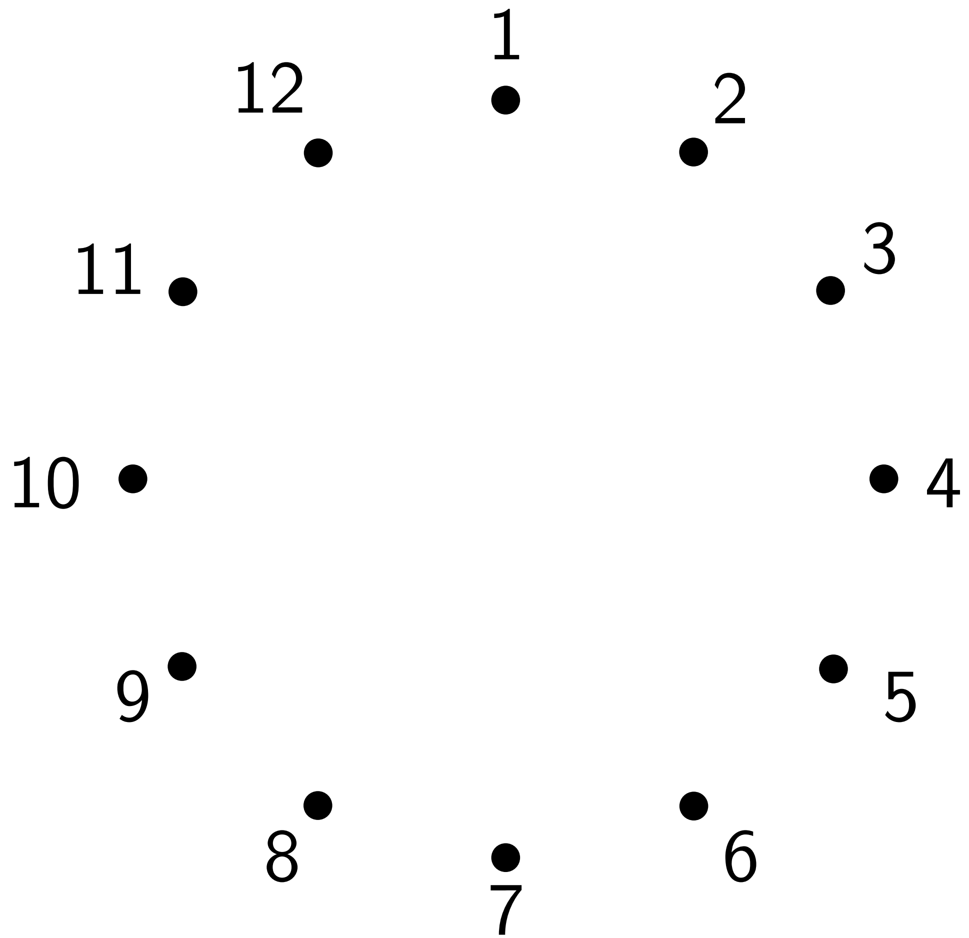
Disjoint compatibility of non-crossing matchings of point sets in convex position

Oswin Aichholzer (TU Graz)
Andrei Asinowski (FU Berlin)
Tillmann Miltzow (FU Berlin)

$2k$ labeled points in convex position in the plane



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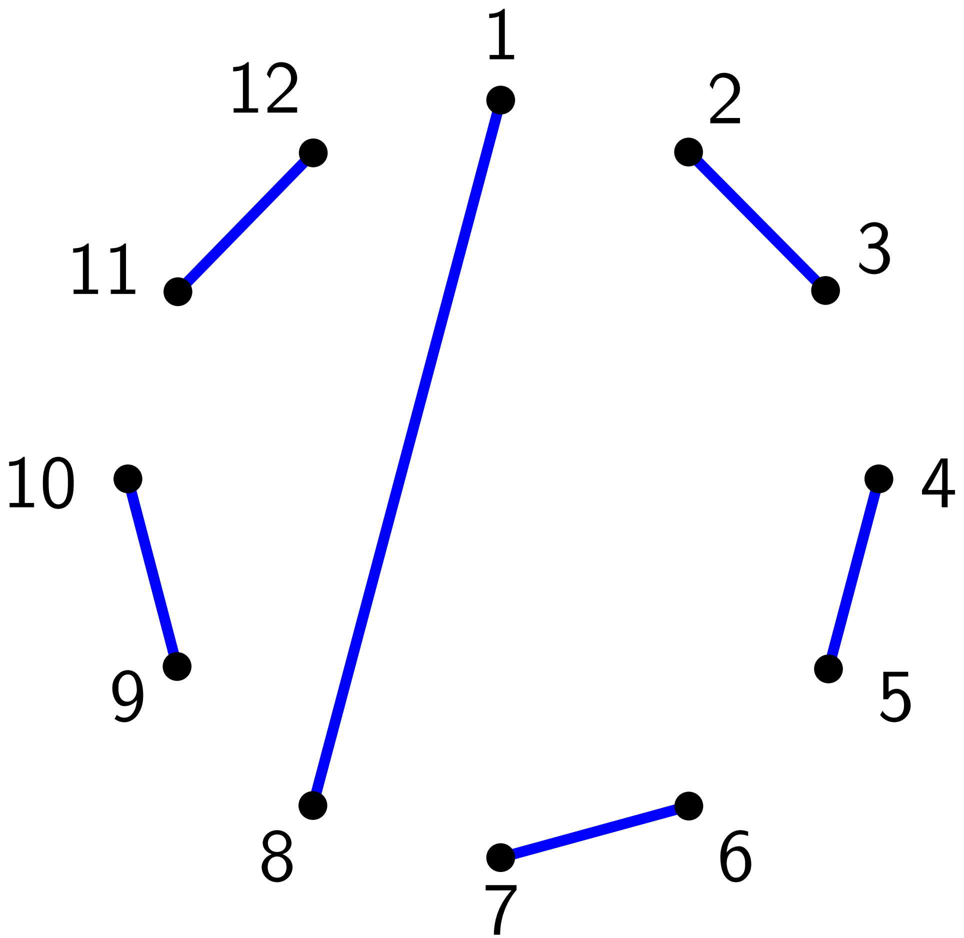
Non-crossing straight-line perfect matchings



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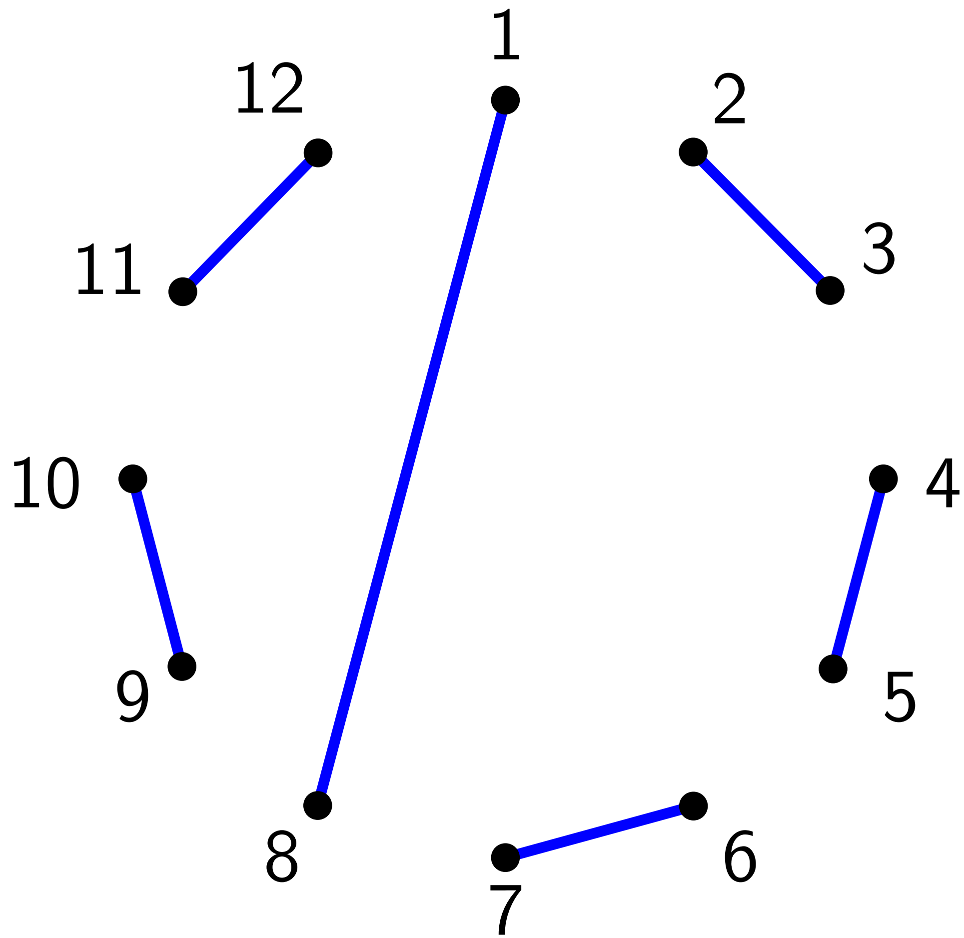
(The number of such matchings is the kth Catalan number)



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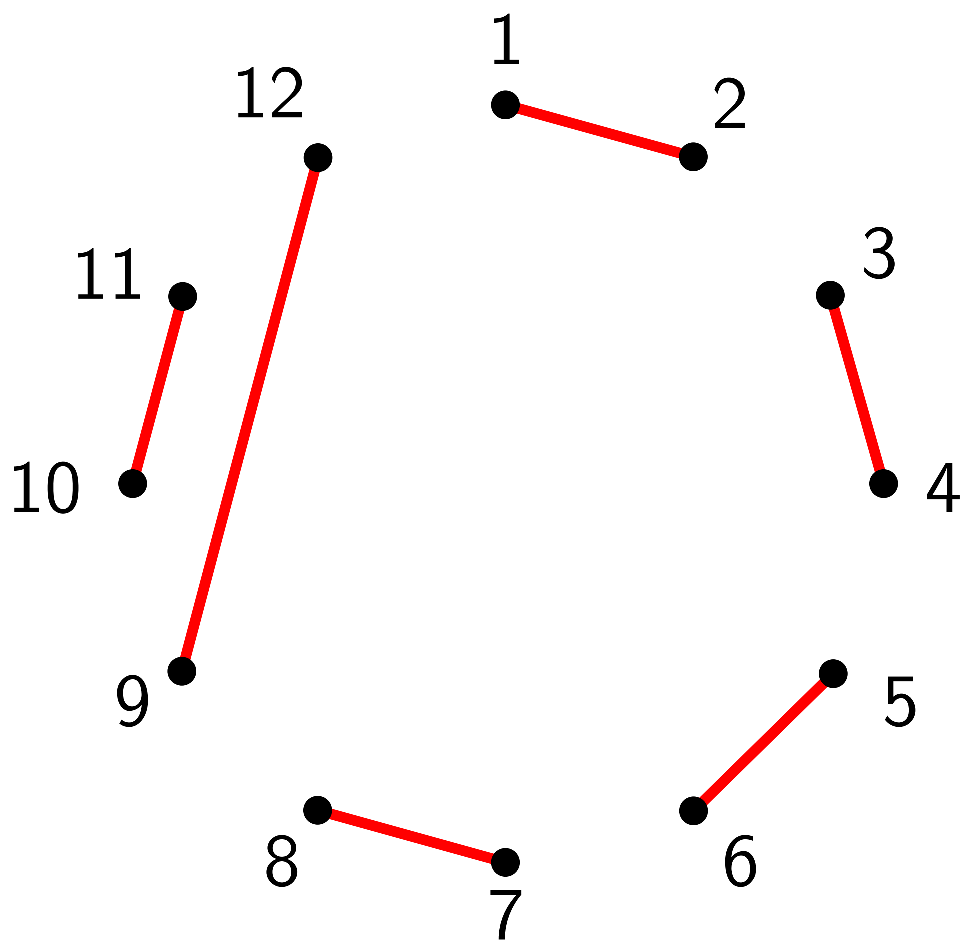
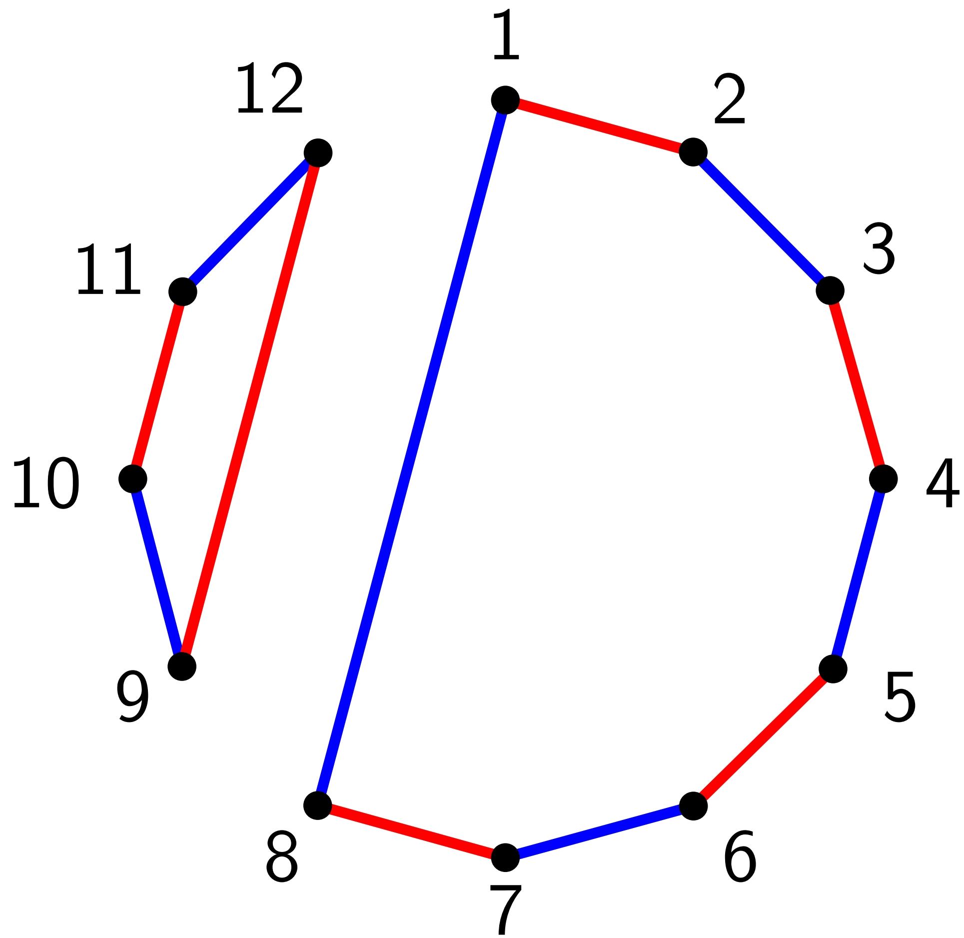


Two matchings, M and L, are *disjoint compatible* if they don't use the same edges, and the edges of M don't cross the edges of L.



The Disjoint Compatibility Graph (DCM_k):

Vertices correspond to matchings; two vertices are adjacent iff the corresponding matchings are disjoint compatible.

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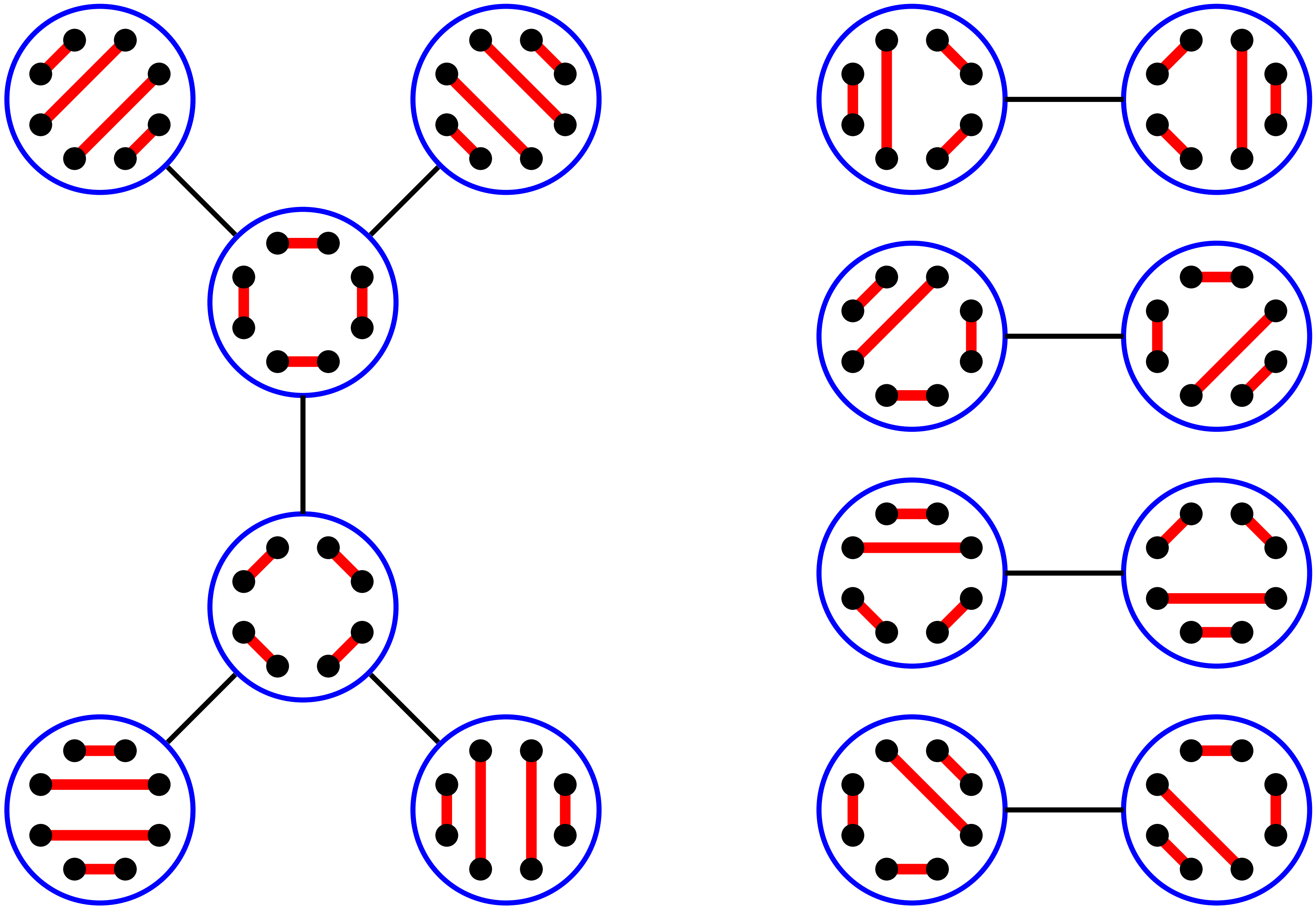

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C. Hernando, F. Hurtado and M.Noy.
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Studied matchings of points in convex position,
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O. Aichholzer, S. Bereg, A. Dumitrescu, A. García, C. Huemer,
F. Hurtado, M. Kano, A. Márquez, D. Rappaport,
S. Smorodinsky, D. L. Souvaine, J. Urrutia, and D. Wood.
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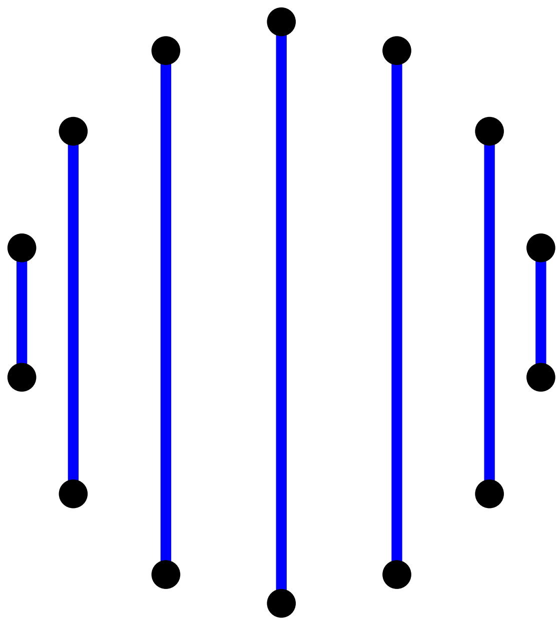

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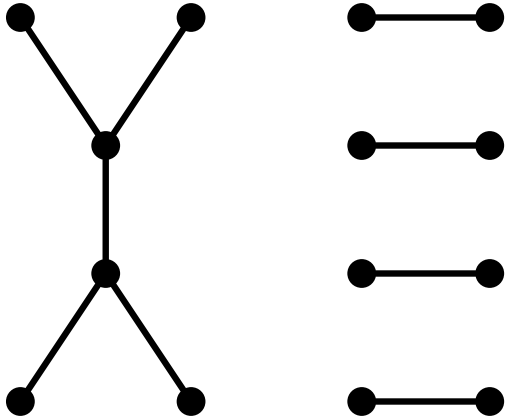
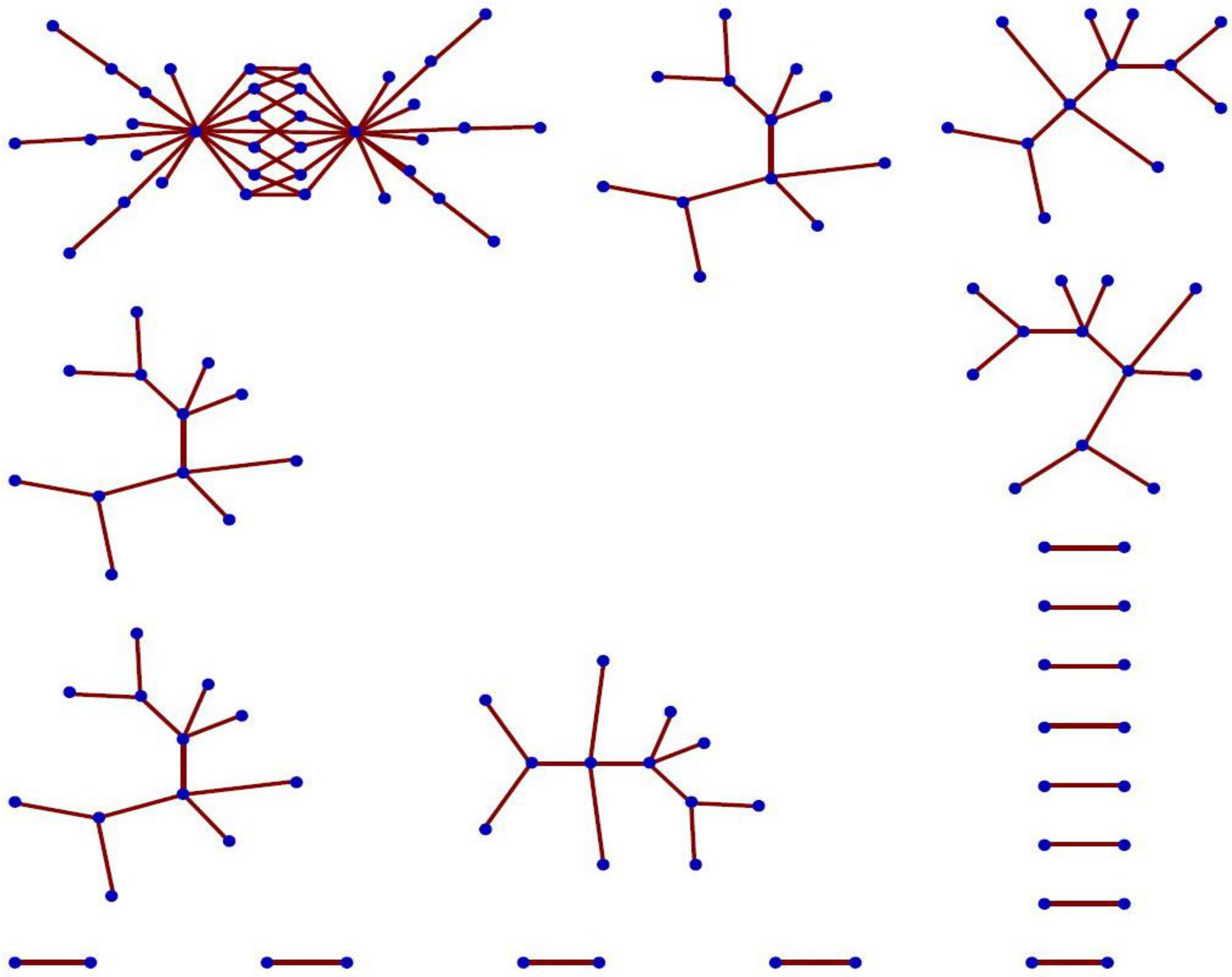
2. "It remains an open problem whether [the disjoint compatibility graph for even k] is always connected".

DCM_2

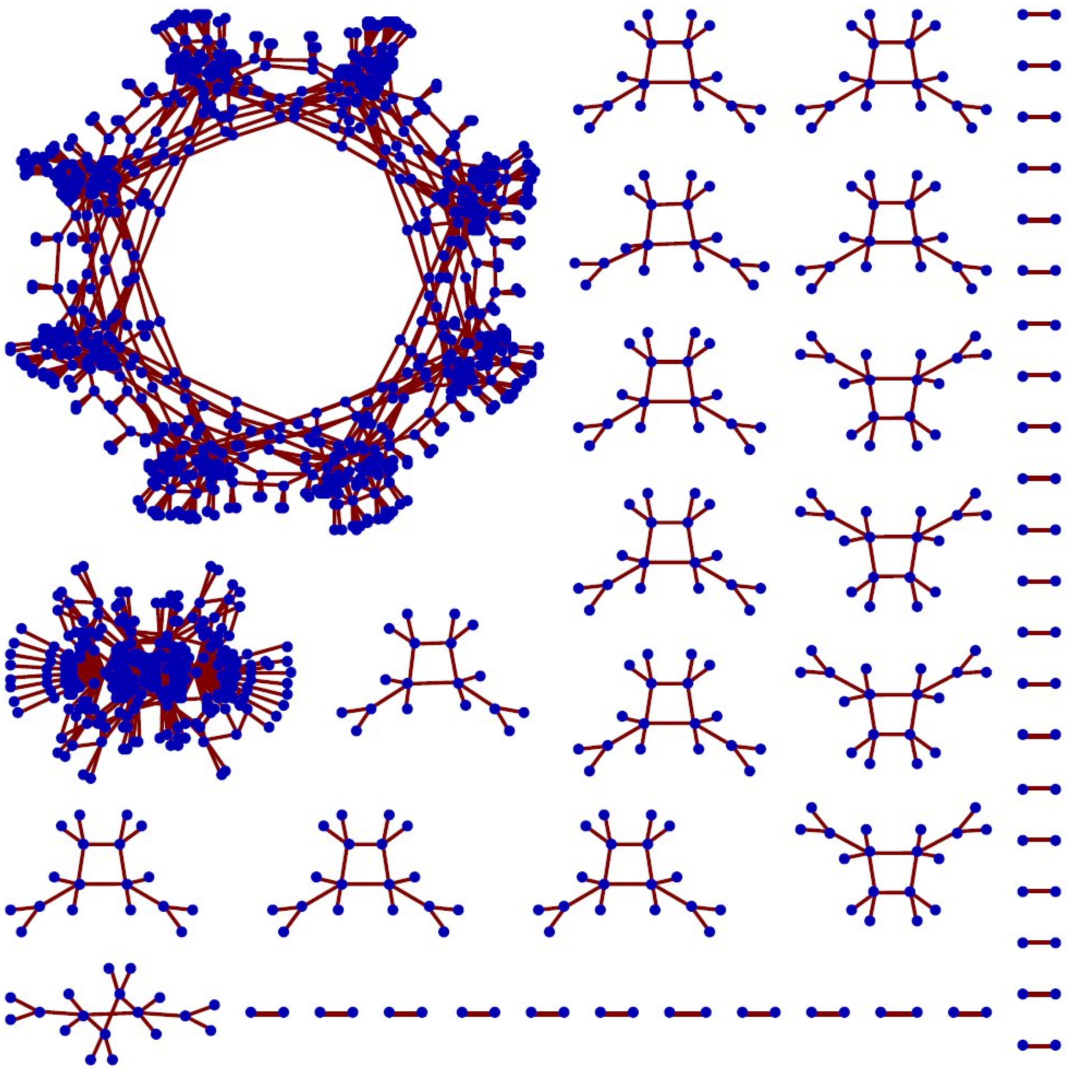
DCM_4



DCM$_6$



DCM_8



For given k, denote by $c(k)$ the number of isomorphism classes of connected components in DCM_k.

k	1	2	3	4	5	6	7	8
$c(k)$	1	1	2	2	3	3	4	4

For given k, denote by $c(k)$ the number of isomorphism classes of connected components in DCM_k.

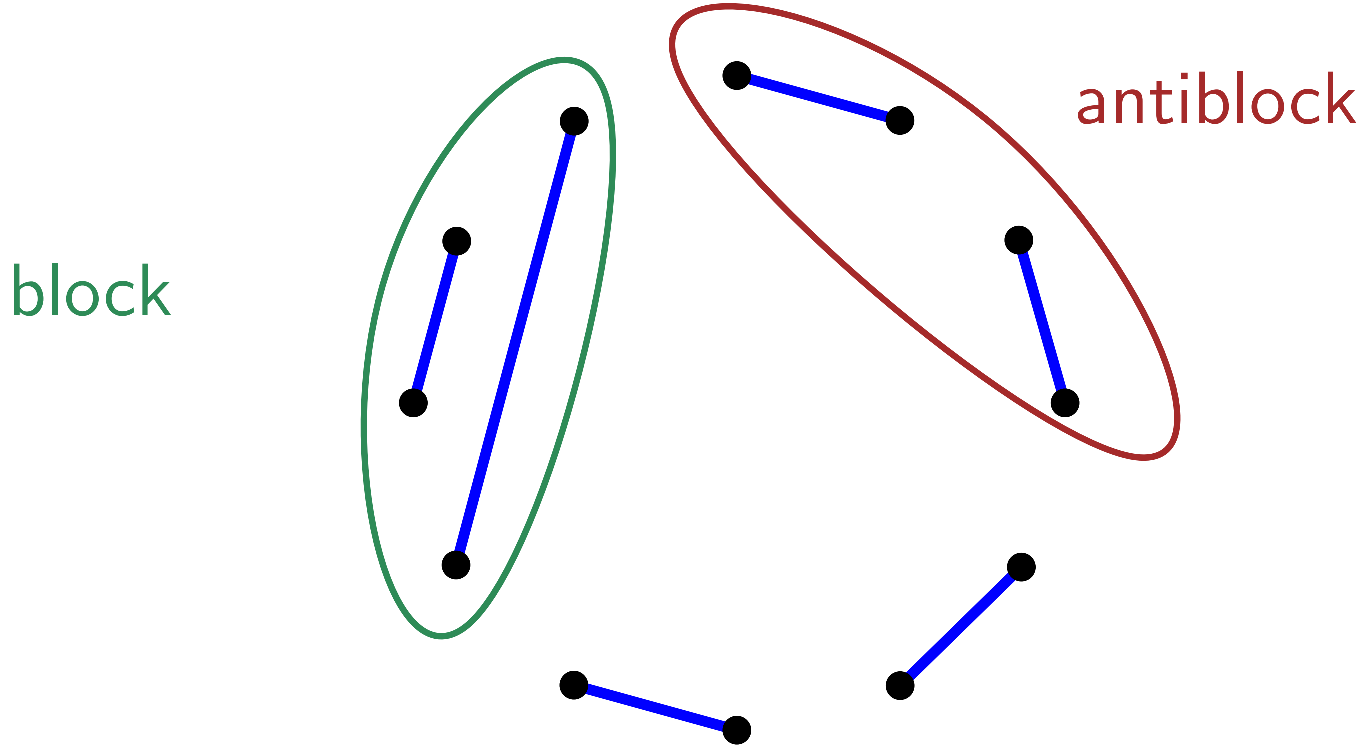
k	1	2	3	4	5	6	7	8	≥ 9
$c(k)$	1	1	2	2	3	3	4	4	3

Theorem: For each $k \geq 9$, the connected components of DCM_k form exactly three isomorphism classes.
Specifically: there are
several components of the smallest size,
several components of the medium size,
and one component of the biggest size.

	odd $k \geq 9$	even $k \geq 10$
	$\ell = \frac{k+1}{2}$	$\ell = \frac{k}{2}$
small: size	1	2
small: number	$\frac{1}{\ell}\binom{4\ell-2}{\ell-1}$	$\ell \cdot 2^{\ell-1}$
medium: size	ℓ	$6\ell - 6$
medium: number	$(2\ell - 1) \cdot 2^{\ell-3}$	$\ell \cdot 2^{\ell-2}$
big: number	1	1

A *block* is four consecutive points $i, i+1, i+2, i+3$ connected by edges $(i, i+3)$ and $(i+1, i+2)$.

An *antiblock* is four consecutive points $i, i+1, i+2, i+3$ connected by edges $(i, i+1)$ and $(i+2, i+3)$.



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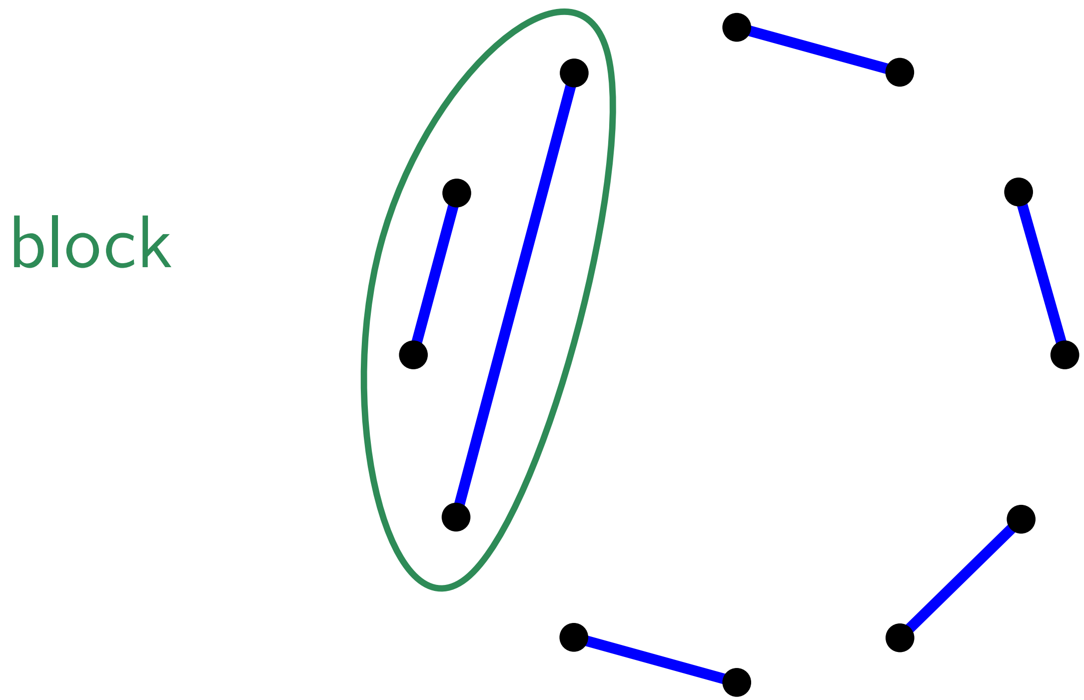
Observation: If we have a block in a matching M, and a matching L is disjoint compatible to M, then in L we have an antiblock on the same points.



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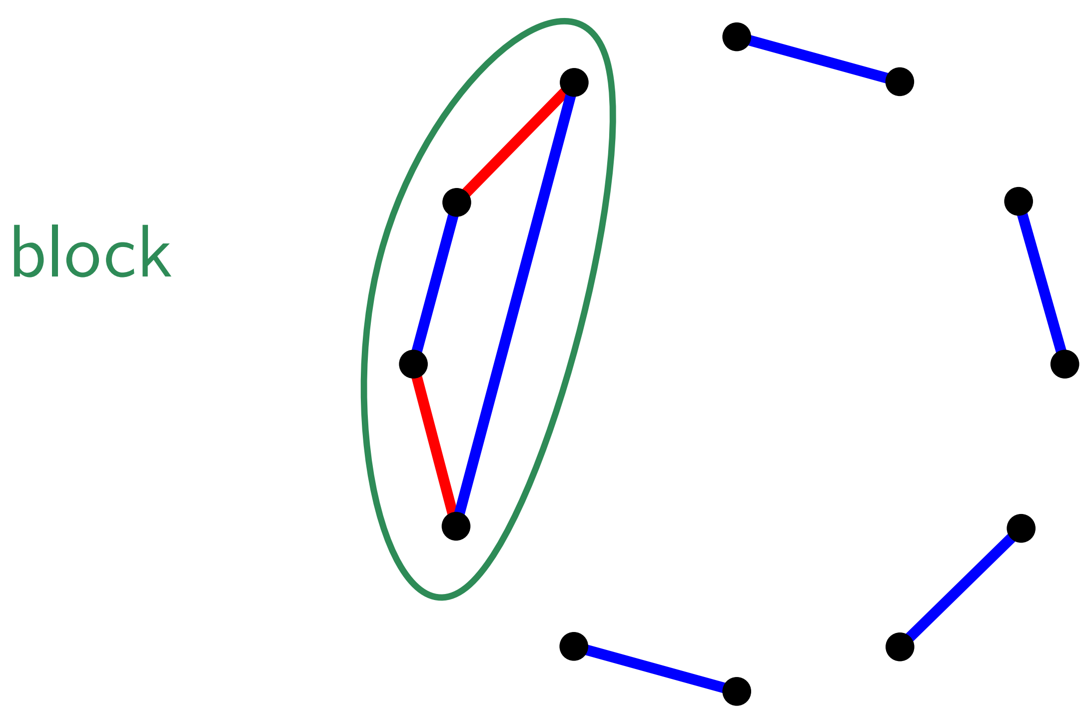
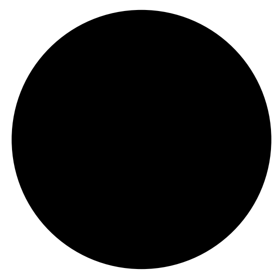


Small components for odd k: "isolated" matchings.



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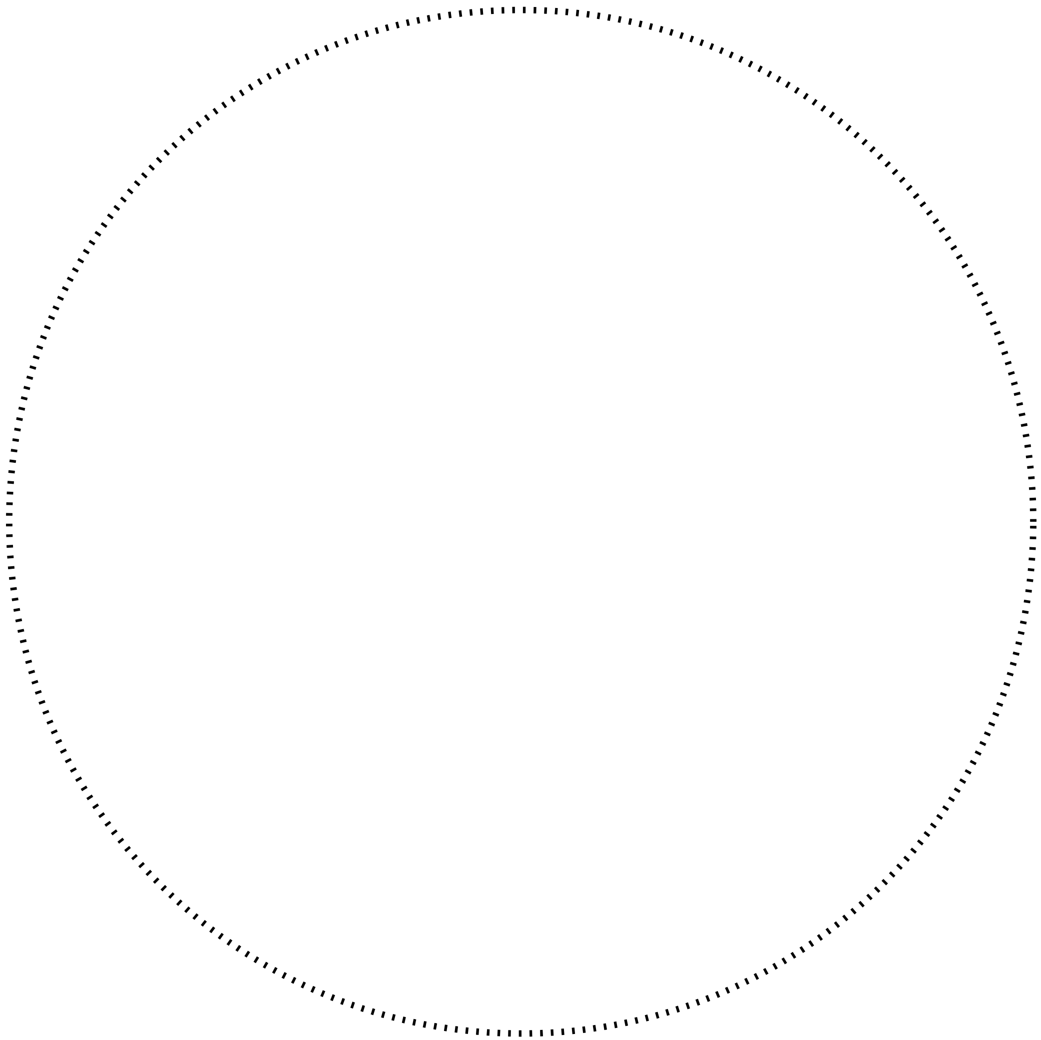

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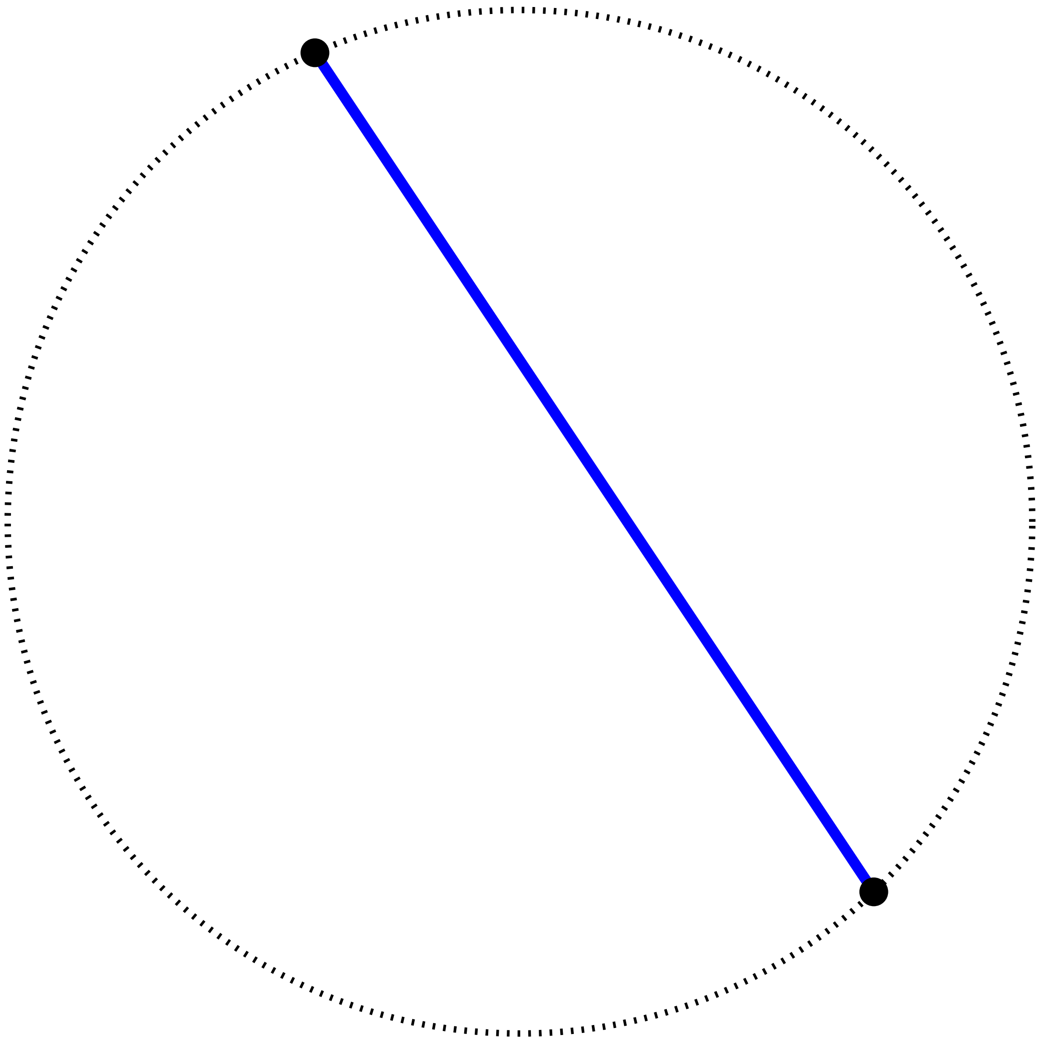

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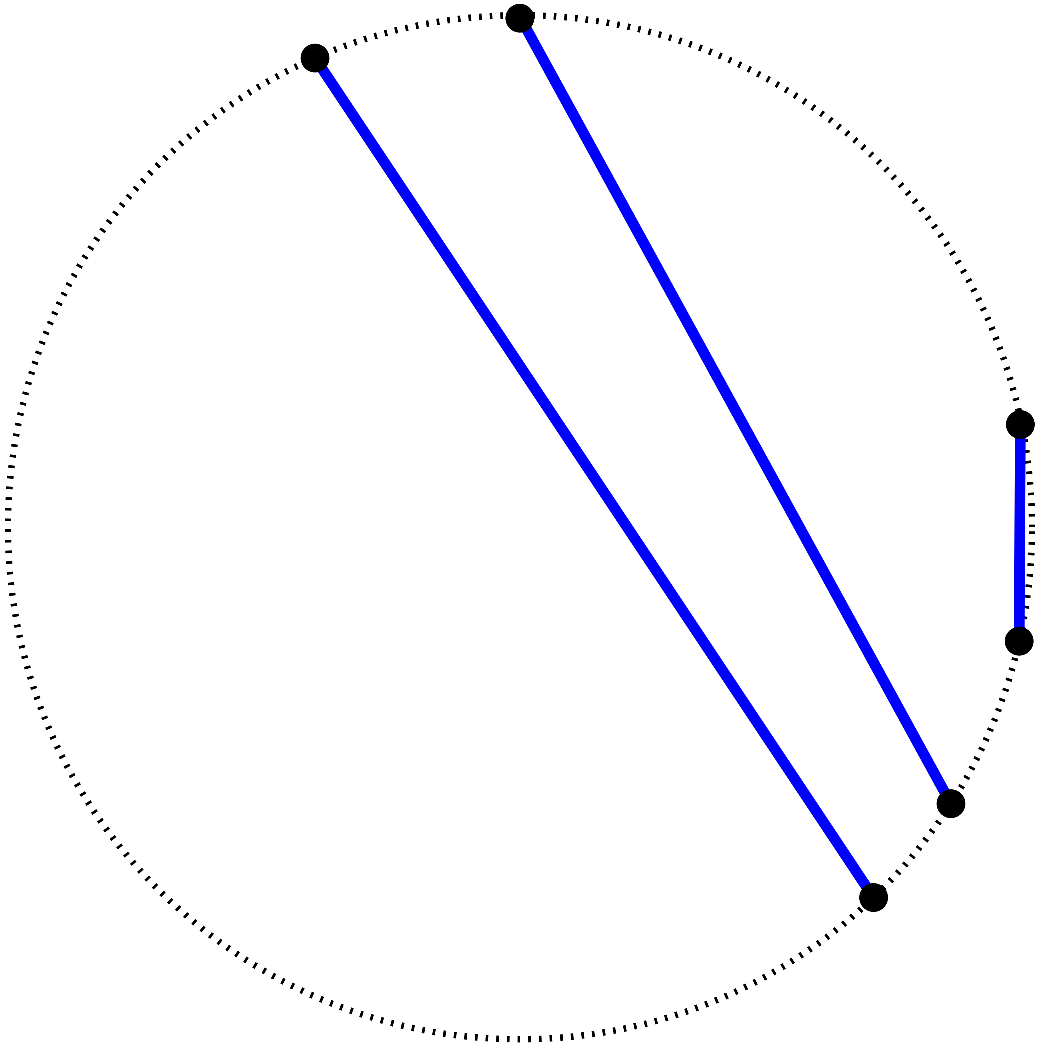

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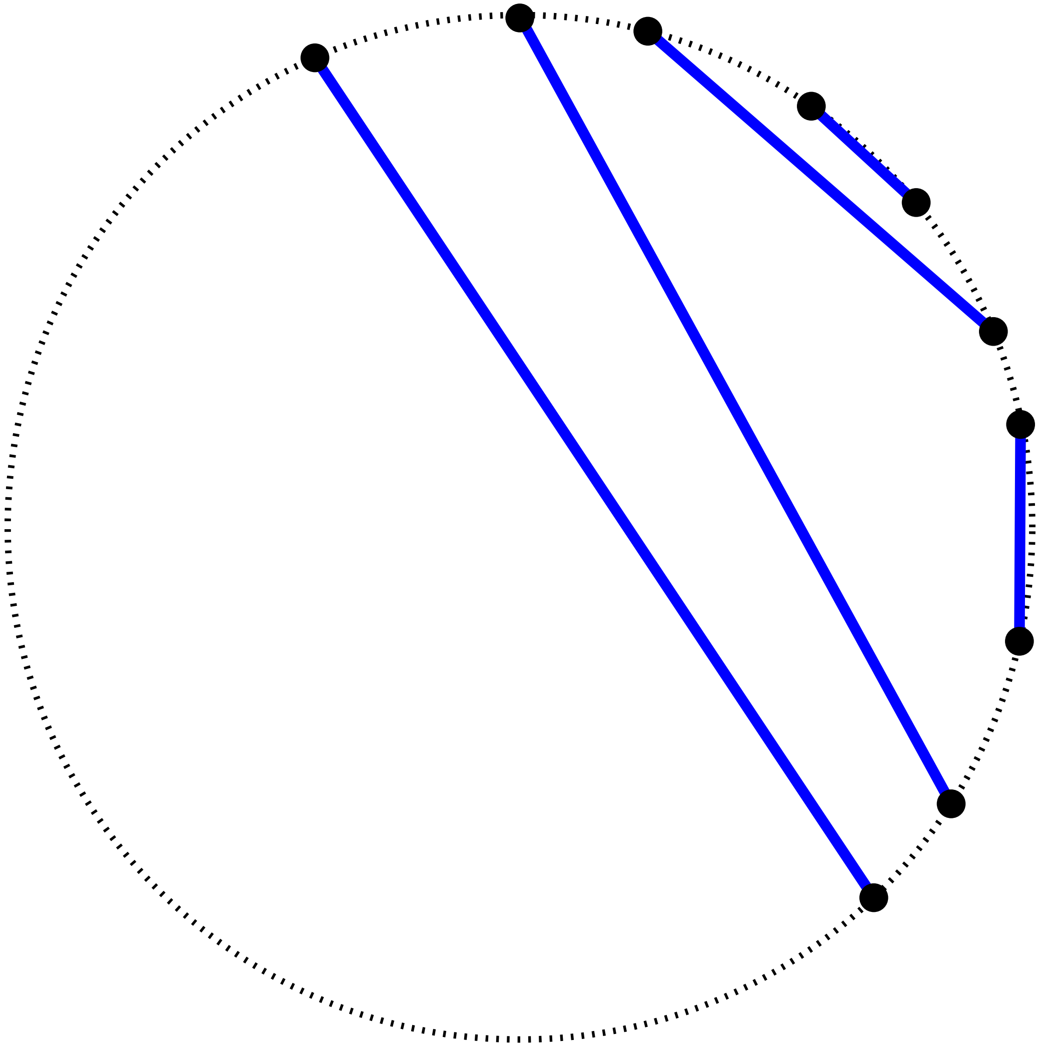

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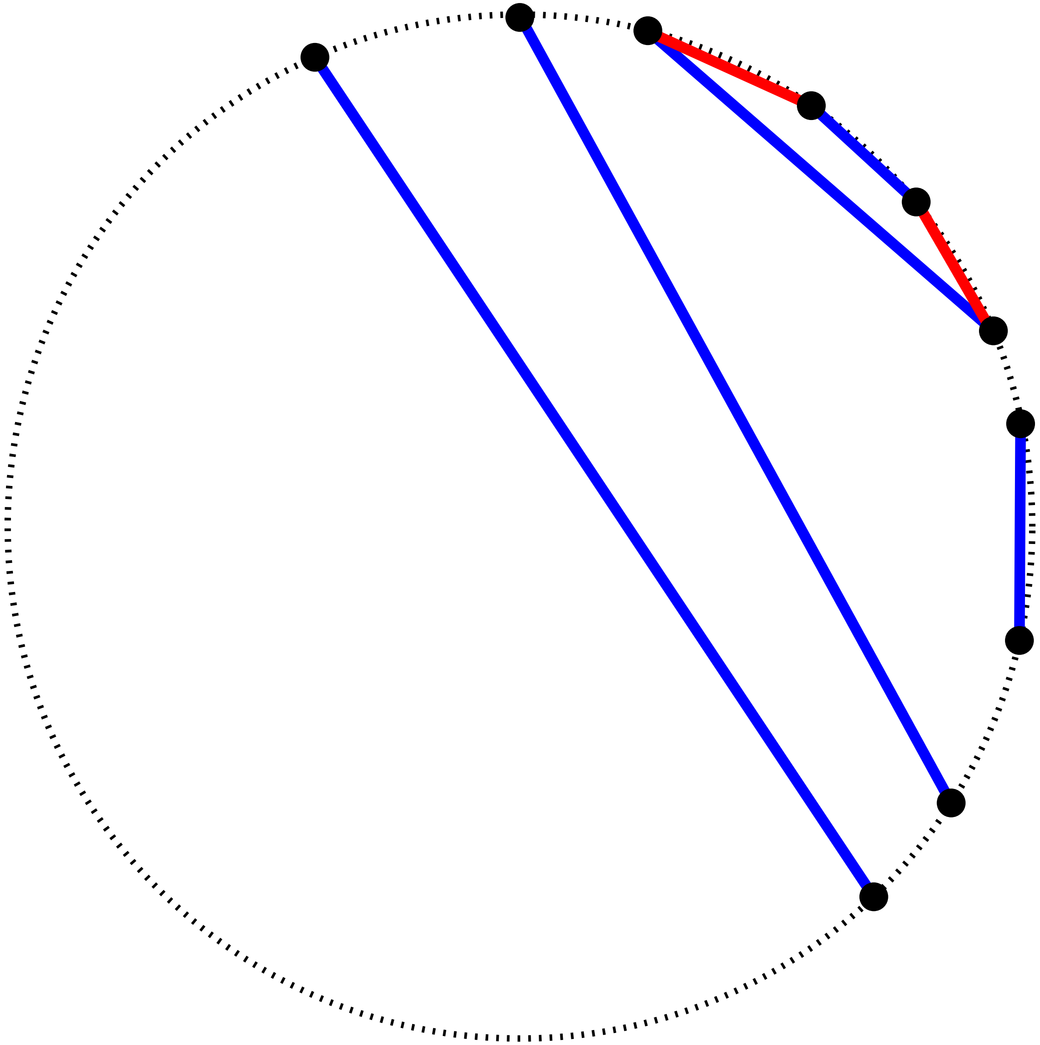

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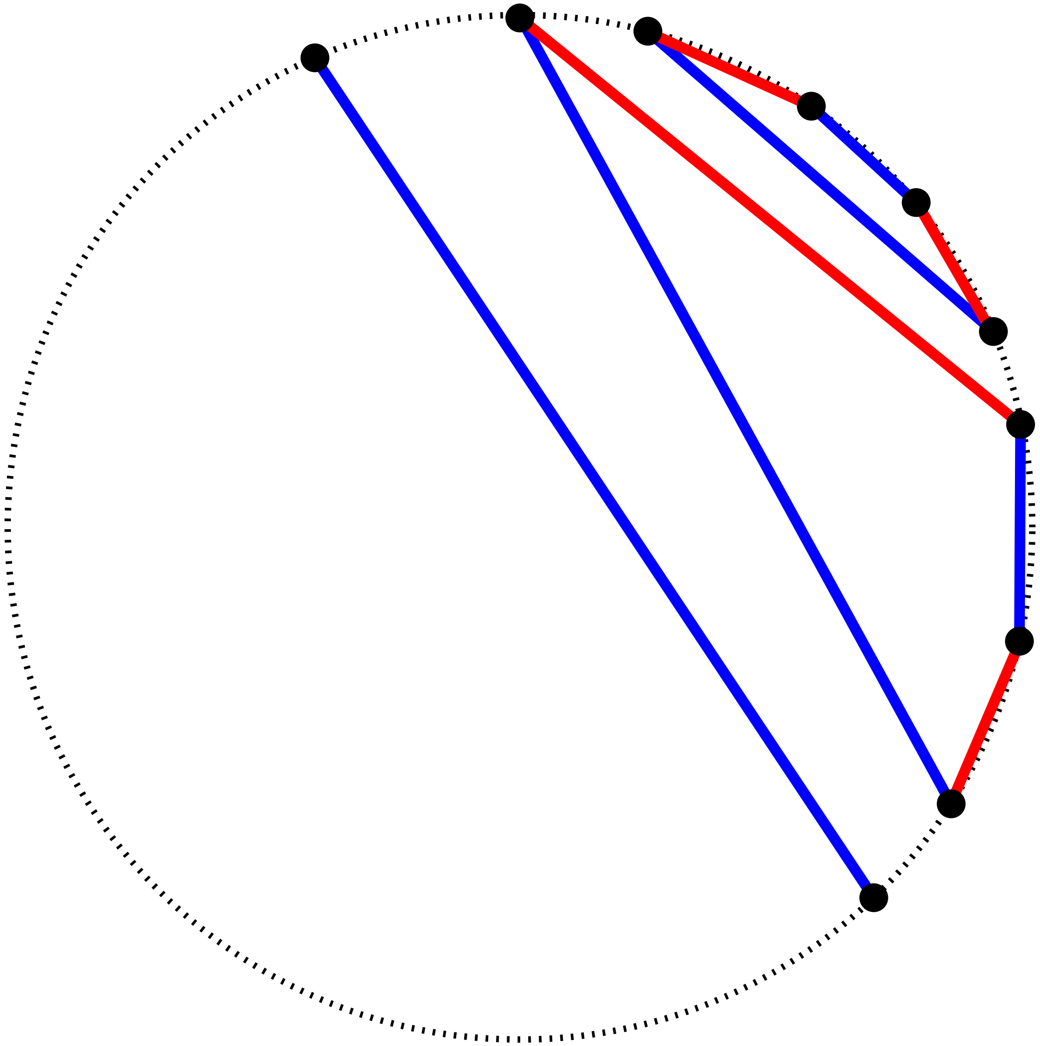

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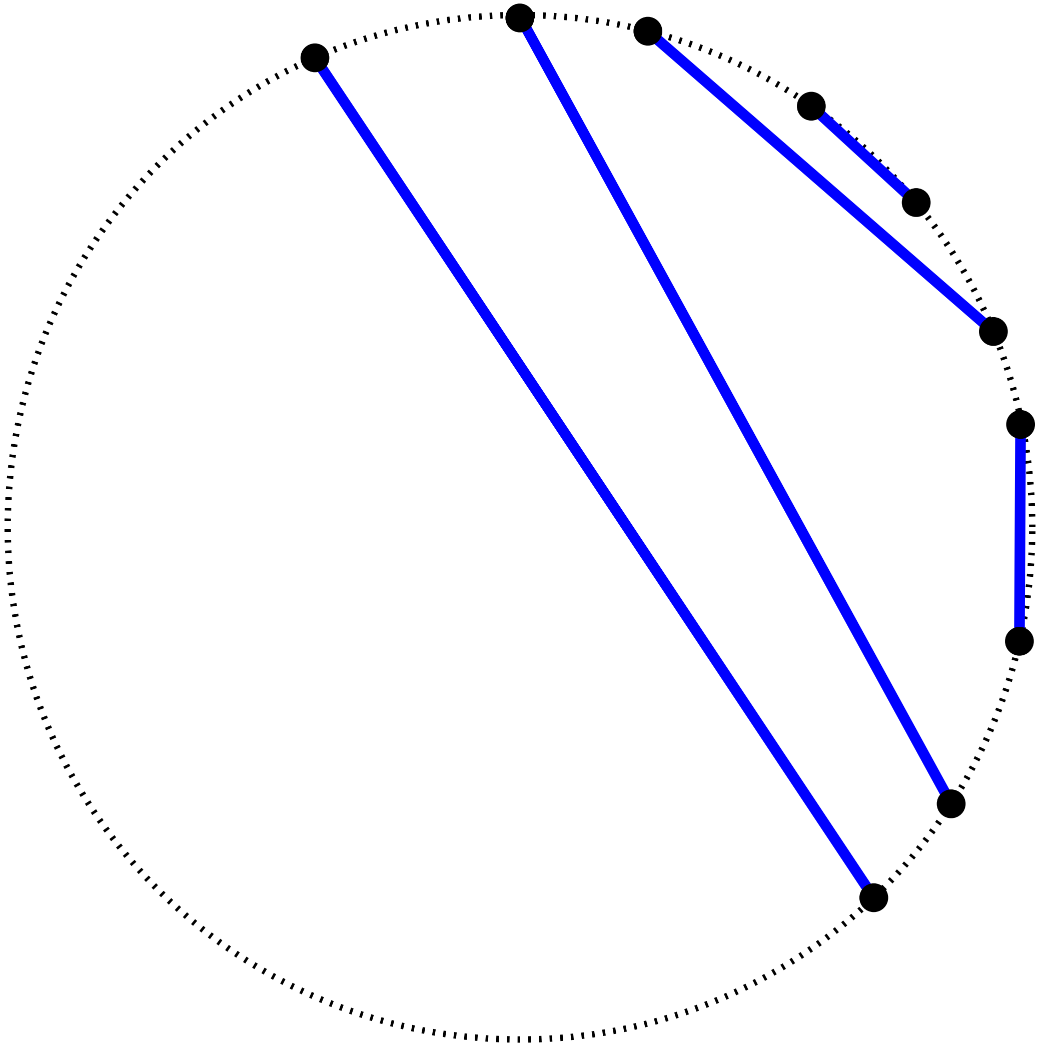

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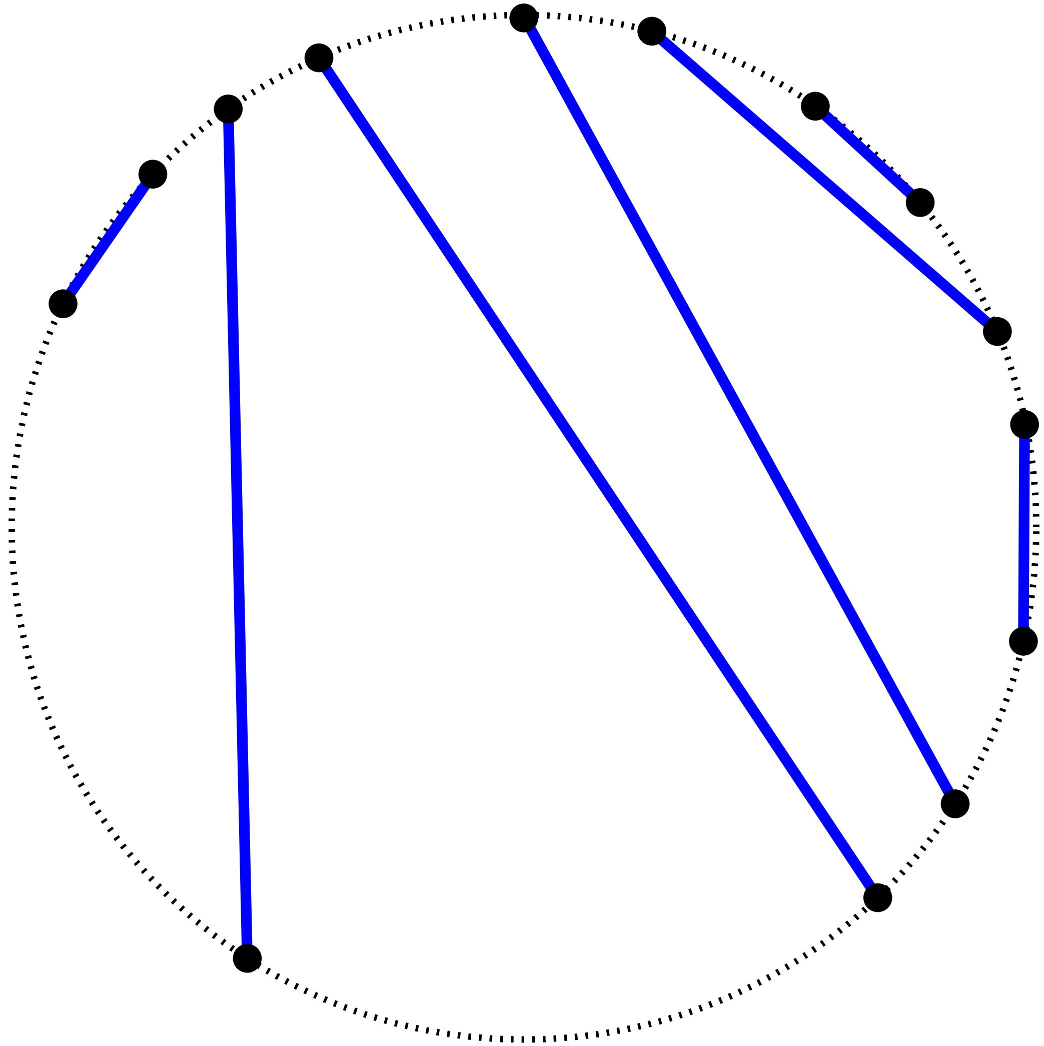

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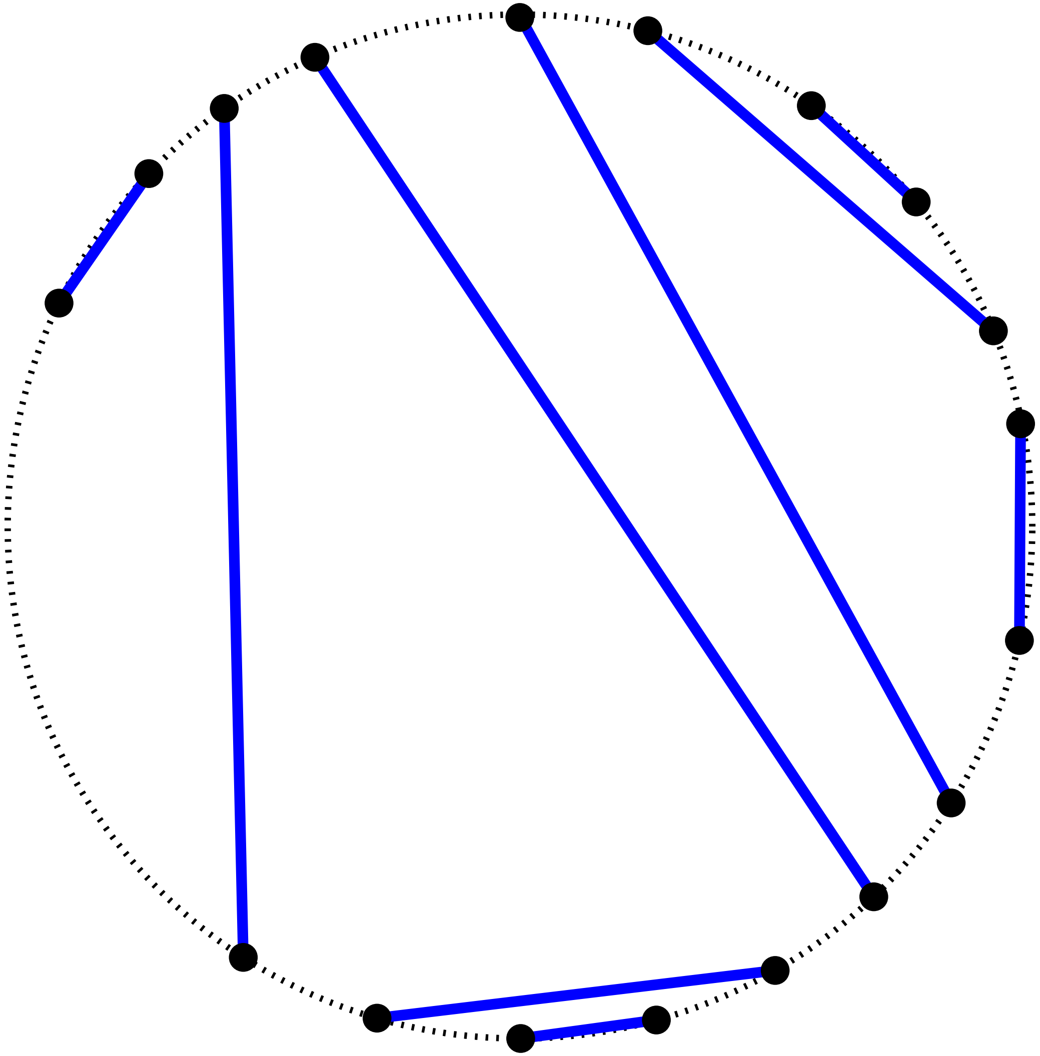

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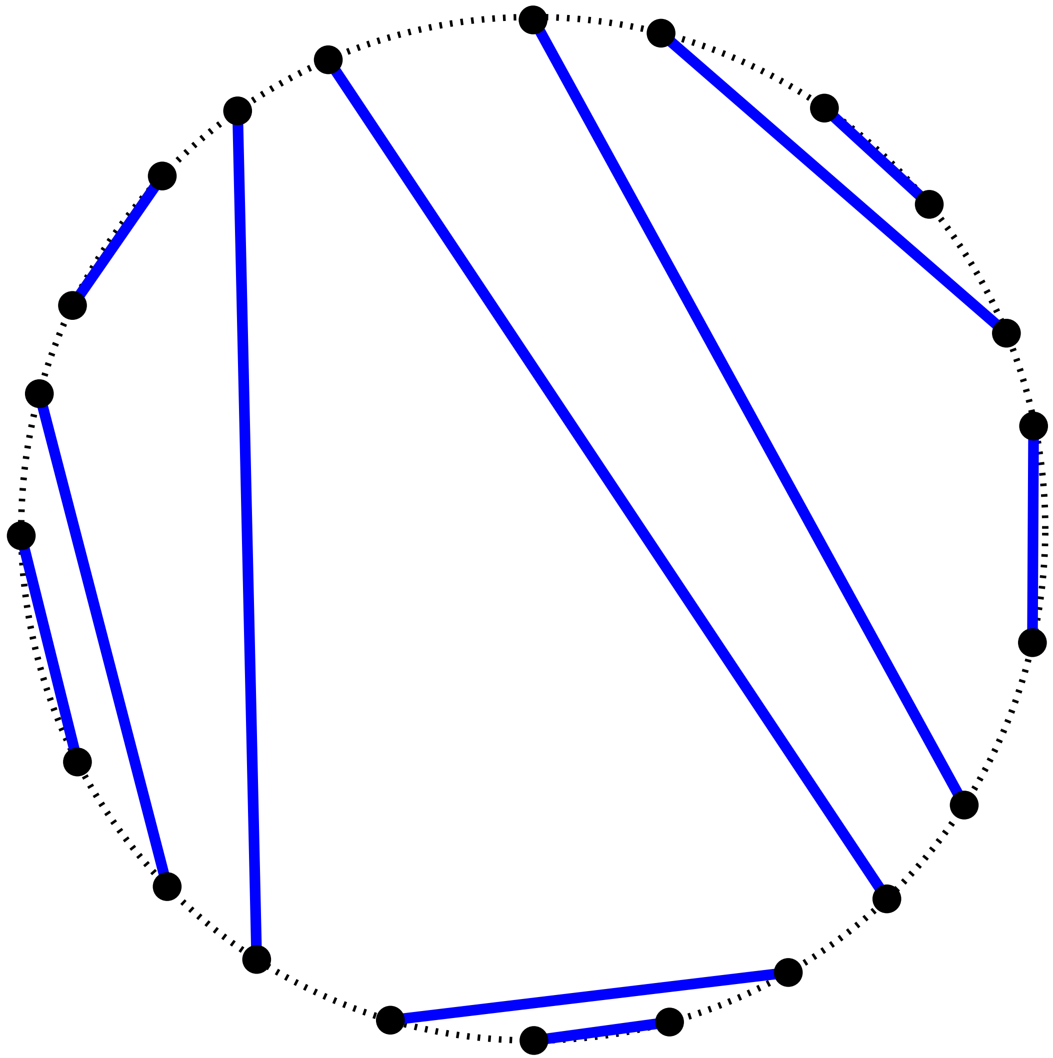

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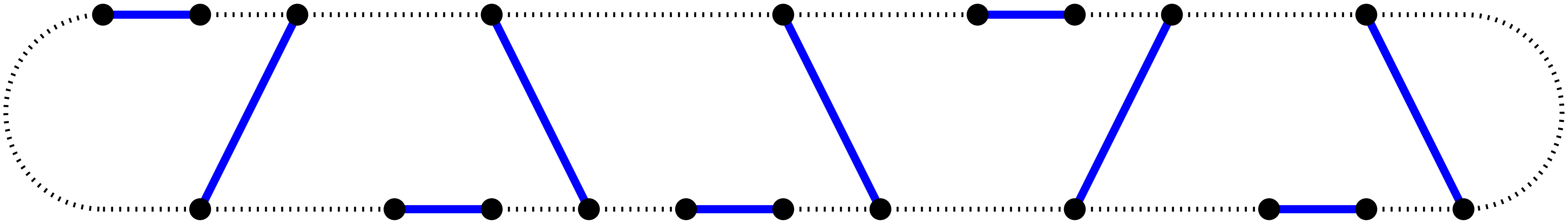

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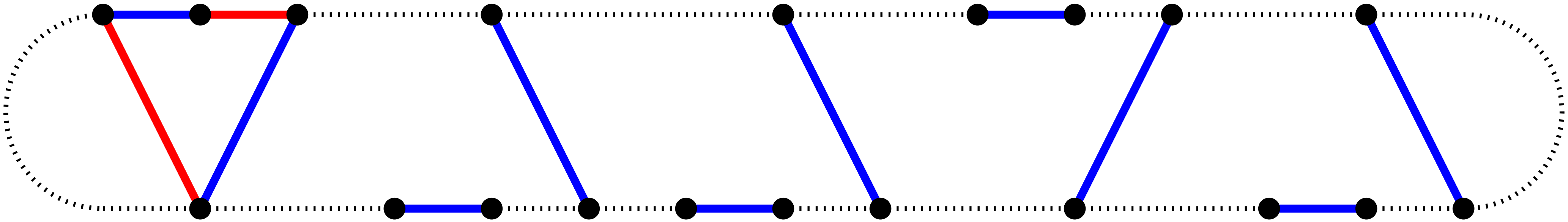

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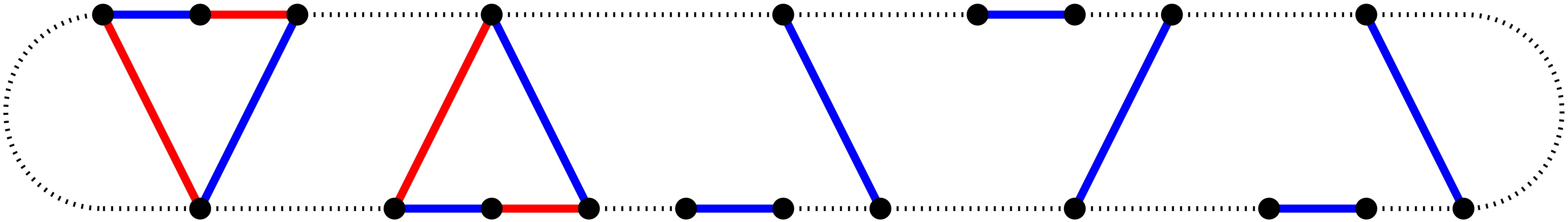

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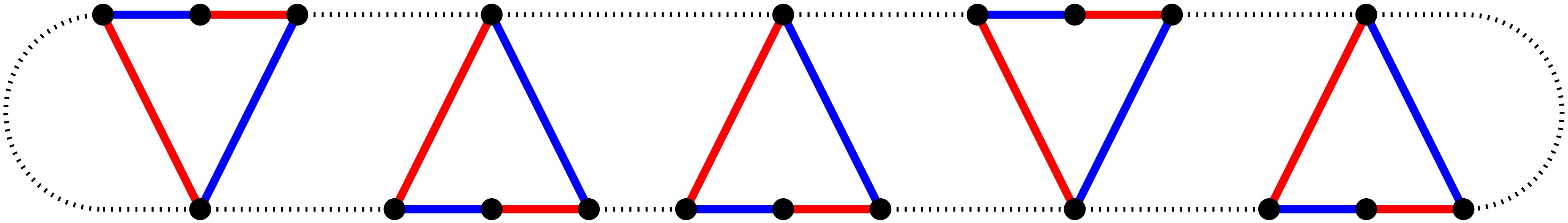

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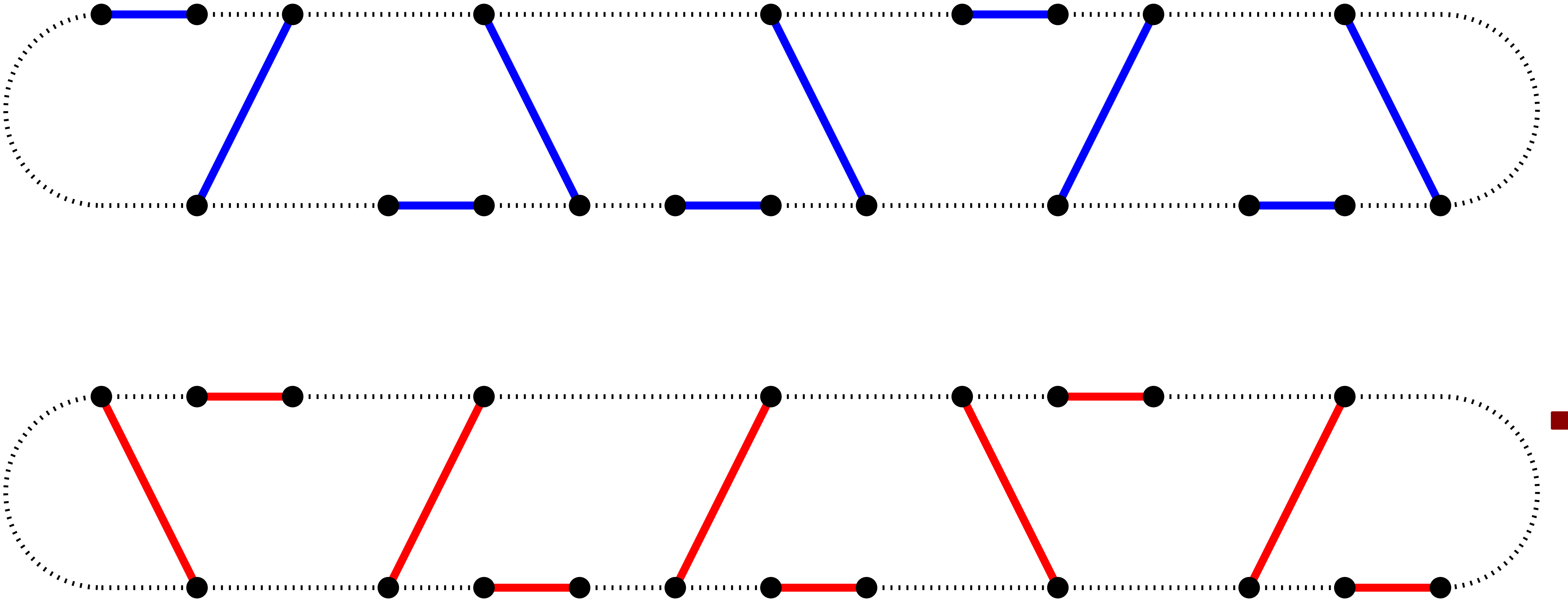
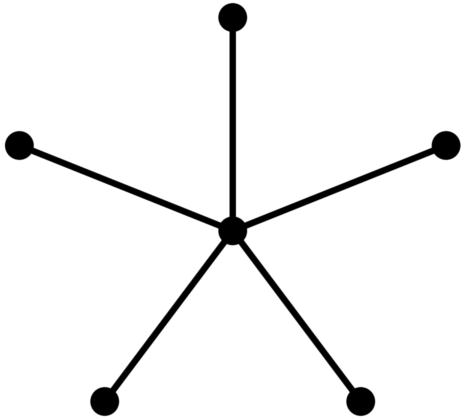


Medium components for odd k: stars with $\frac{k-1}{2}$ leaves.



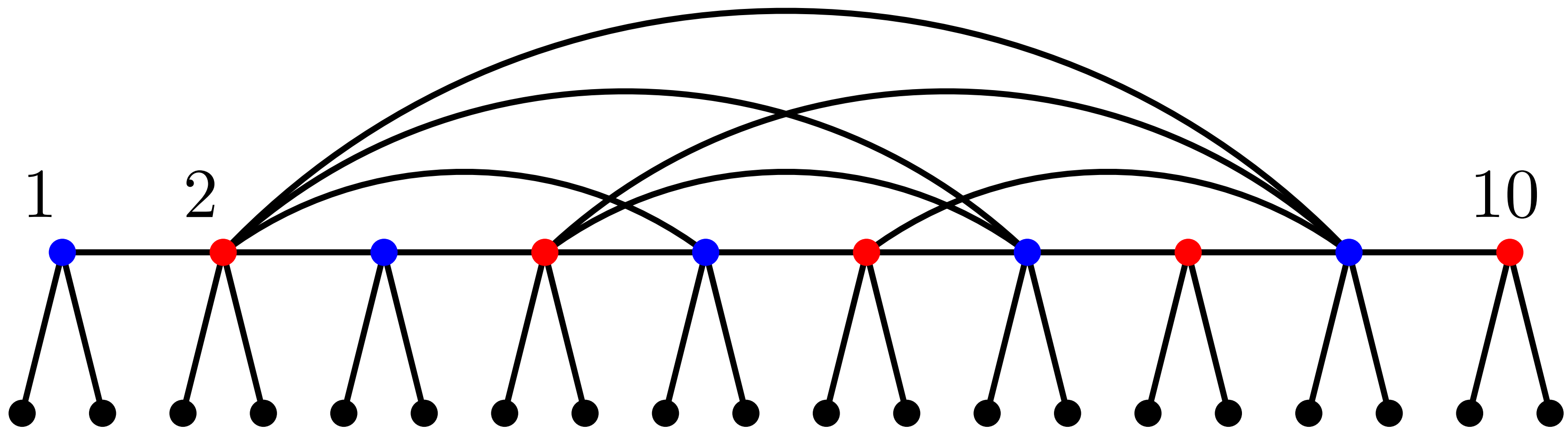
$k = 11$

Medium components for even k:

A path P with $(k-2)$ vertices (labeled $1, 2, \ldots, k-2$),

Each vertex of P has two adjacent leaves,

In addition, all the pairs (a, b), where a, b are on P, $a < b$, a even, b odd, are connected by an edge.



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A matching is *special* if it belongs to the types described above. Otherwise, a matching is *regular*.

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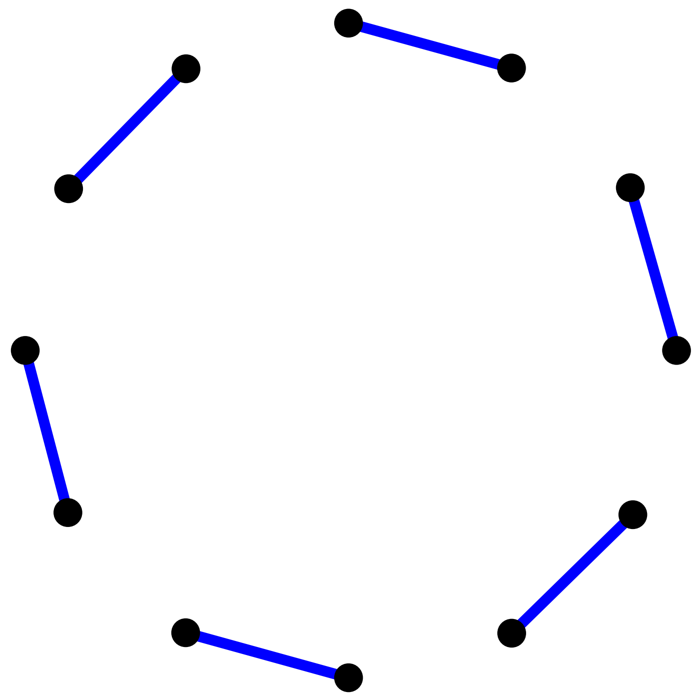

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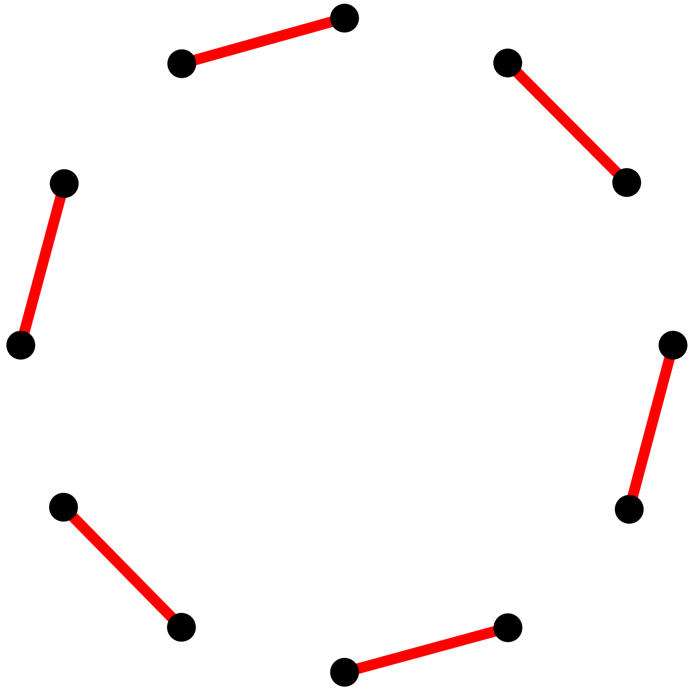

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Lemma. For $k \geq 9$, the component of DCM_k that contains the rings, is not bipartite.

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Proof. For $k = 9, 10$, this was verified directly.

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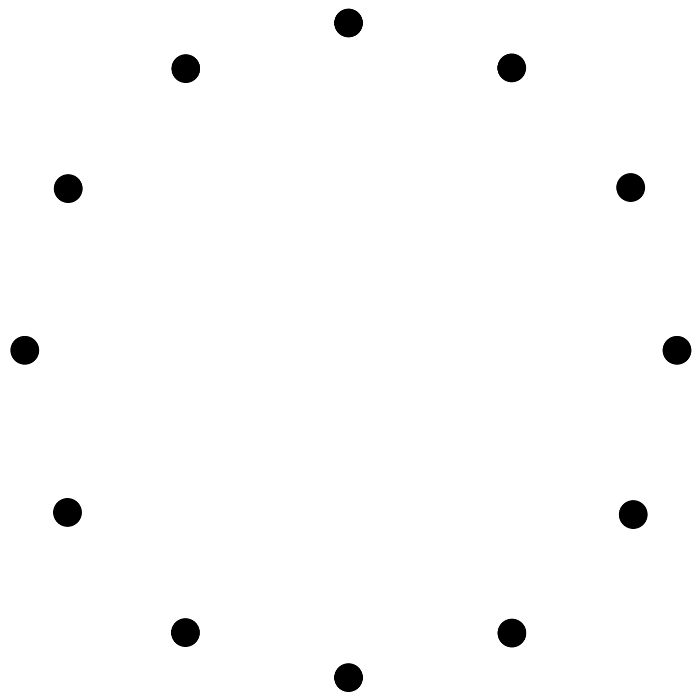

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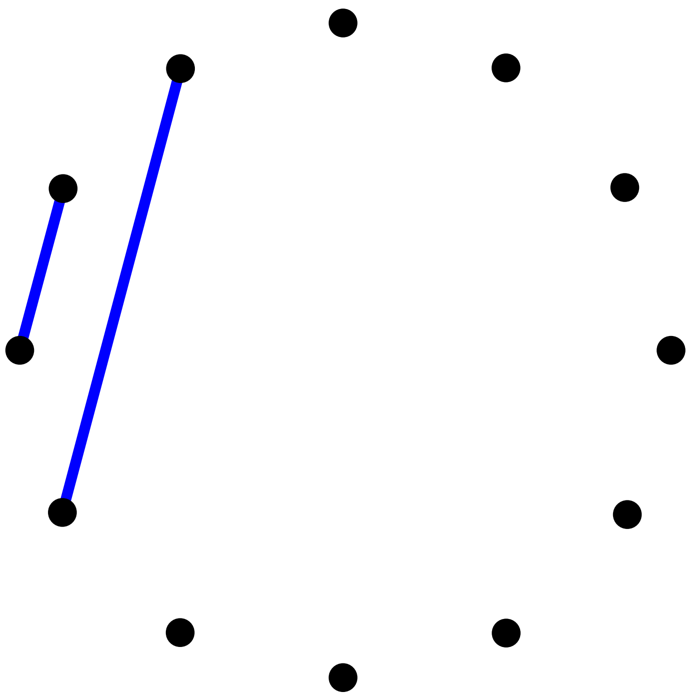
Find a block or an antiblock



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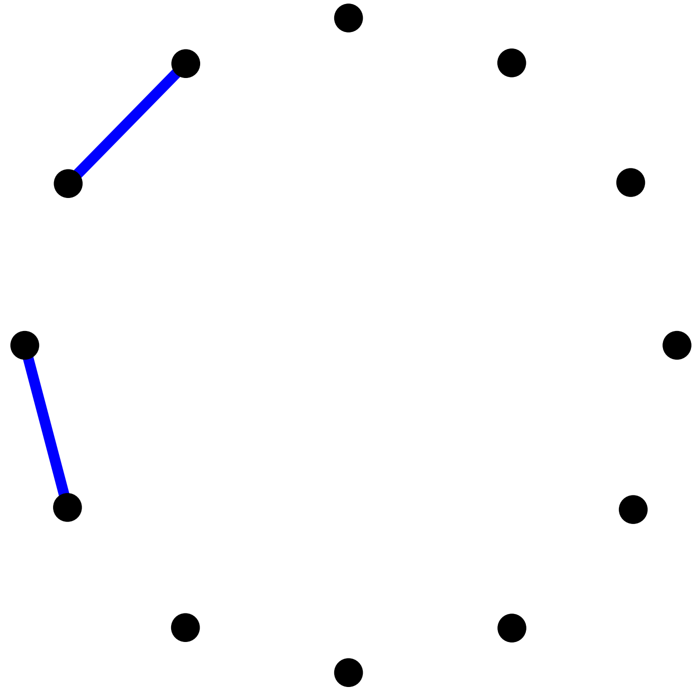

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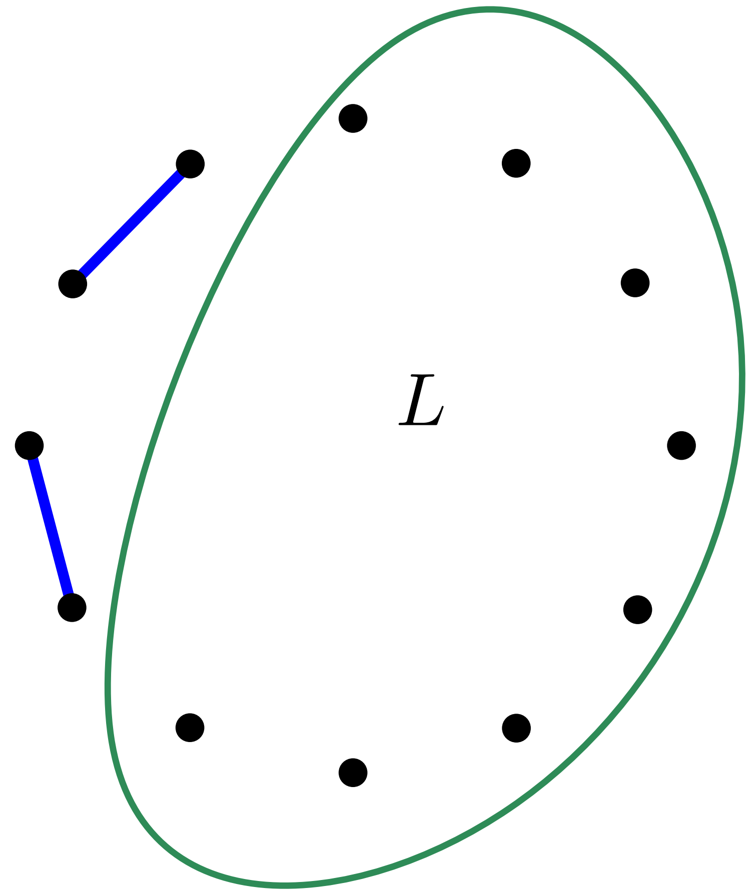
Find a block or an antiblock

and look at the remainig part (L).



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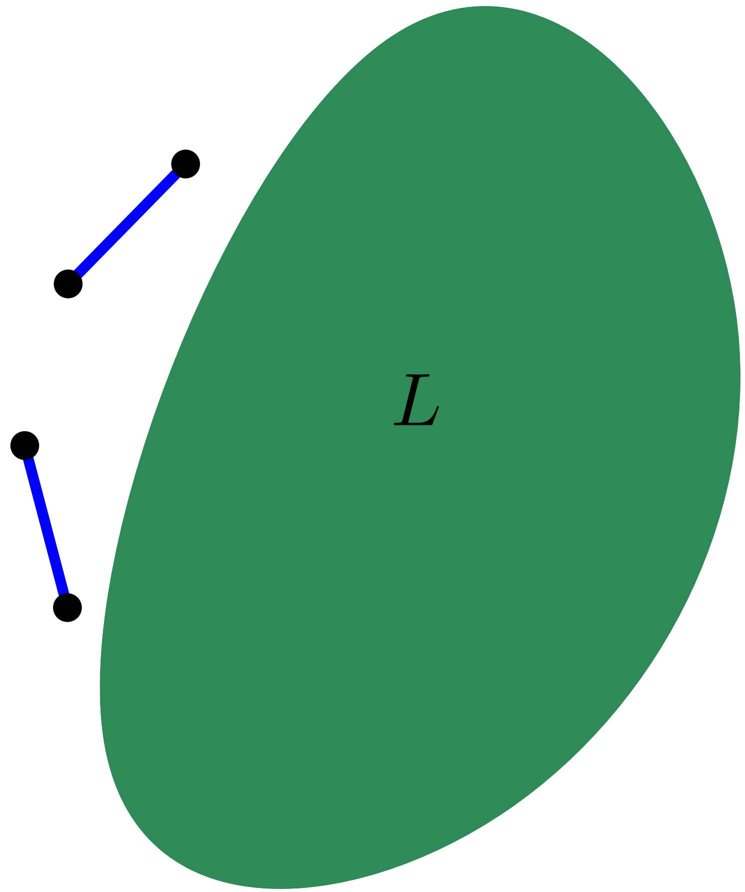
If L is regular, induction applies:
we transfer L to a ring,
while the (anti)block “oscillates”.



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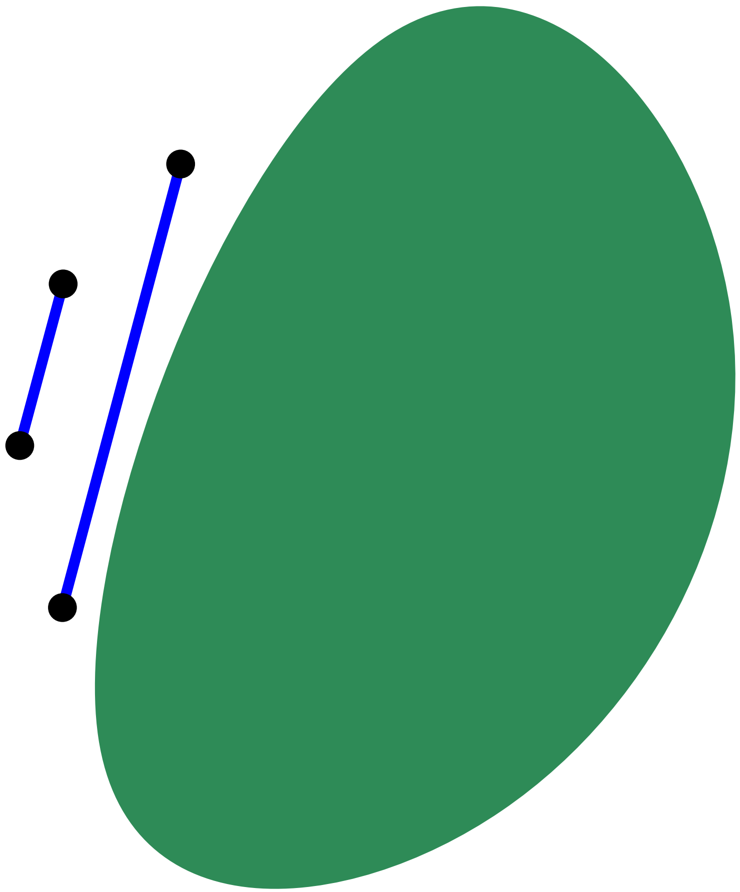

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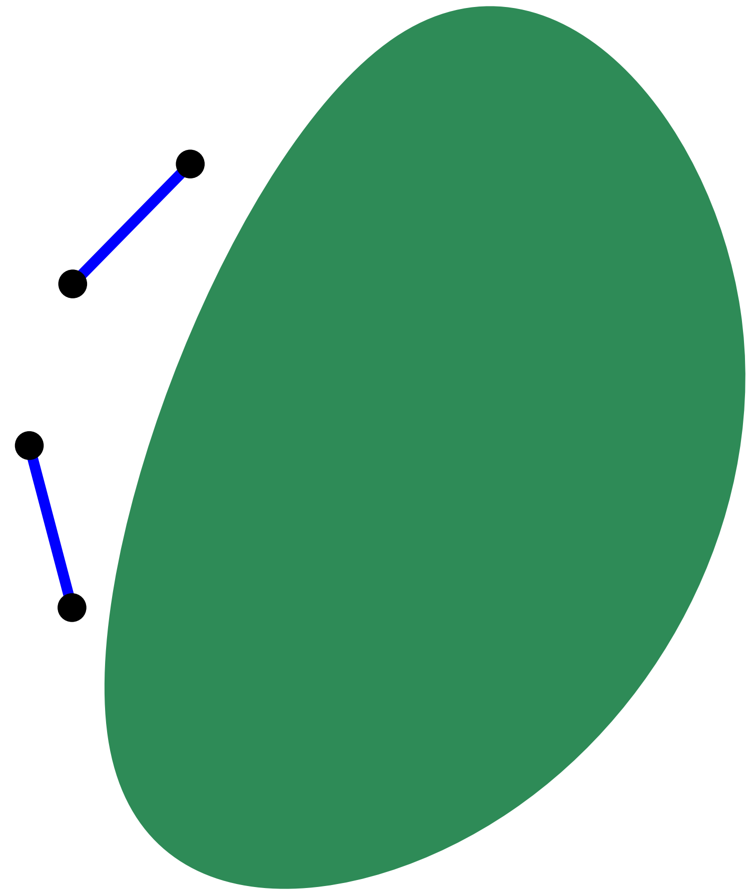

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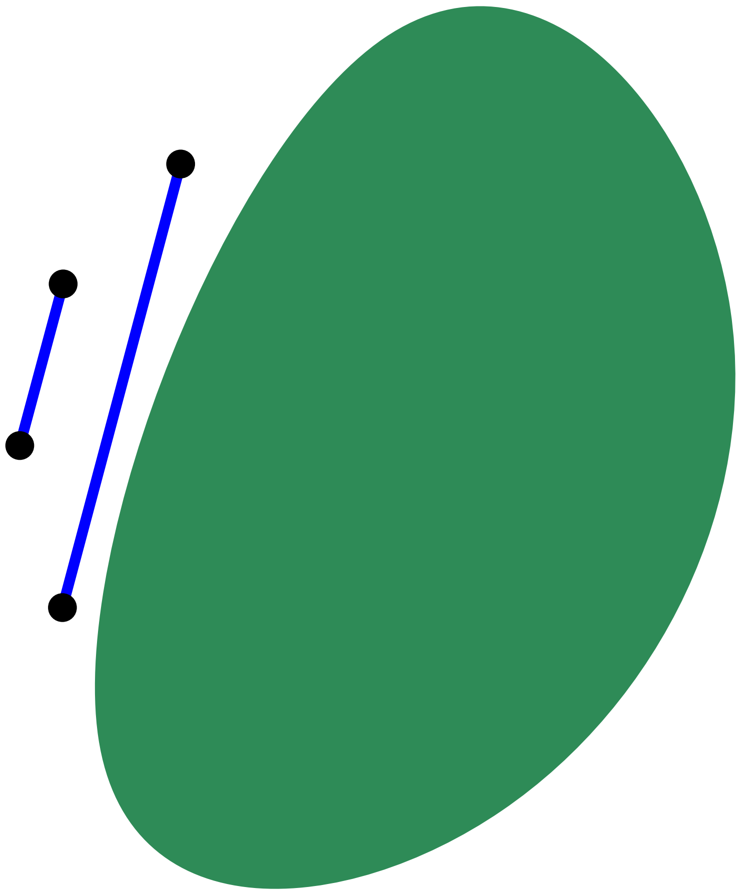

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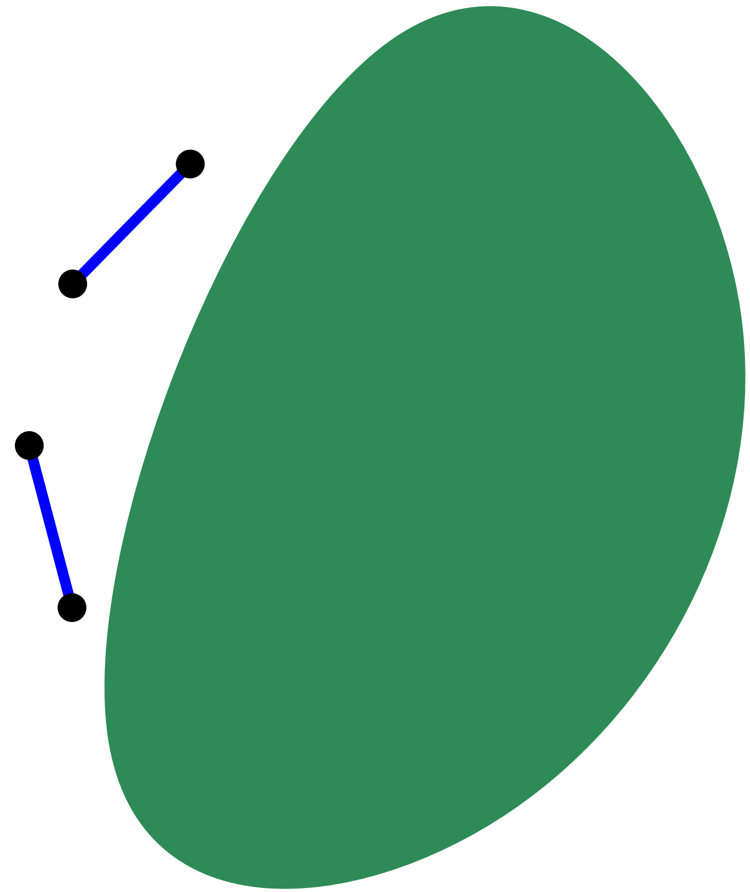

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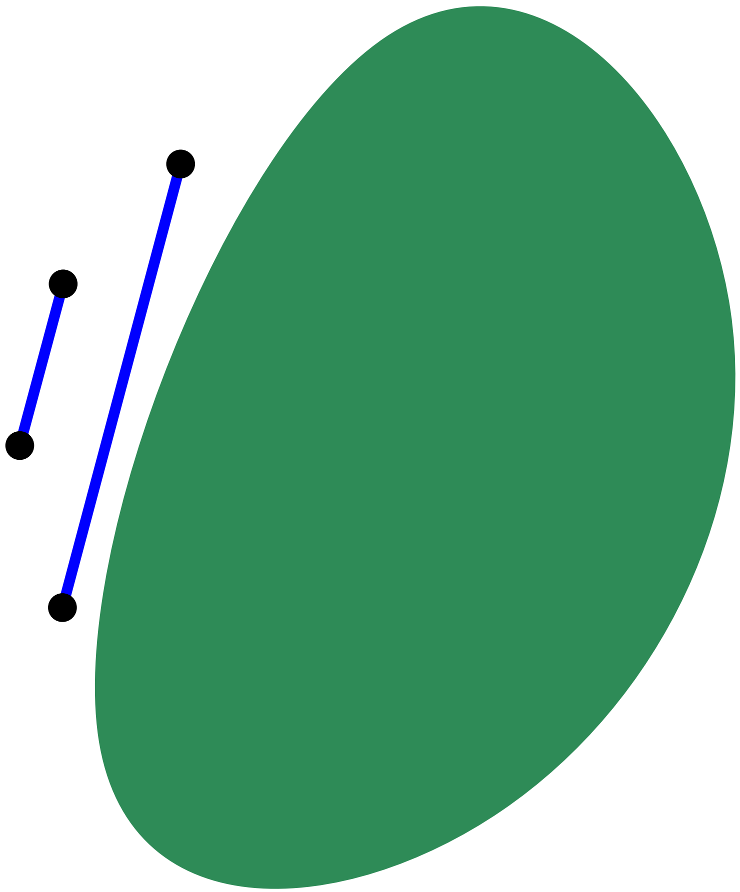

