Disjoint compatibility of non-crossing matchings of point sets in convex position

Oswin Aichholzer (TU Graz)
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$2k$ labeled points in convex position in the plane
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Non-crossing straight-line perfect matchings
$2k$ labeled points in convex position in the plane

Non-crossing straight-line perfect matchings

(The number of such matchings is the $k$th Catalan number)
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Non-crossing straight-line perfect matchings

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Two matchings, $M$ and $L$, are *disjoint compatible* if they don’t use the same edges, and the edges of $M$ don’t cross the edges of $L$. 

![Diagram showing disjoint compatible matchings](image)
The Disjoint Compatibility Graph (DCM<sub>k</sub>):
Vertices correspond to matchings; two vertices are adjacent iff the corresponding matchings are disjoint compatible.
The Disjoint Compatibility Graph ($\text{DCM}_k$):
Vertices correspond to matchings; two vertices are adjacent iff the corresponding matchings are disjoint compatible.

$k = 4$
Background:
C. Hernando, F. Hurtado and M. Noy.
Graph of non-crossing perfect matchings. (2002):
  Studied matchings of points in convex position,
  but with respect to a different kind of reconfiguration.
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O. Aichholzer, S. Bereg, A. Dumitrescu, A. García, C. Huemer,
F. Hurtado, M. Kano, A. Márquez, D. Rappaport,
Compatible geometric matchings (2009):
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M. Ishaque, D. Souvaine and C. Tóth.

Disjoint compatible geometric matchings (2013):

1. Proved the conjecture.
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M. Ishaque, D. Souvaine and C. Tóth.
Disjoint compatible geometric matchings (2013):

1. Proved the conjecture.
2. “It remains an open problem whether [the disjoint compatibility graph for even $k$] is always connected”.

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$DCM_2$

$DCM_4$
$DCM_6$
DCM_8
For given $k$, denote by $c(k)$ the number of isomorphism classes of connected components in $\text{DCM}_k$. 

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<td>$c(k)$</td>
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<td>$c(k)$</td>
<td>1</td>
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<td>3</td>
<td>4</td>
<td>4</td>
<td>3</td>
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**Theorem:** For each $k \geq 9$, the connected components of $\text{DCM}_k$ form exactly three isomorphism classes. Specifically: there are several components of the smallest size, several components of the medium size, and one component of the biggest size.

<table>
<thead>
<tr>
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<th>odd $k \geq 9$</th>
<th>even $k \geq 10$</th>
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<tbody>
<tr>
<td>$\ell$</td>
<td>$\frac{k+1}{2}$</td>
<td>$\frac{k}{2}$</td>
</tr>
<tr>
<td>small: size</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>small: number</td>
<td>$\frac{1}{\ell} \left(\frac{4\ell-2}{\ell-1}\right)$</td>
<td>$\ell \cdot 2^{\ell-1}$</td>
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<tr>
<td>medium: size</td>
<td>$\ell$</td>
<td>$6\ell - 6$</td>
</tr>
<tr>
<td>medium: number</td>
<td>$(2\ell - 1) \cdot 2^{\ell-3}$</td>
<td>$\ell \cdot 2^{\ell-2}$</td>
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<tr>
<td>big: number</td>
<td>1</td>
<td>1</td>
</tr>
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A *block* is four consecutive points $i, i + 1, i + 2, i + 3$ connected by edges $(i, i + 3)$ and $(i + 1, i + 2)$.

An *antiblock* is four consecutive points $i, i + 1, i + 2, i + 3$ connected by edges $(i, i + 1)$ and $(i + 2, i + 3)$. 

![Diagram of a block and an antiblock](image-url)
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Observation: If we have a block in a matching $M$, and a matching $L$ is disjoint compatible to $M$, then in $L$ we have an antiblock on the same points.
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\[ \begin{array}{cccccccc}
\text{V} & \text{A} & \text{A} & \text{A} & \text{A} & \text{V} & \text{A} & \text{A} \\
\end{array} \]
Small components for even $k$: “pairs”.

![Graph with blue and red lines showing pairs for even $k$.]
Medium components for odd $k$: stars with $\frac{k-1}{2}$ leaves.

$k = 11$
Medium components for even \( k \):

A path \( P \) with \( (k-2) \) vertices (labeled 1, 2, \ldots, k-2),

Each vertex of \( P \) has two adjacent leaves,

In addition, all the pairs \((a, b)\), where \( a, b \) are on \( P \), \( a < b \), \( a \) even, \( b \) odd, are connected by an edge.

\[ k = 12 \]
A matching is *special* if it belongs to the types described above. Otherwise, a matching is *regular*.

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![Diagram of a ring structure with red lines connecting the points]
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**Lemma.** For \( k \geq 9 \), the component of \( DCM_k \) that contains the rings, is not bipartite.
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Proof. For $k = 9, 10$, this was verified directly.
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Find a block or an antiblock and look at the remaining part ($L$).
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If $L$ is regular, induction applies: we transfer $L$ to a ring, while the (anti)block “oscillates”.

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Characterize isolated matchings for point sets in \textit{general} position (for odd $k$).
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A sufficient (but not necessary) condition: