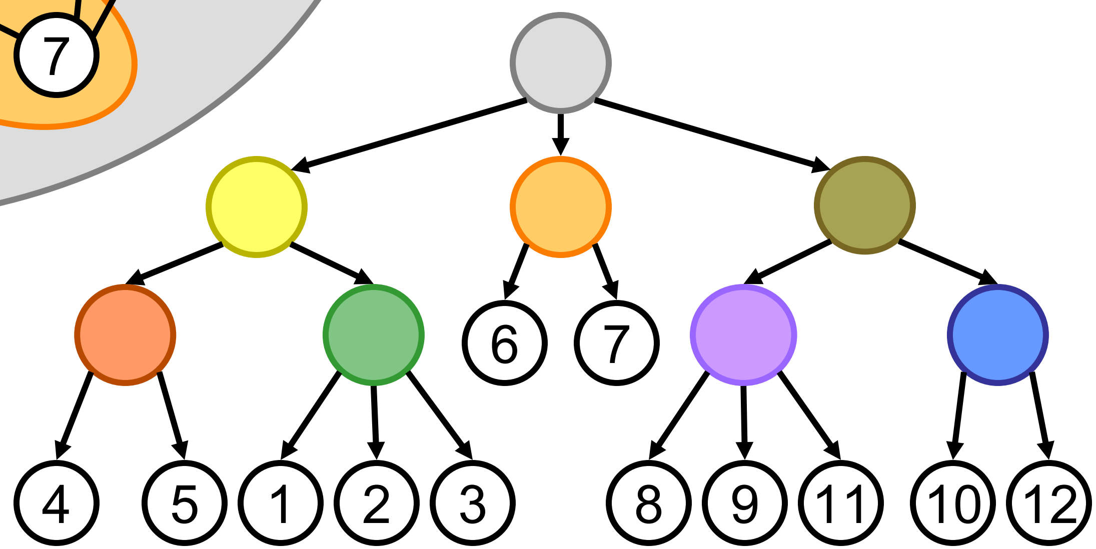

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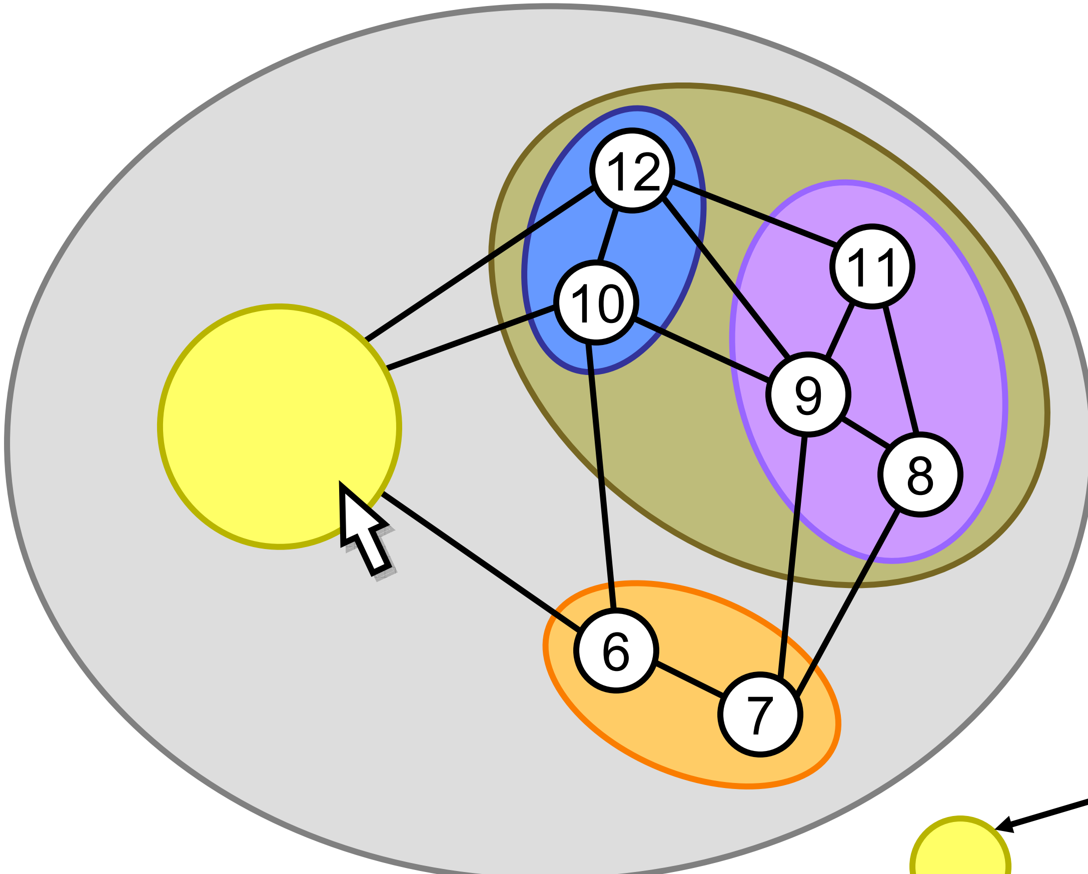

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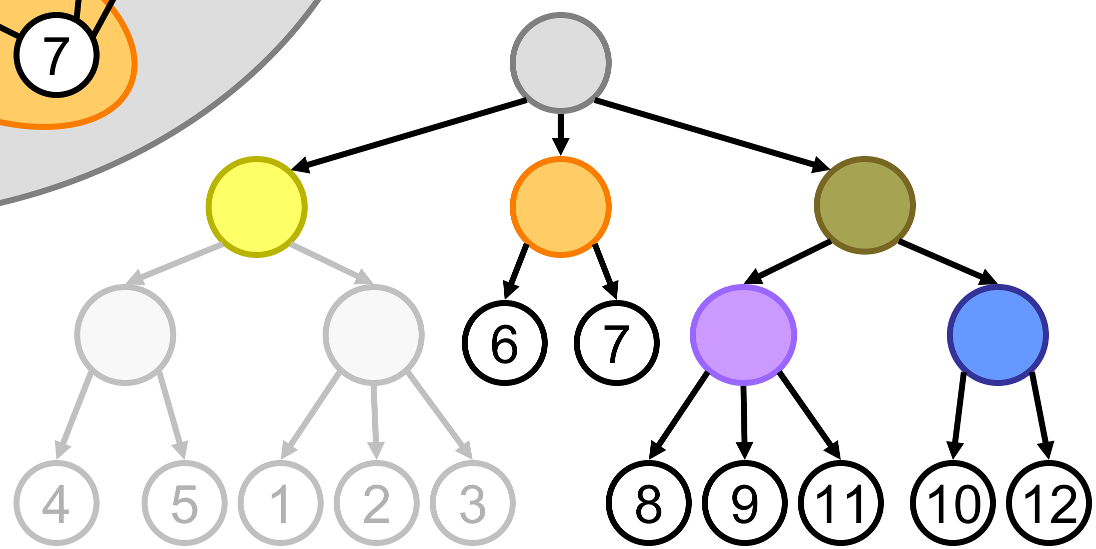
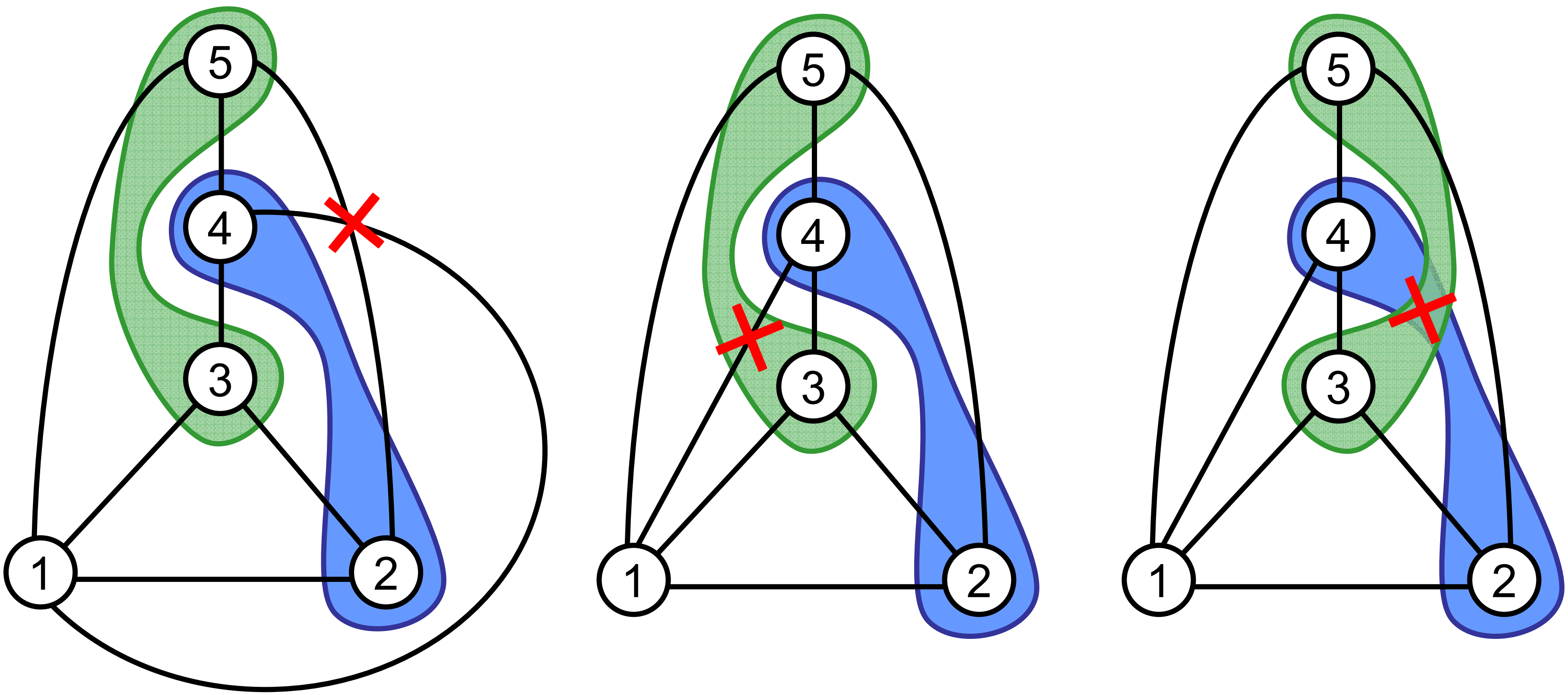


C-Planar Drawings of C-Graphs

- No crossing allowed



edge-edge crossings

edge-region crossings

region-region crossings

C-Planarity Problems

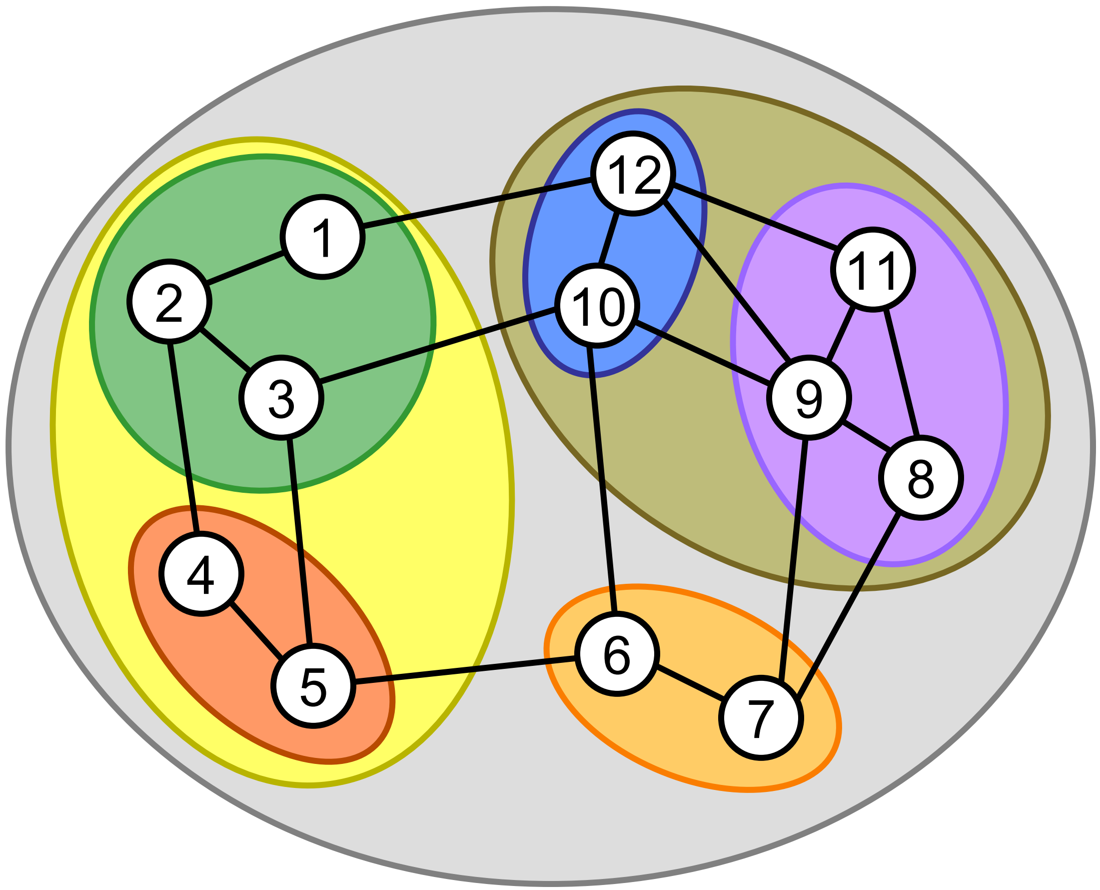
- Decide if a c-graph is c-planar
  - unknown complexity
- Find a c-planar drawing of a planar c-graph
  - unknown complexity
- Both problems were posed in the nineties!
  - [Feng, Cohen, Eades, '95]

Common Restrictions of C-Planarity

- Clusters induce a small number of connected components
- The underlying graph has a fixed embedding
- The underlying graph belongs to a particular graph family
- The hierarchy has few levels
- The instance has reduced size

C-Connected Clustered Graphs

- The subgraph induced by each cluster is connected



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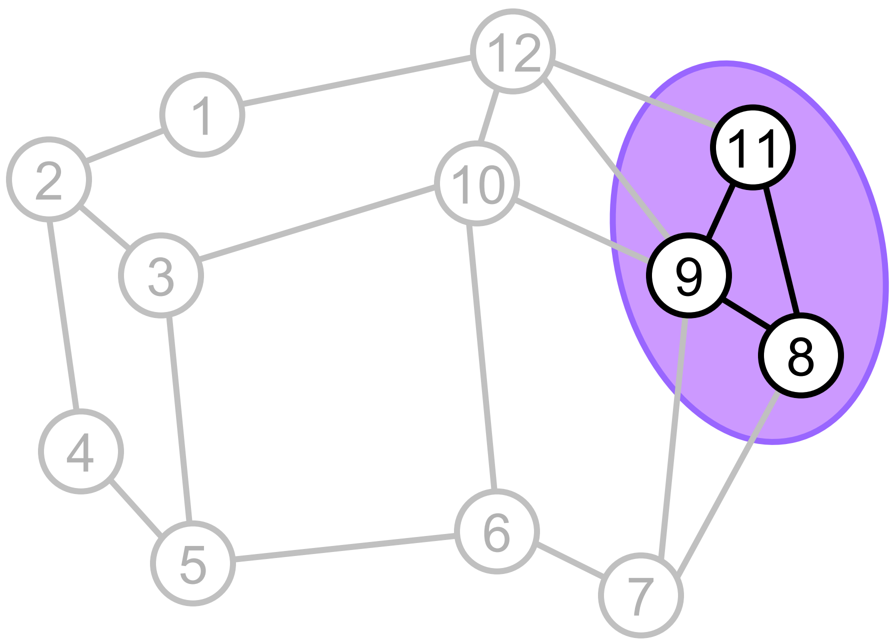

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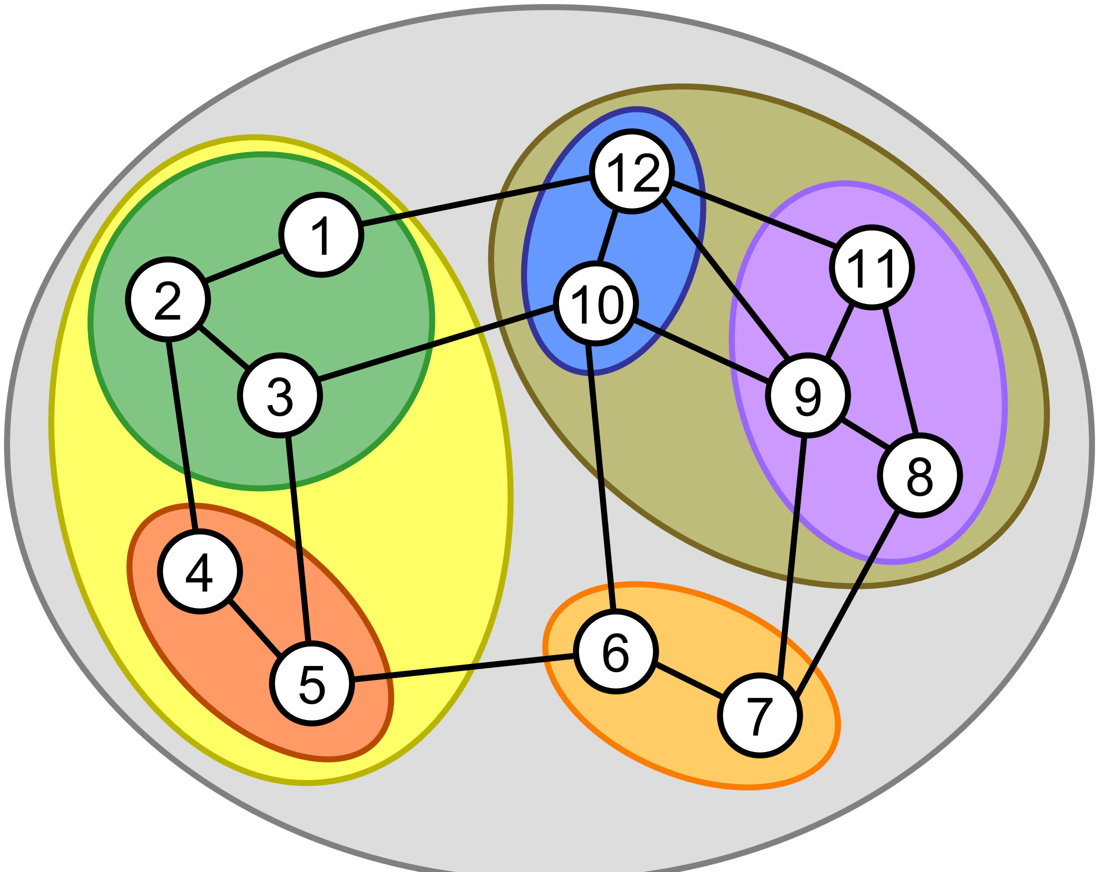

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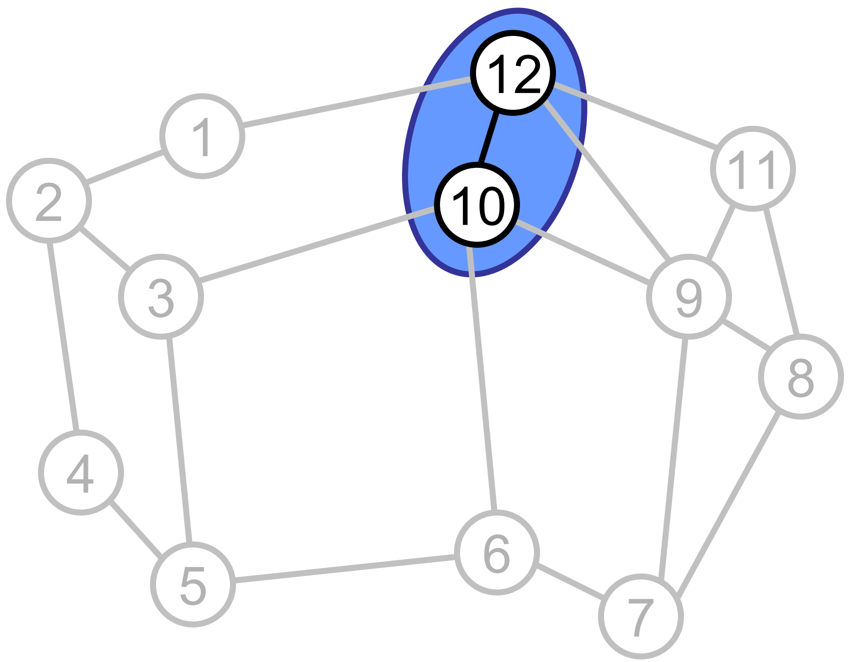

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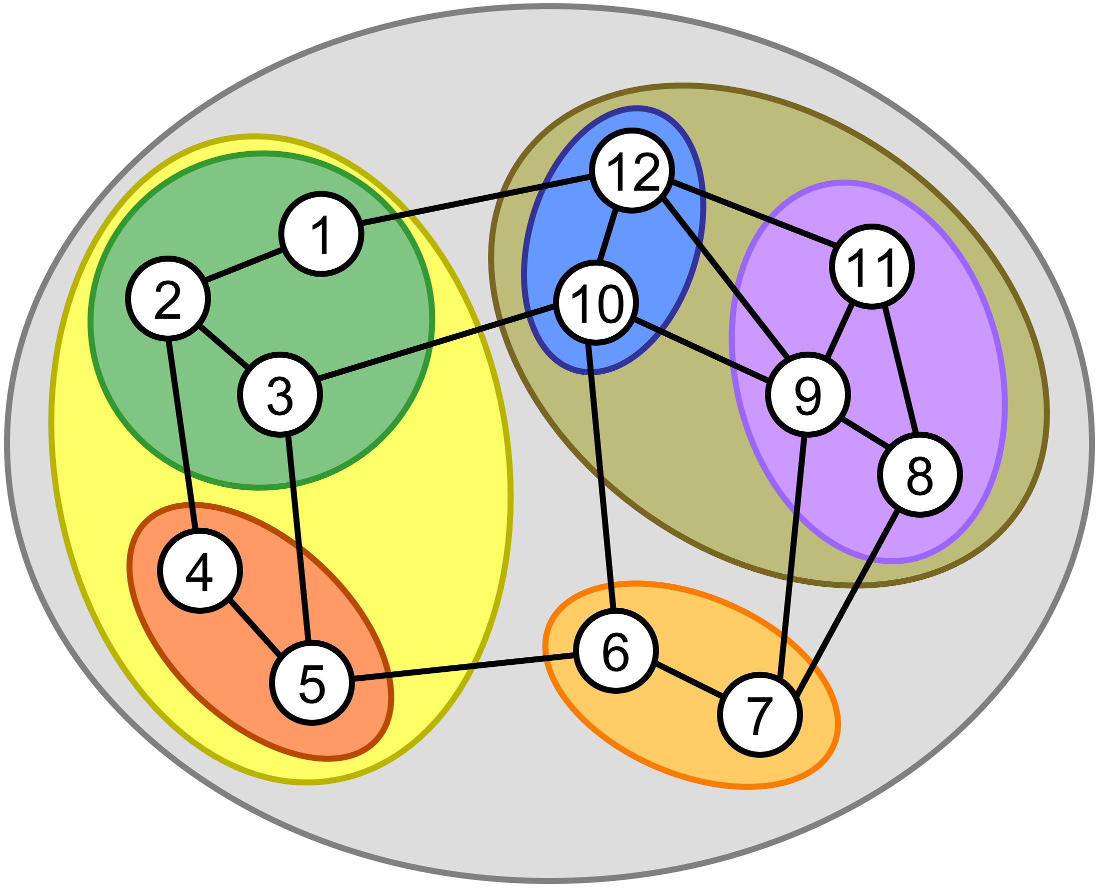

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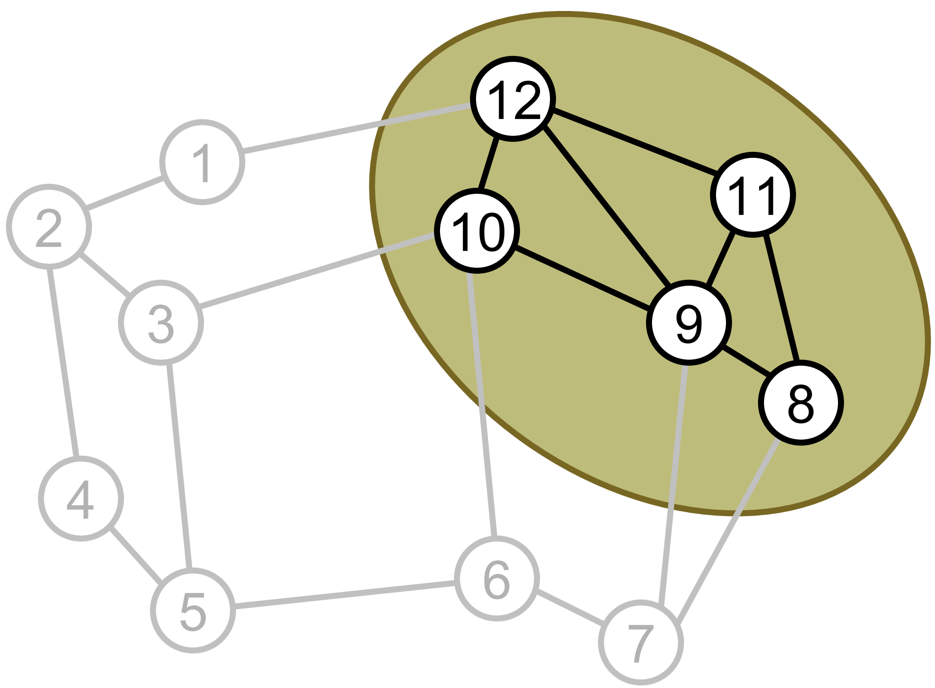

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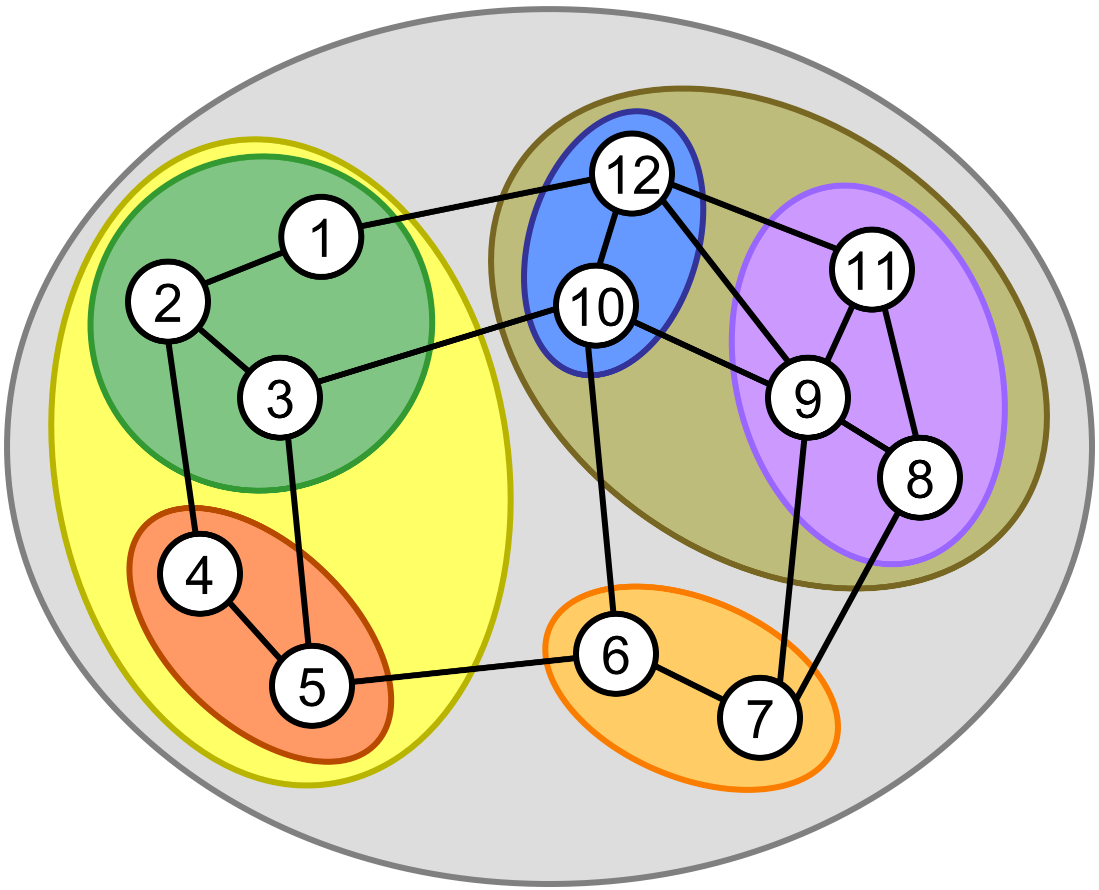


C-Connected Clustered Graphs

- Polynomial
  - recursively replace clusters with suitable gadgets
    - [Lengauer, '89] [Feng *et al.*, '95]
  - improved to linear
    - [Dahlhaus, '98] [Cortese *et al.*, '08]

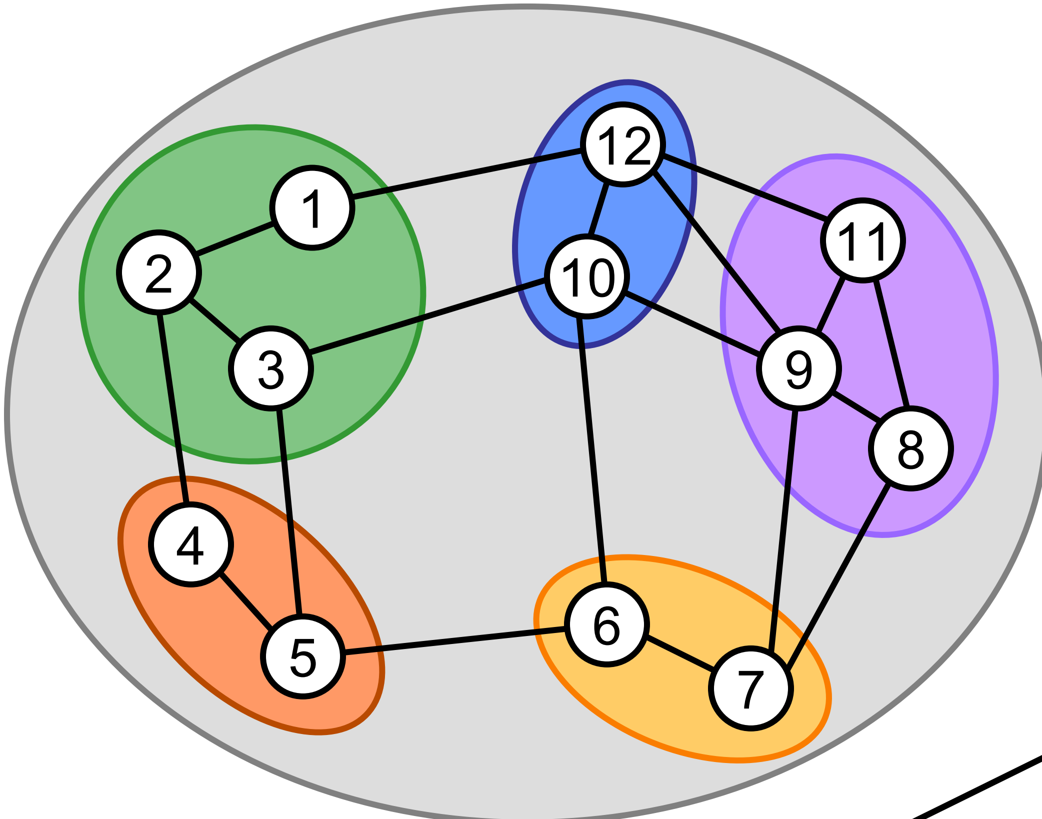
Other Kinds of C-Connectivity (1/2)

- *Completely connected* c-graphs [Cornelsen, Wagner, '03]
  - both the subgraph *inside* each cluster and the subgraph *outside* each cluster are connected
  - a completerly connected c-graph is c-planar if and only if its underlying graph is planar
    - linear-time recognition and embedding algorithms
- Two components c-graphs
  - each cluster induces a subgraph with at most two connected components
    - linear time testing algorithm if the embedding is given [Jelínek *et al.*, '08]

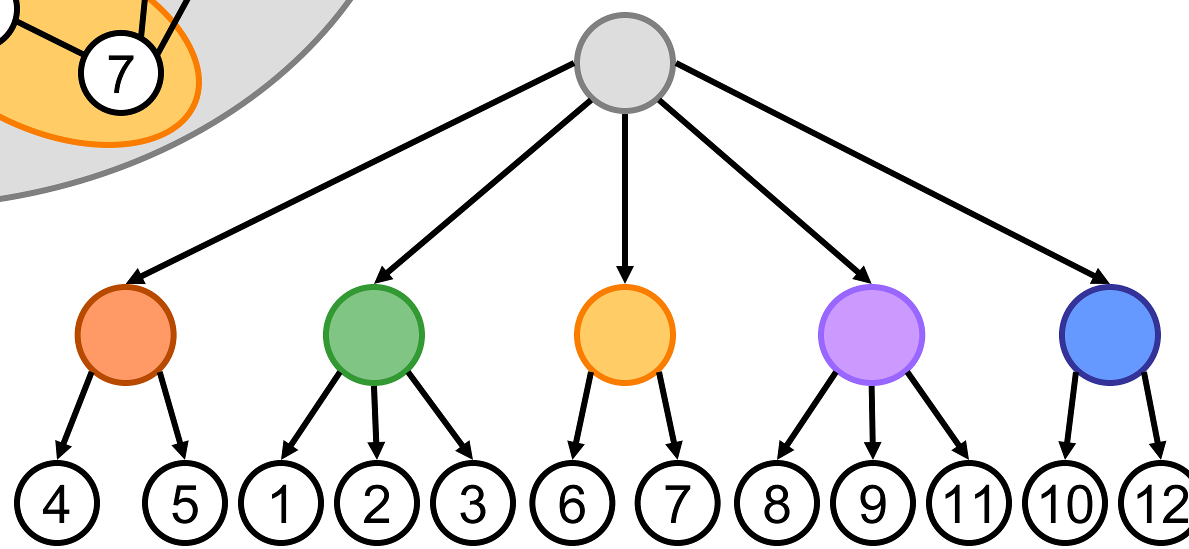
Other Kinds of C-Connectivity (2/2)

- *Almost c-connected* c-graphs
  - all non-connected clusters are in the same path starting at the root of the hierarchy
    - quadratic-time test [Gutwenger *et al.*, ‘02]
  - each non-connected cluster has c-connected parent and siblings
    - quadratic-time test [Gutwenger *et al.*, ‘02]
- *Extrovert* c-graphs
  - each non-connected cluster has c-connected parent and its connected components are linked outside the parent
    - polynomial-time testing and embedding [Goodrich *et al.*, ‘05]

Flat Clustered Graph



all clusters different from the root are children of the root



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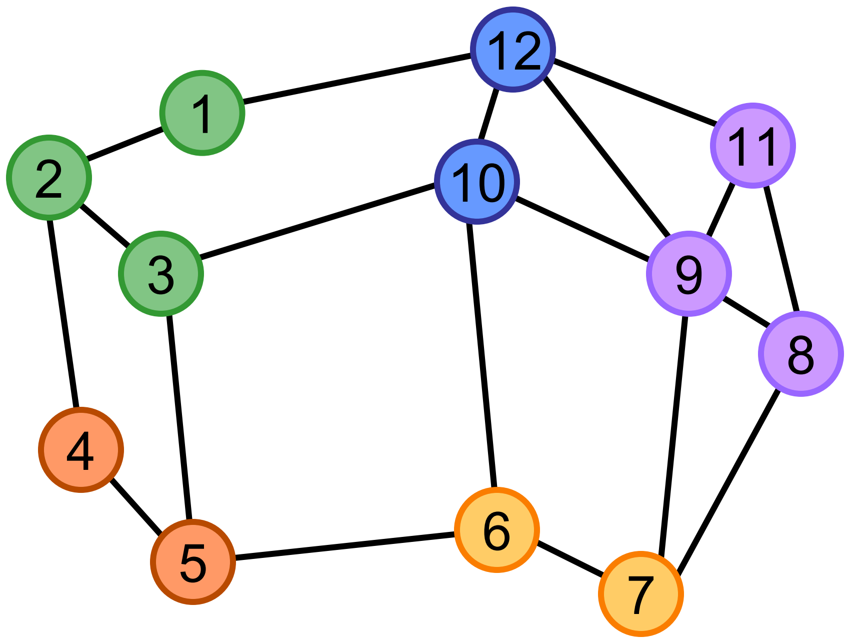
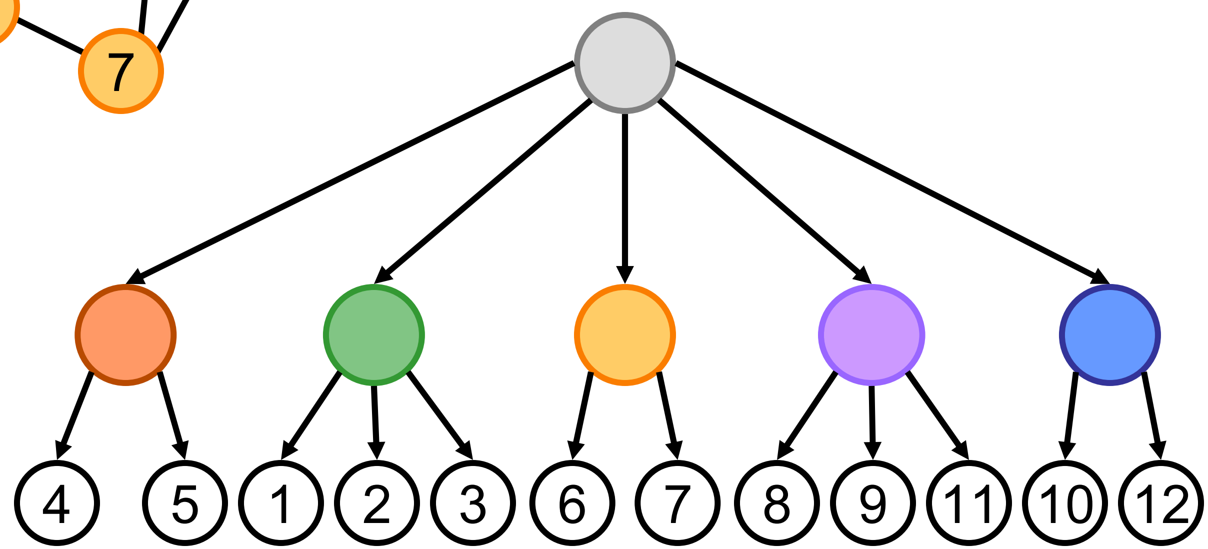
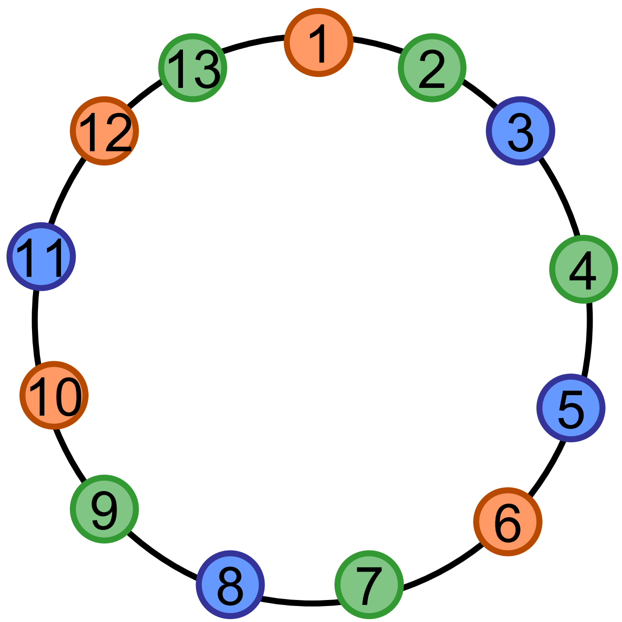


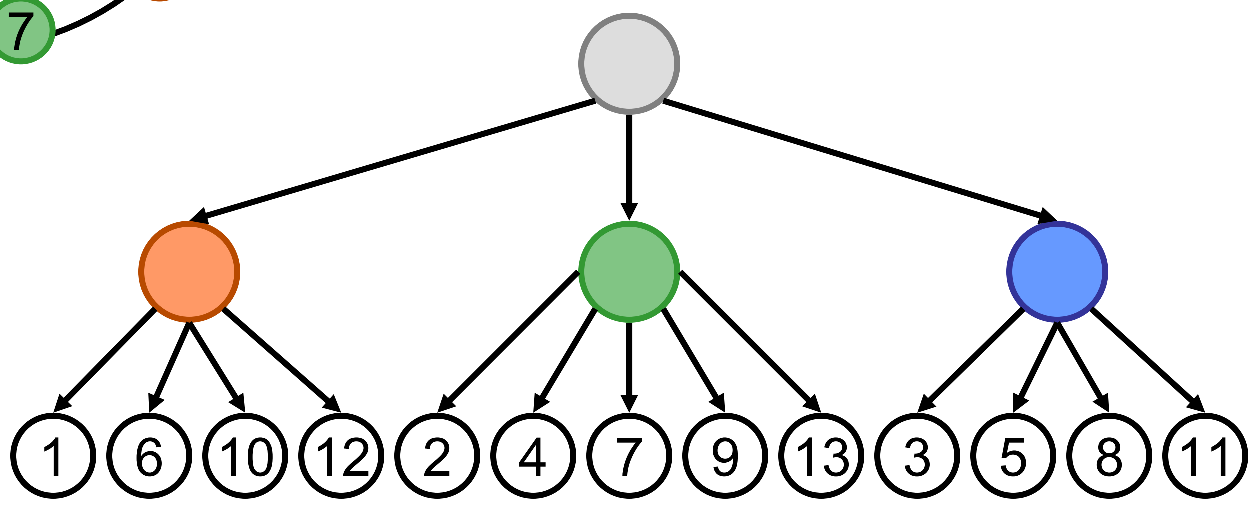
all clusters different from the root are children of the root

equivalent to a coloring of the vertices of the underlying graph



Flat Clustered Cycles





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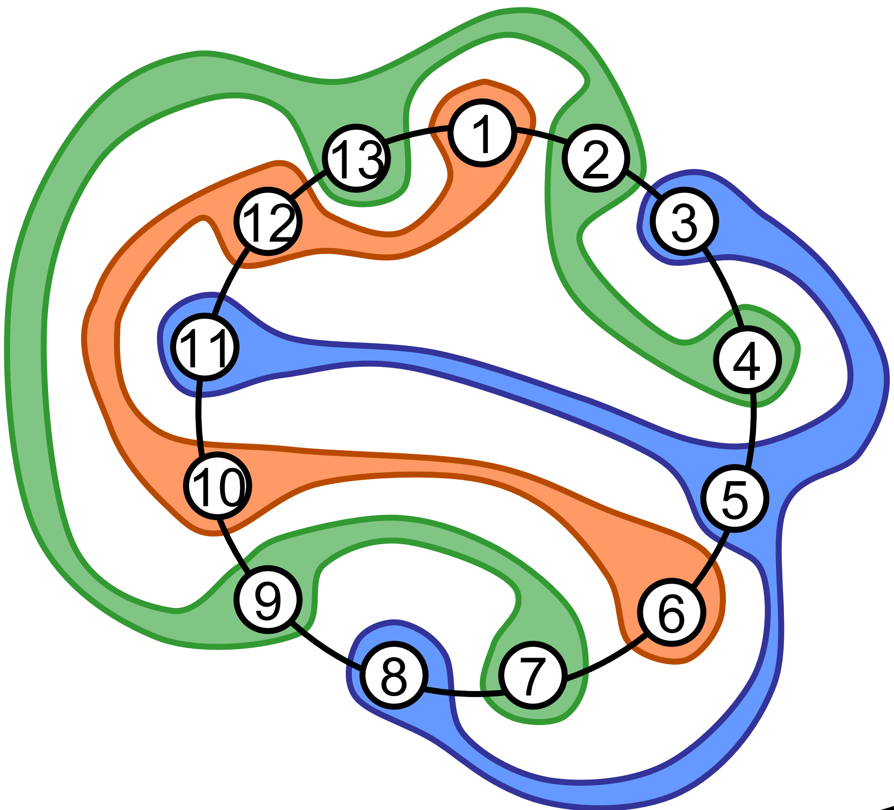

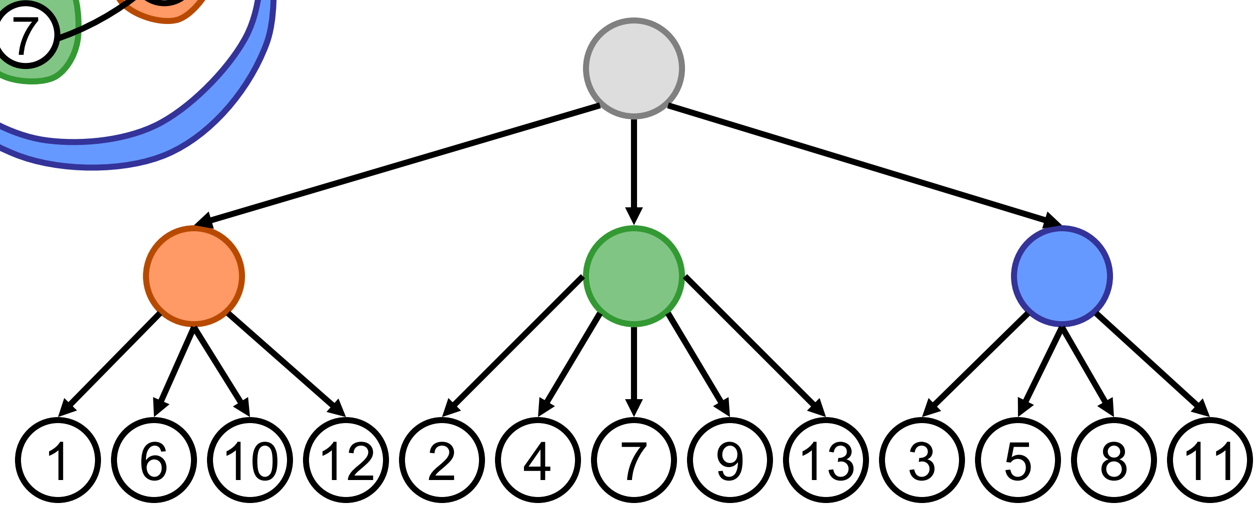

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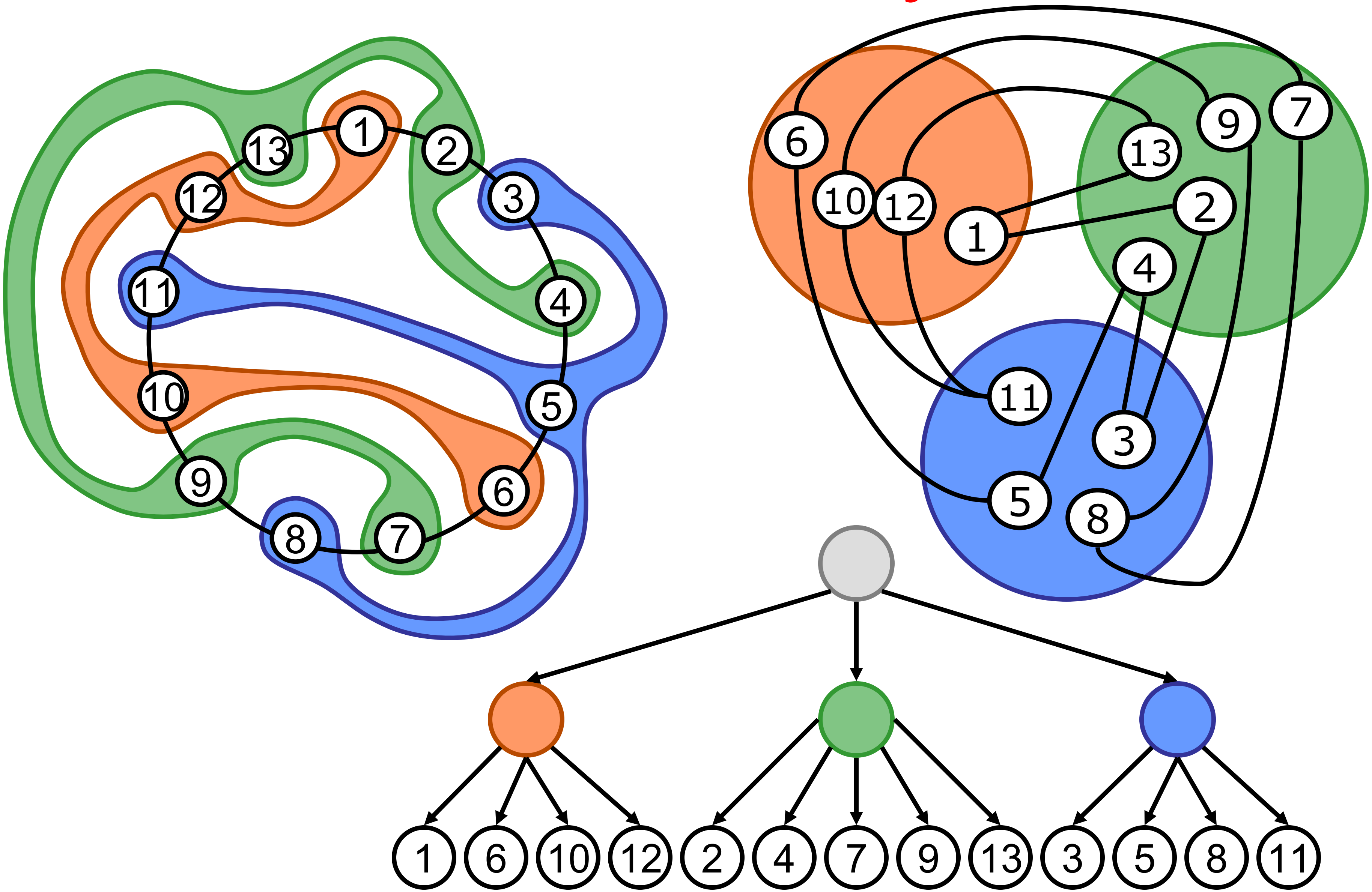


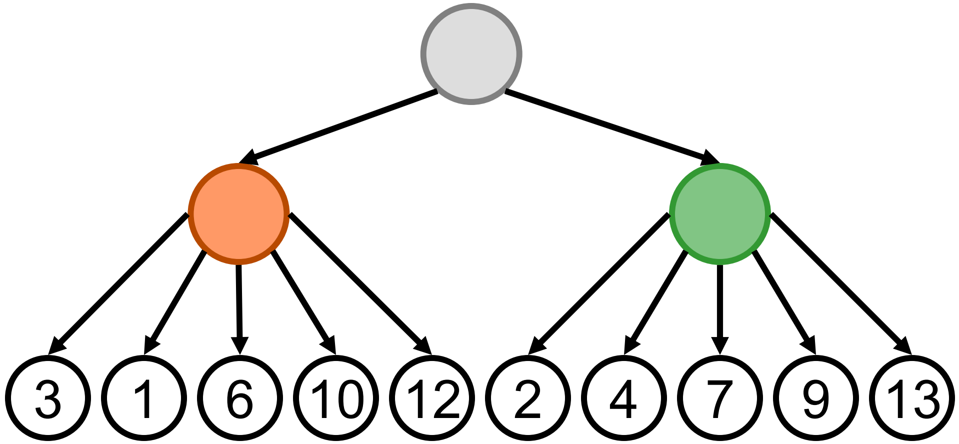
Flat Clustered Cycles

- Polynomial if
  - each cluster contains at most three vertices
    - [Jelínková *et al.*, ‘07]
  - clusters are arranged into a cycle or path
    - [Cortese *et al.*, ‘04]
  - clusters are arranged into an embedded plane graph
    - [Cortese *et al.*, ‘09]

Flat C-Graphs with Small Faces

- Polynomial if three-connected and all faces of size at most four
  - [Jelínková *et al.*, ‘07]
- Polynomial if the embedding is fixed and all faces have size at most five
  - [Di Battista, Frati, ‘07]

Flat C-Graphs with Two Clusters



- Linear if the embedding is fixed
  - HH-drawings [Biedl *et al.*, '98]
- Linear in the variable embedding setting
  - consequence of two-page book embeddings with fixed edge partitions [Hong, Nagamochi, '09]

Clusters of Reduced Size

- Clusters of size at most 3
  - polynomial for vertex-3-connected graphs
    - [Jelínková *et al.*, 07]
  - extended to Rib-Eulerian graphs
    - can be obtained from vertex-3-connected graphs of fixed size by cloning and subdividing edges

Recent Results and Open Problems

For Both the Problems

C-Planarity Reduces to $SEFE_2$

- Key result of [Schaefer, JGAA '13]

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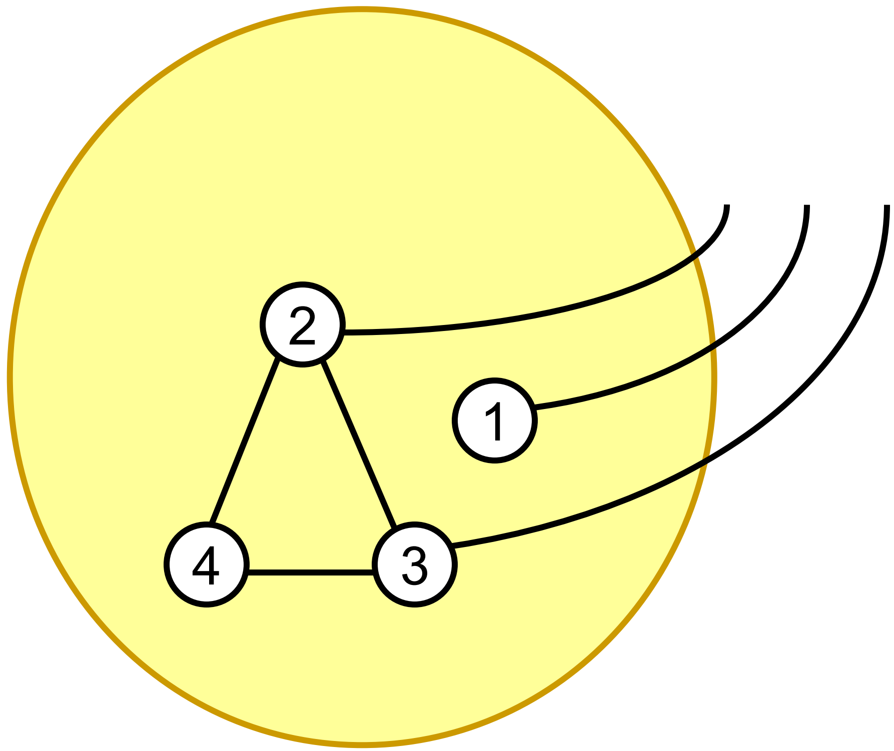

C-Planarity Reduces to $SEFE_2$

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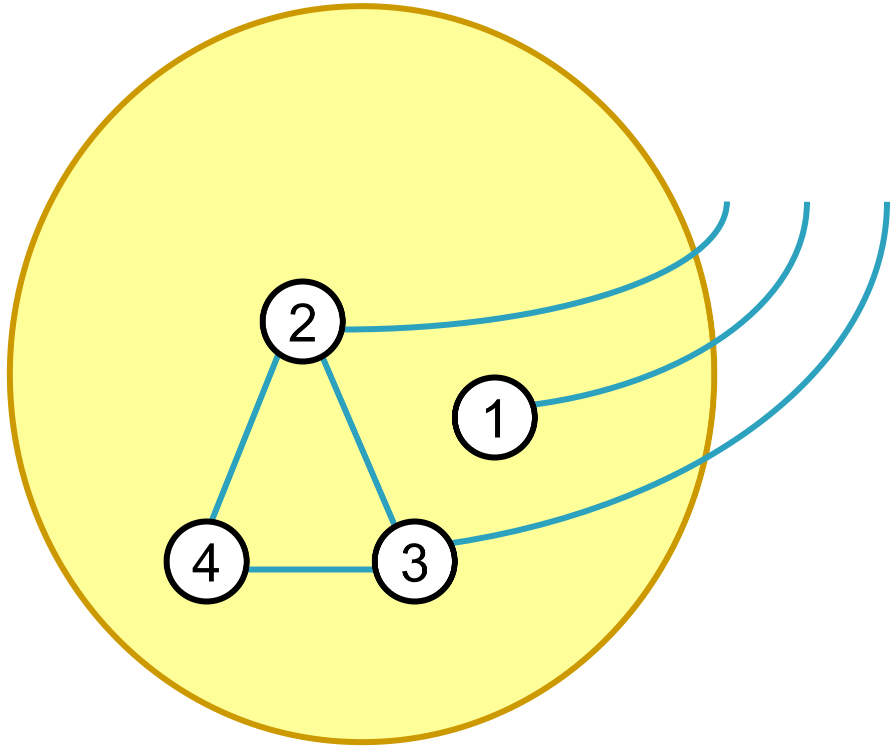

C-Planarity Reduces to $SEFE_2$

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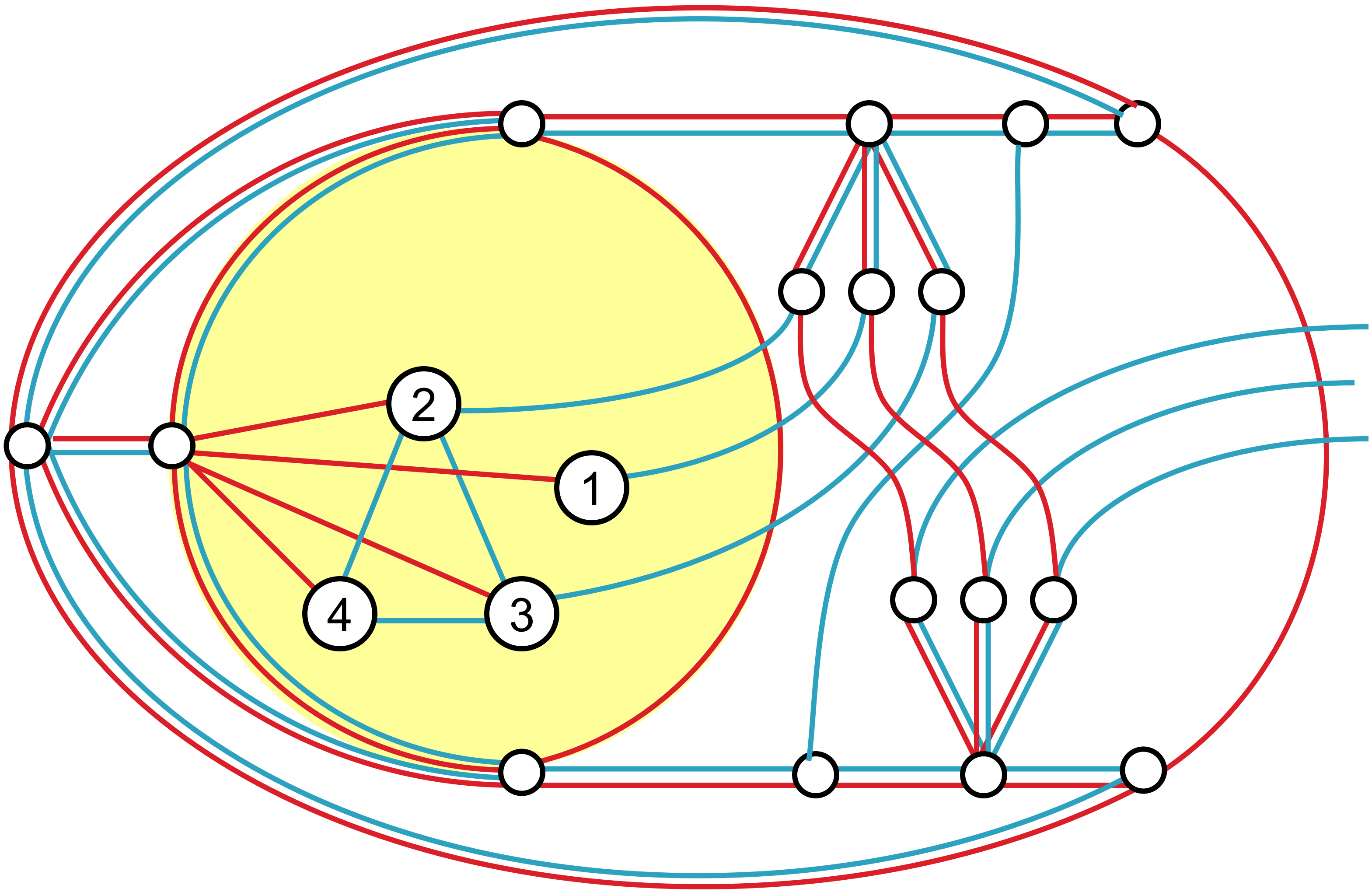


Clustered Planarity ≤ $SEFE_2$

- If c-planarity is NP-hard, $SEFE_2$ is also NP-hard
- If $SEFE_2$ is polynomial, c-planarity also is polynomial

- Can we reduce $SEFE_2$ to clustered planarity?
  - at least for some special cases?

Problem Relaxations

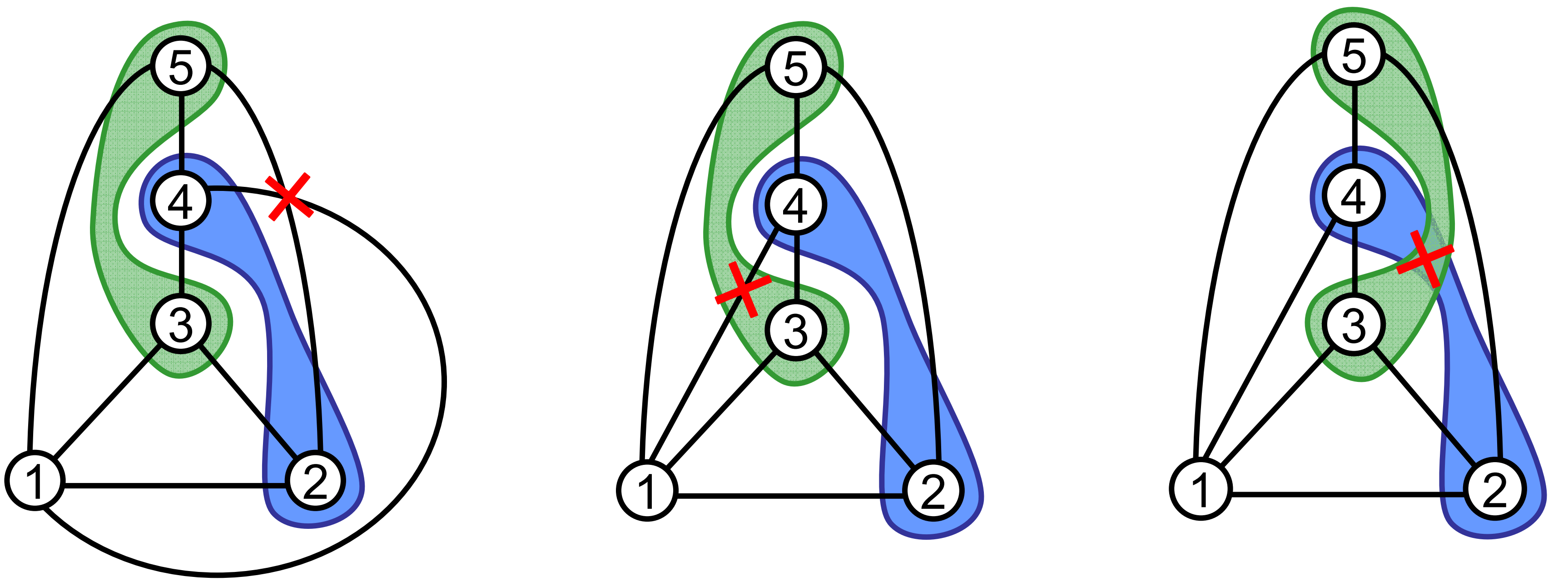
- Relax SEFE
  - by allowing some common edges to be drawn differently
- Relax clustered planarity
  - by allowing (a certain kind of) crossings

Relaxing SEFE

- Max-SEFE definition
  - Maximize the number of edges that are drawn the same in a simultaneous drawing of k graphs
    - Open Problem 9 of Chapter 11 of Tamassia's Handbook of Graph Drawing and Visualization [Bläsius, Kobourov, Rutter, '13]
- Max-SEFE is NP-complete for k=2
  - [Angelini, Da Lozzo, Neuwirth, WALCOM '14]
    - hard both in the fixed and in the variable embedding scenario
    - in the variable embedding scenario it is hard even if the intersection graph has degree at most two

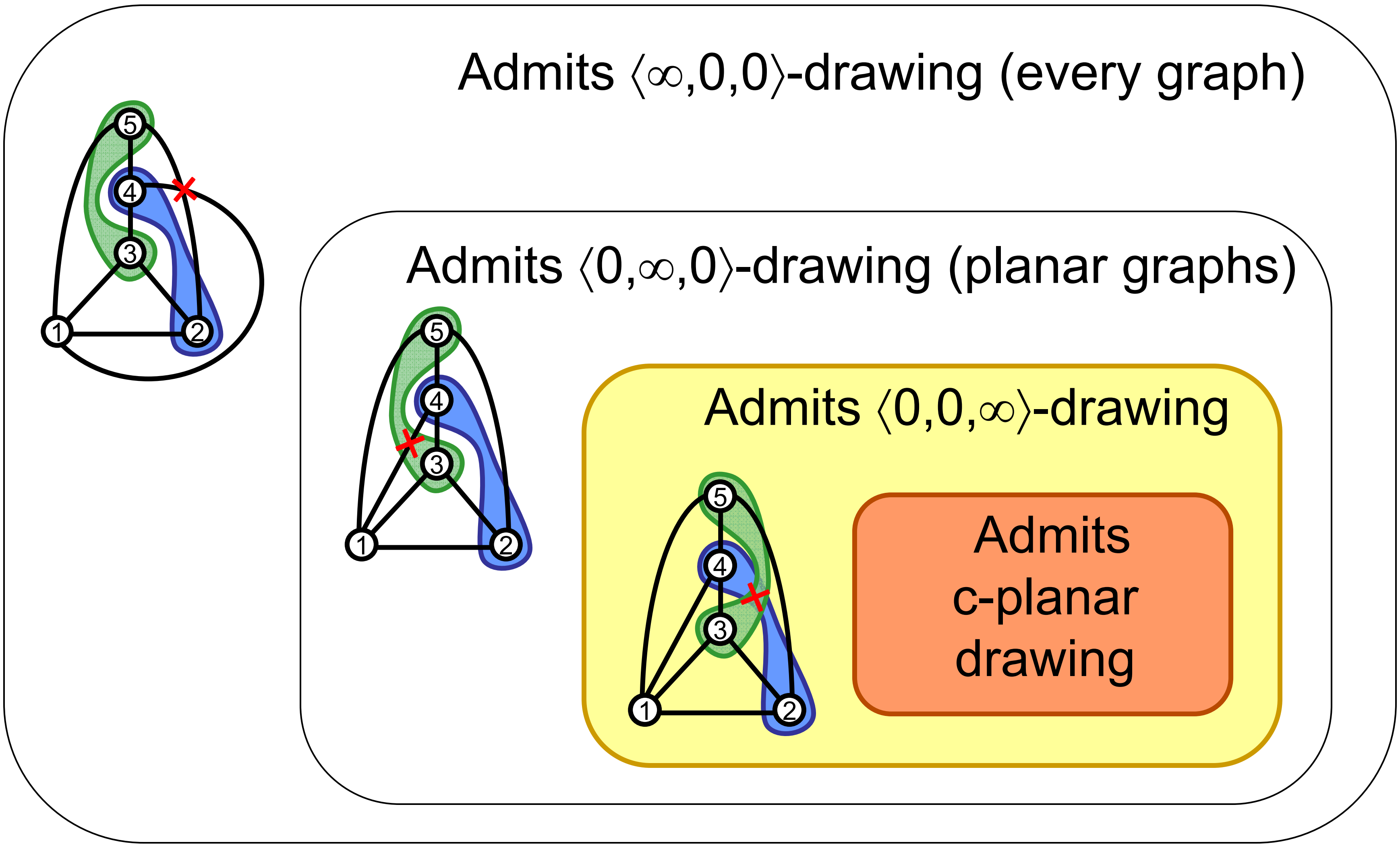
A Relaxed Model of C-Planarity

- Let an *⟨α,β,γ⟩-drawing* be a drawing of a c-graph such that the number of edge-edge, edge-region, and region-region crossings is equal to α, β, and γ, respectively



edge-edge crossings | edge-region crossings | region-region crossings

C-Graphs Hierarchy



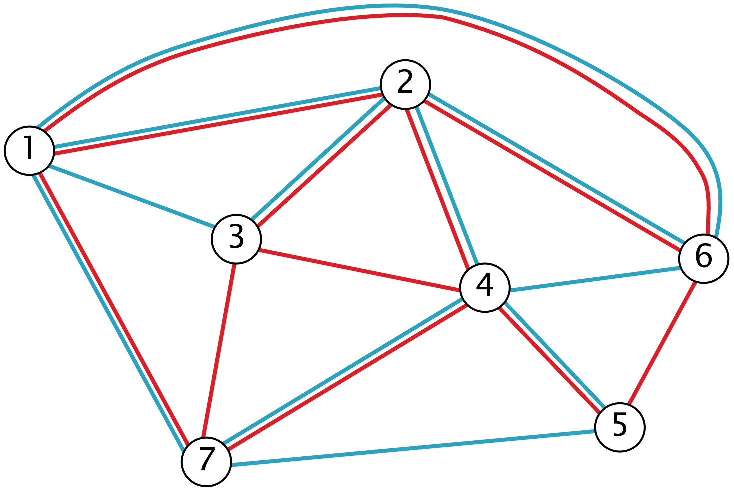
Relaxed C-Planarity

- Polynomial-time testing algorithm for the existence of $\langle 0,0,\infty \rangle$-drawings when G is biconnected
  - first non-trivial necessary condition for the existence of a c-planar drawing
    - [Angelini *et al.*, arxiv, submitted to journal]
- NP-hardness of minimizing the total number of crossings in
  - $\langle \alpha,\beta,\gamma \rangle$-drawings, $\langle \alpha,0,0 \rangle$-drawings, $\langle 0,\beta,0 \rangle$-drawings, and $\langle 0,0,\gamma \rangle$-drawings

$SEFE_2$ with Fixed Embedding

What if the embedding of the input graphs is fixed?

- If we have only two graphs, the problem is easy
  - you only have to check that the embeddings of G_1 and G_2 restricted to the intersection graph is the same



$SEFE_3$ with Fixed Embedding

- [Angelini *et al.*, '13]
  - Polynomial-time algorithm for testing the existence of a SEFE-FE of three graphs
  - Proof that SEFE-FE is NP-Hard for 14 graphs
  - Proof that Geometric Simultaneous Embedding-Fixed Embedding is NP-Hard for 13 graphs

SEFE with Sunflower Intersection

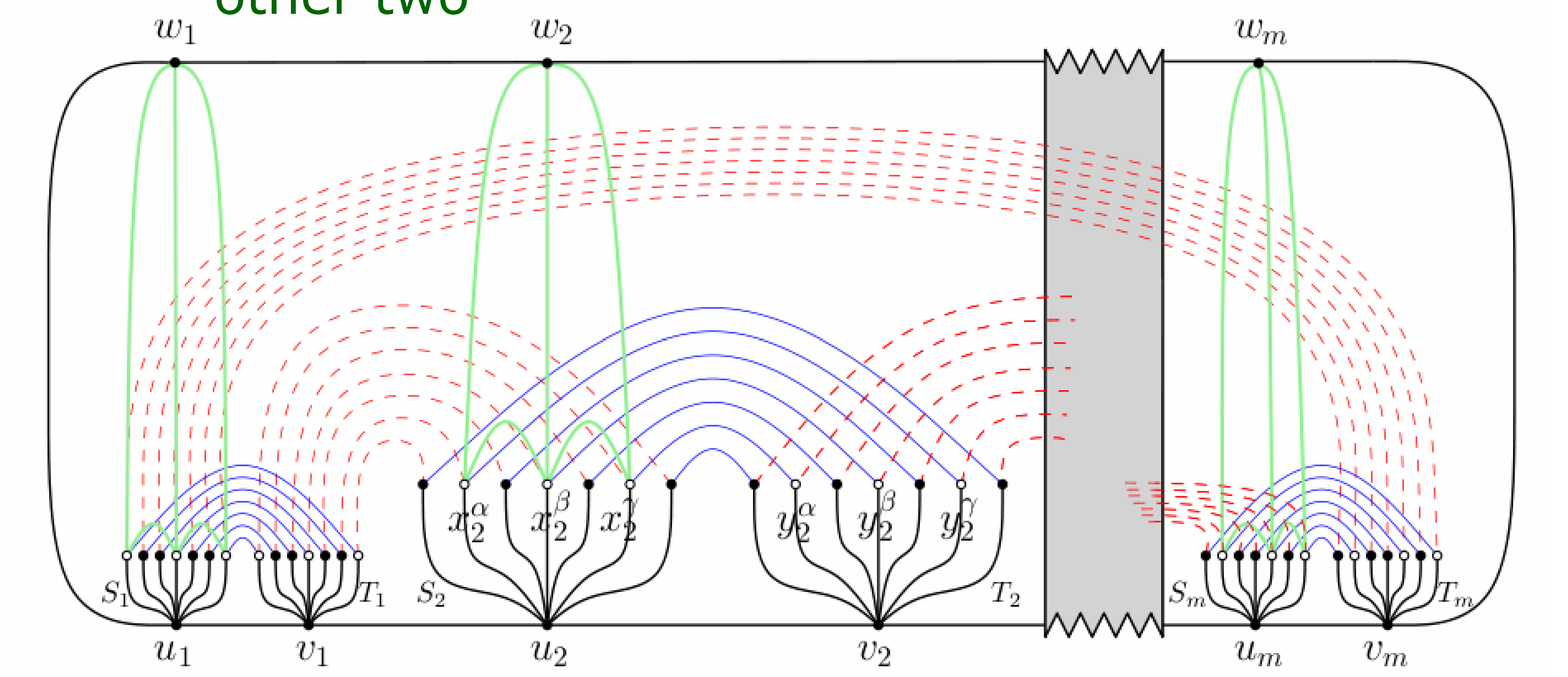
- Sunflower SEFE
  - if an edge belongs to two input graphs then it belongs to all the input graphs
- for k=2, any SEFE has sunflower intersection
  - all results (and open problems) for $SEFE_2$ also apply to this case

Sunflower SEFE Complexity

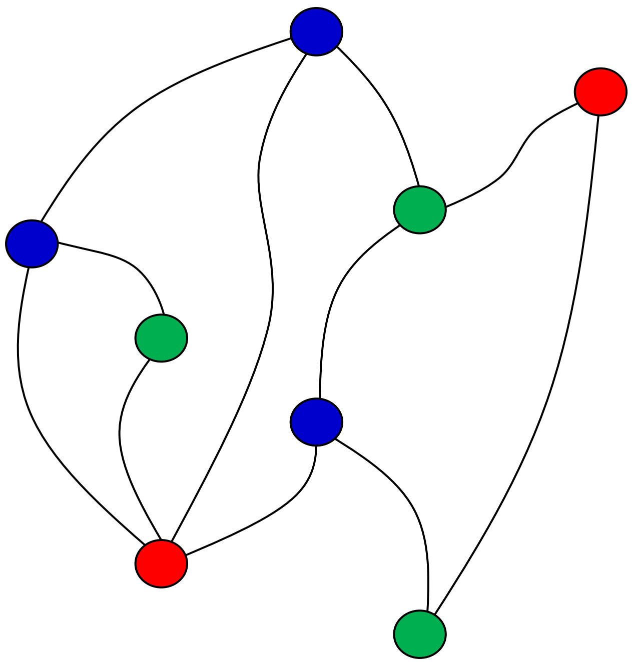
- NP-complete for $k \geq 3$
  - [Schaefer, JGAA '13]
    - instances in which the intersection graph is a spanning forest composed of an unbounded number of star graphs
  - [Angelini, Da Lozzo, Neuwirth, WALCOM '14]
    - the intersection graph is connected
    - $k=3$ and two graphs are biconnected

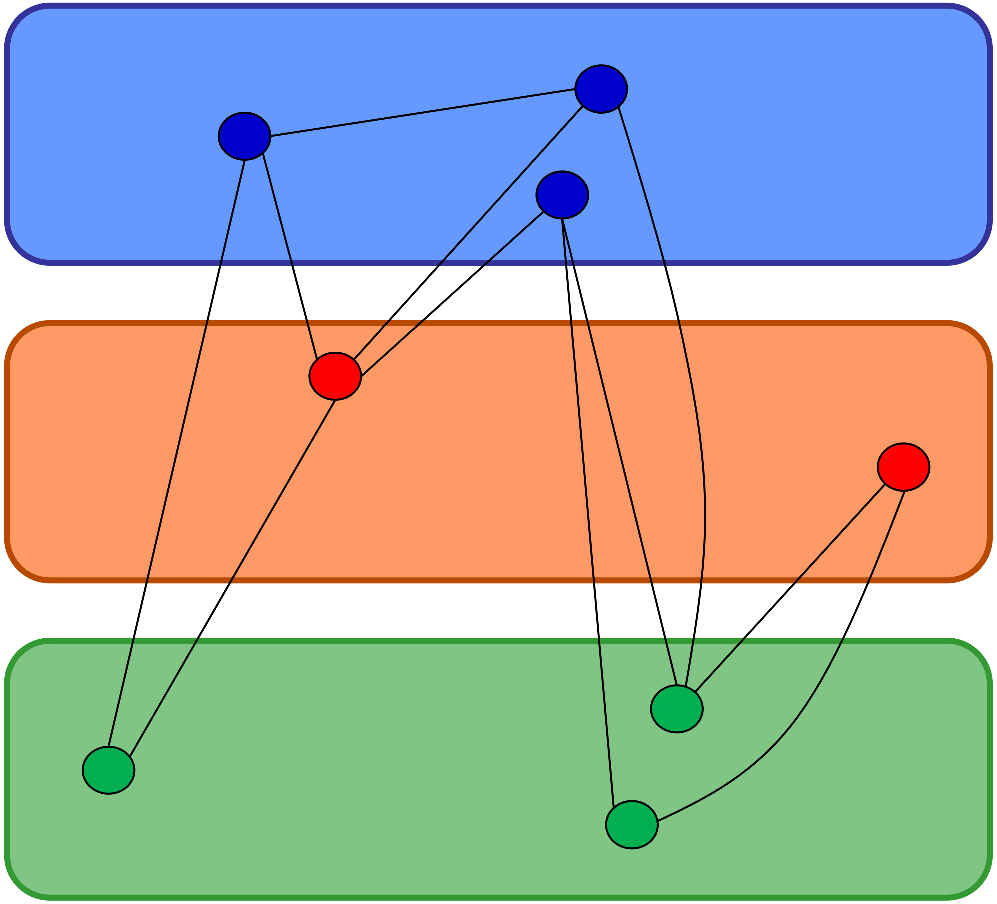
Hint of the Proof

- Reduction from Betweenness:
  - given a finite set of triplets of distinct elements
  - find a linear ordering where the central element of each triplet is in the middle of the other two

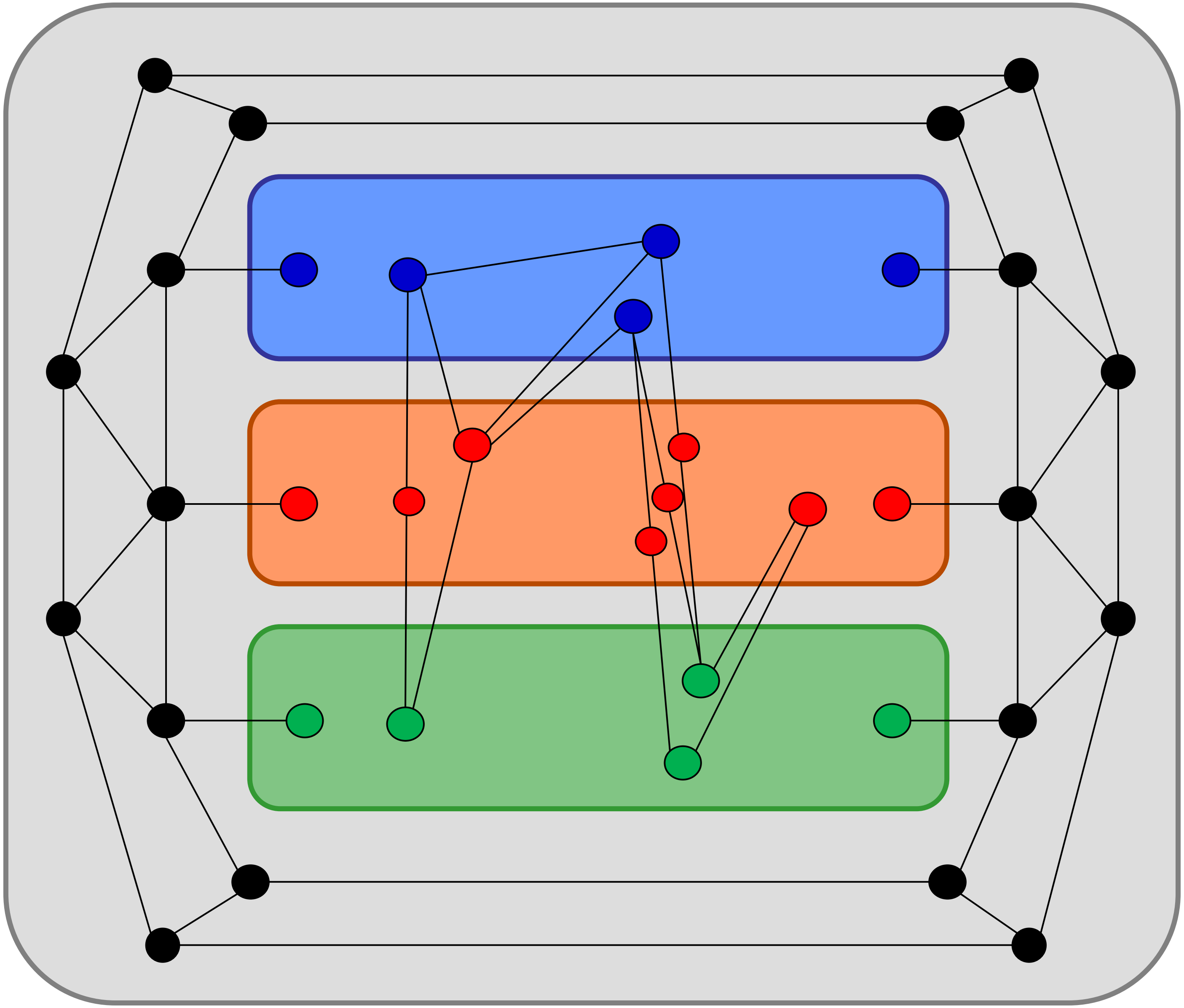


Strip Planarity



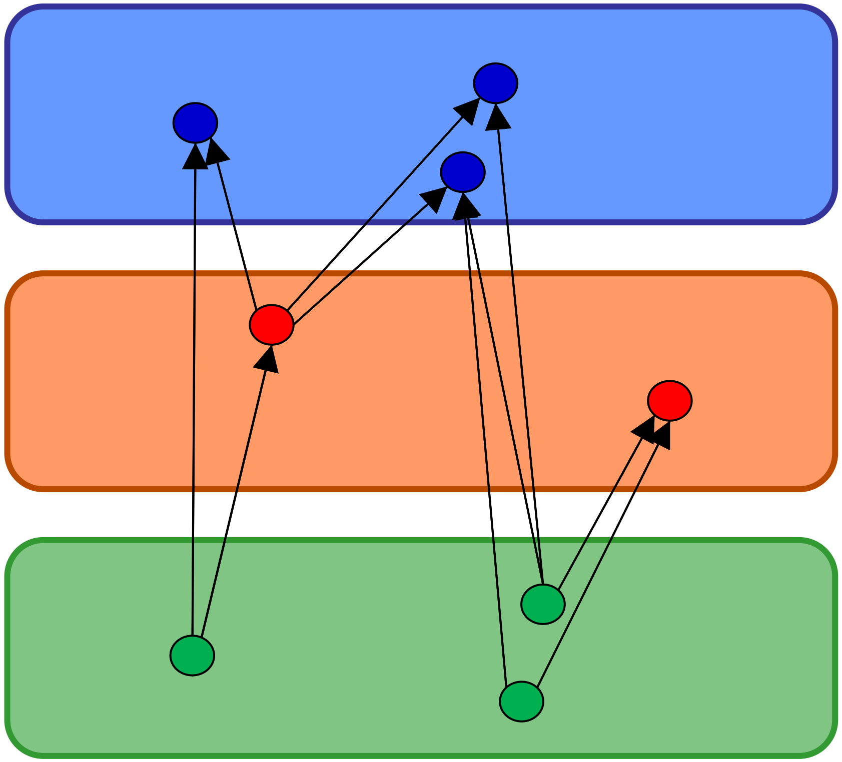


Strip Planarity Reduces to C-Planarity



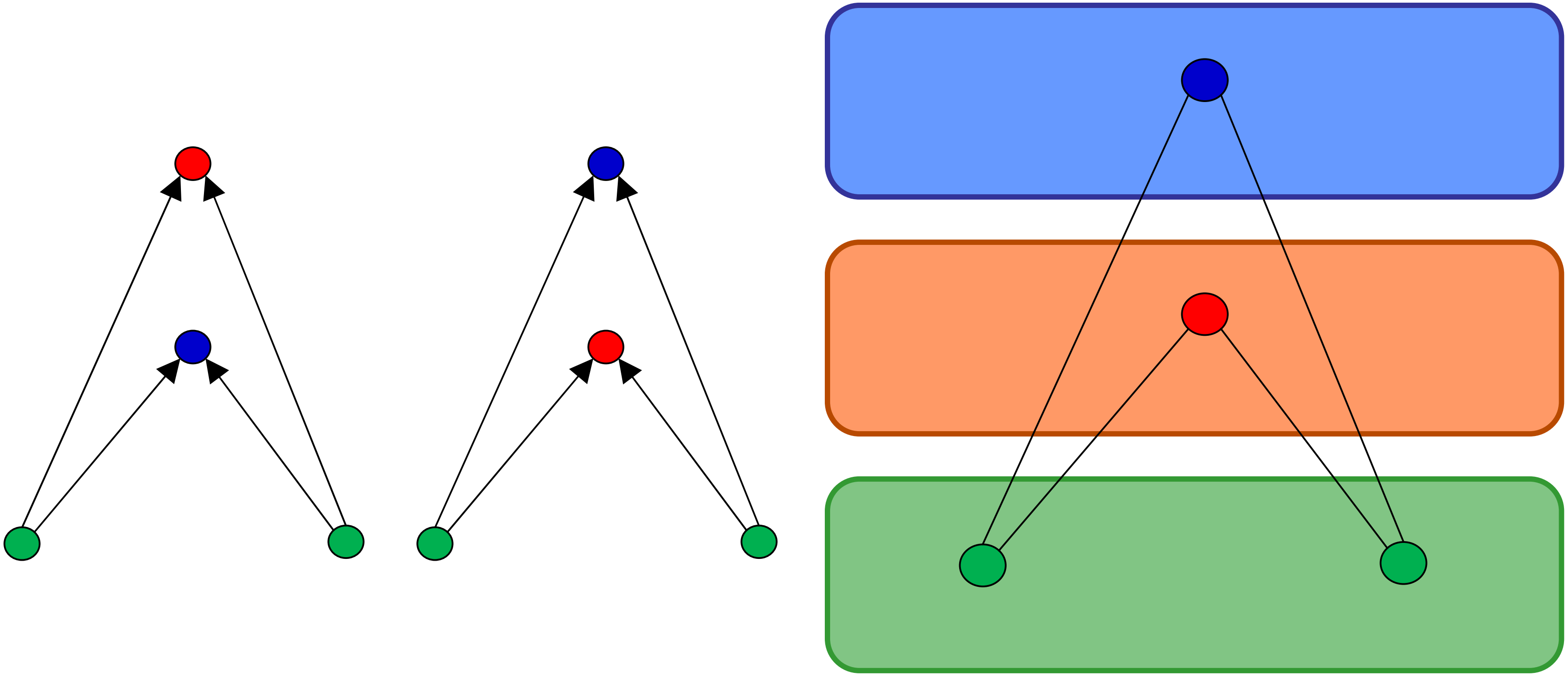
Strip Planarity and Upward Planarity

- Upward planarity
  - NP-complete in the variable embedding setting [Garg, Tamassia, ‘01]
  - Polynomial with fixed embedding [Bertolazzi *et al.*, ‘94]



Strip Planarity and Upward Planarity

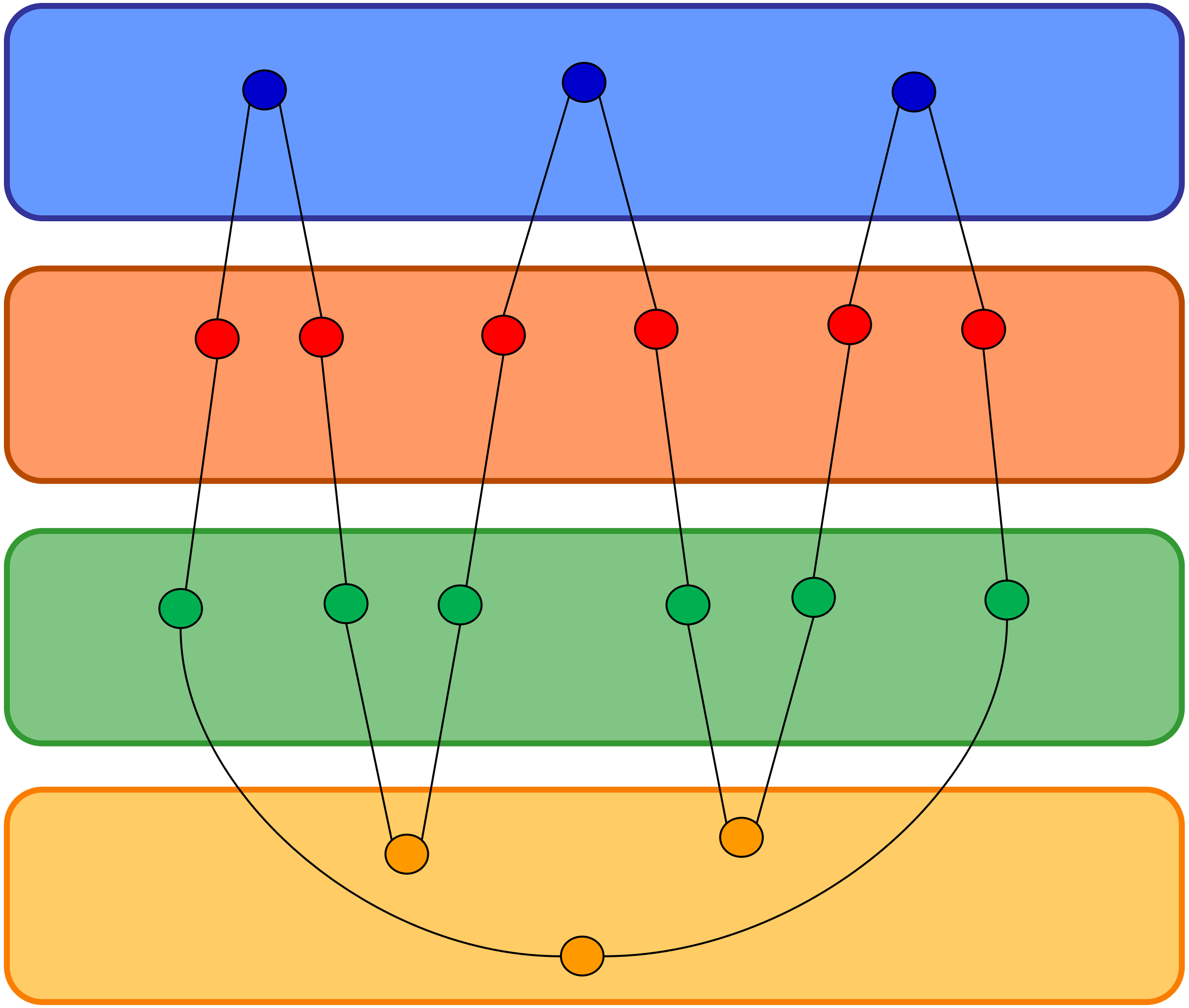
- Two different problems



Complexity of Testing Strip Planarity

- Given an embedded planar graph G, it is possible to test in polynomial time whether G admits a strip planar drawing preserving the given embedding
  - [Angelini, Da Lozzo, Di Battista, Frati, GD '13]

Proof Hint: Jagged Faces



Proof Strategy

- Lemma 1: A *jagged instance* is strip planar if and only if its associated directed graph is upward planar
- Lemma 2: Given a general instance of strip planarity, transform it into an equivalent jagged instance
  - proved by a sequence of reductions
    - general strip planarity instance
    - proper and strict equivalent instance
    - quasi-jagged instance
    - jagged instance

Open Problems (SEFE)

- Complexity of deciding SEFE with k=2
  - for example, when the intersection graph is not biconnected nor a star
- Complexity of Sunflower SEFE when the intersection graph is a star graph and k is bounded by a constant

Open Problems (Clustered Planarity)

- Flat clustered graphs with specific families of underlying graphs
  - set of disjoint cycles, set of disjoint lines, trees, series-parallel graphs, etc
- Is Strip Planarity NP-complete in the variable embedding setting?
  - strip reduces to clustered planarity
  - strip planarity NP-complete would imply the NP-completeness of clustered planarity

Open Problems (Both)

- Reduce clustered planarity to $SEFE_2$
- Show that the two problems are equivalent in some setting

Thank you!