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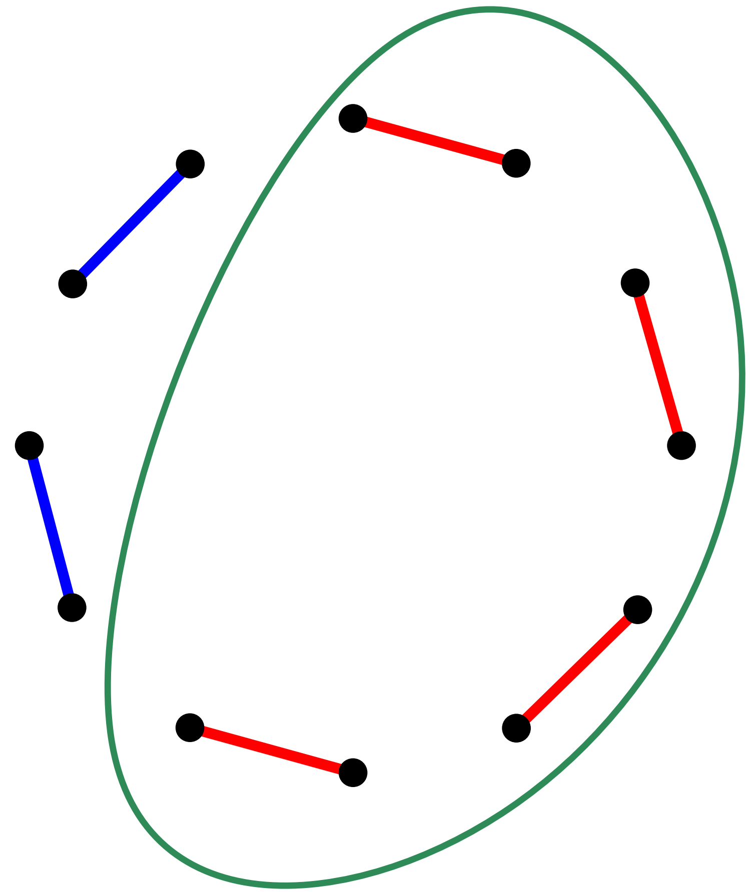

Find a block or an antiblock

and look at the remainig part (L).

If L is regular, induction applies:
we transfer L to a ring,
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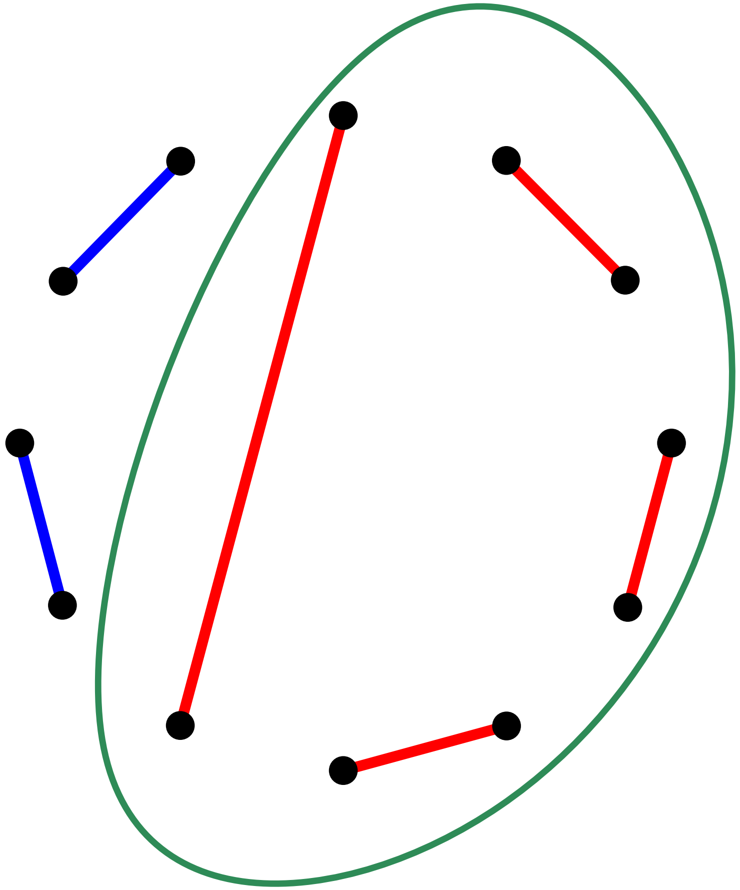

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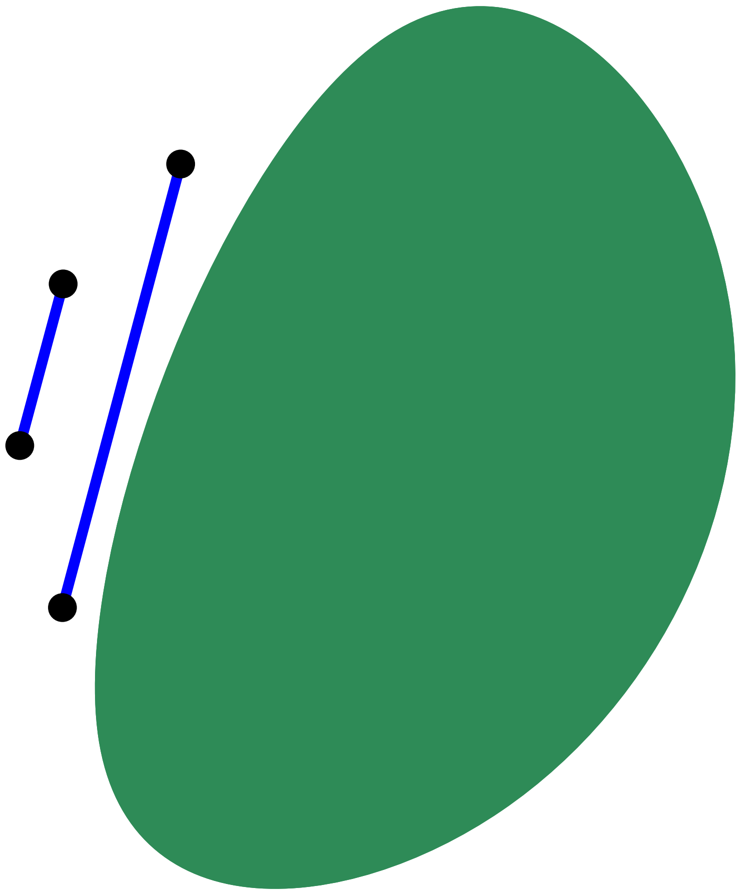

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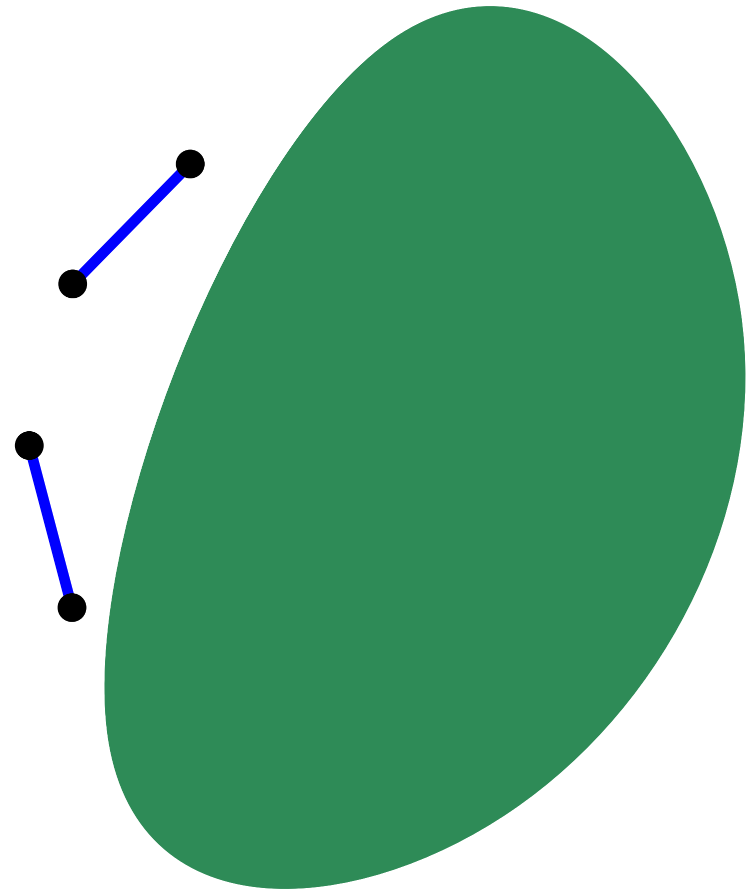

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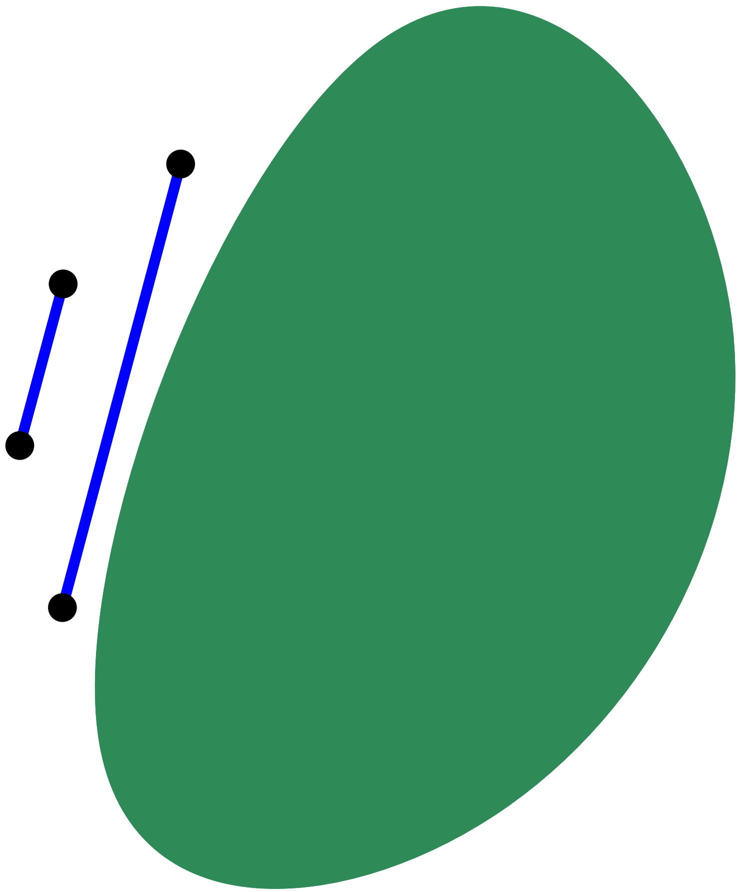

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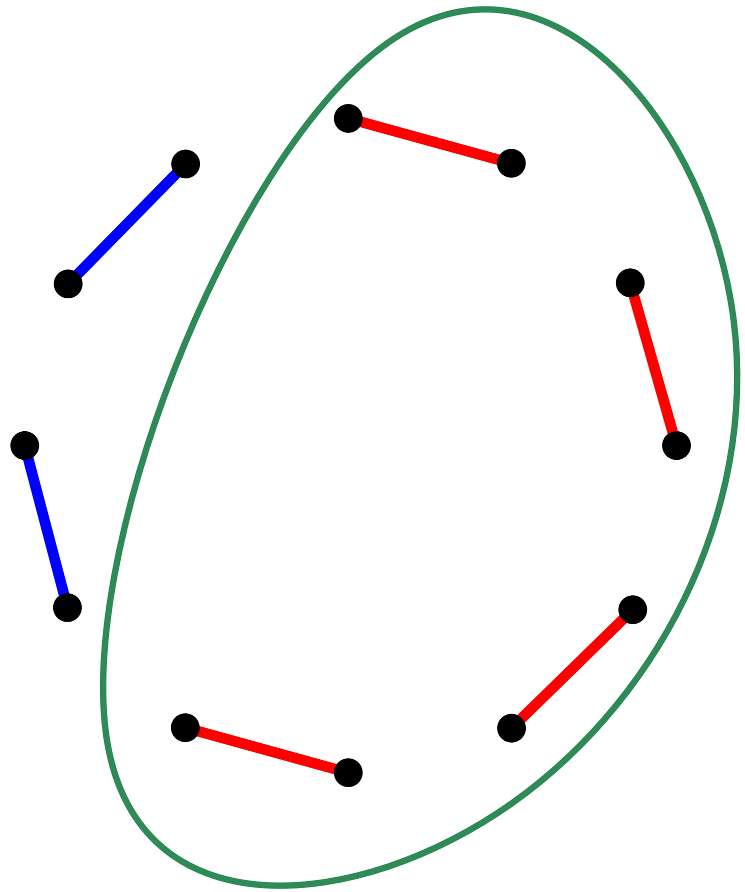

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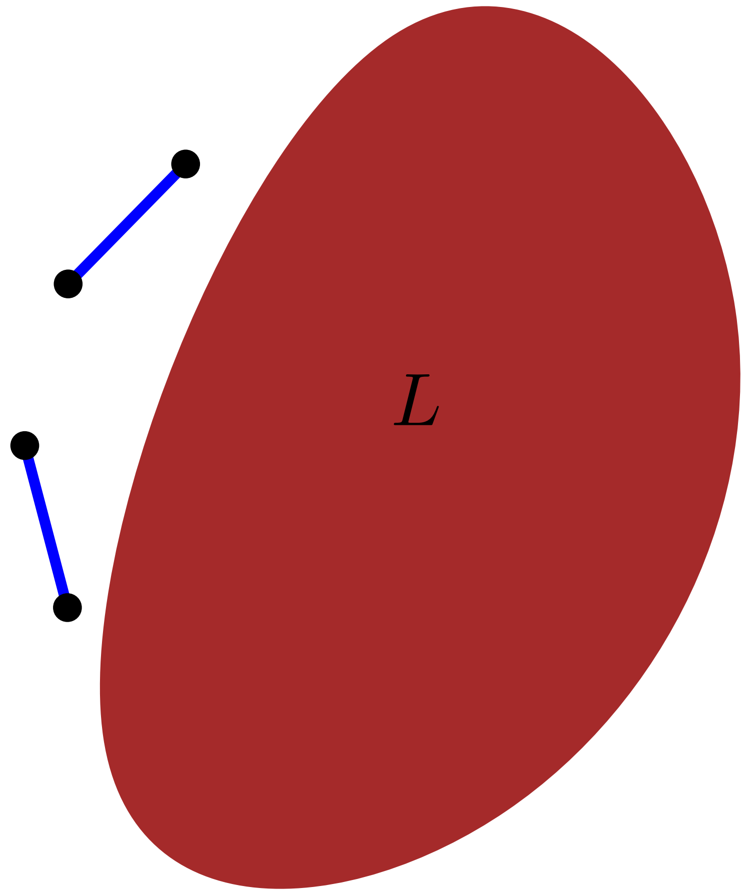

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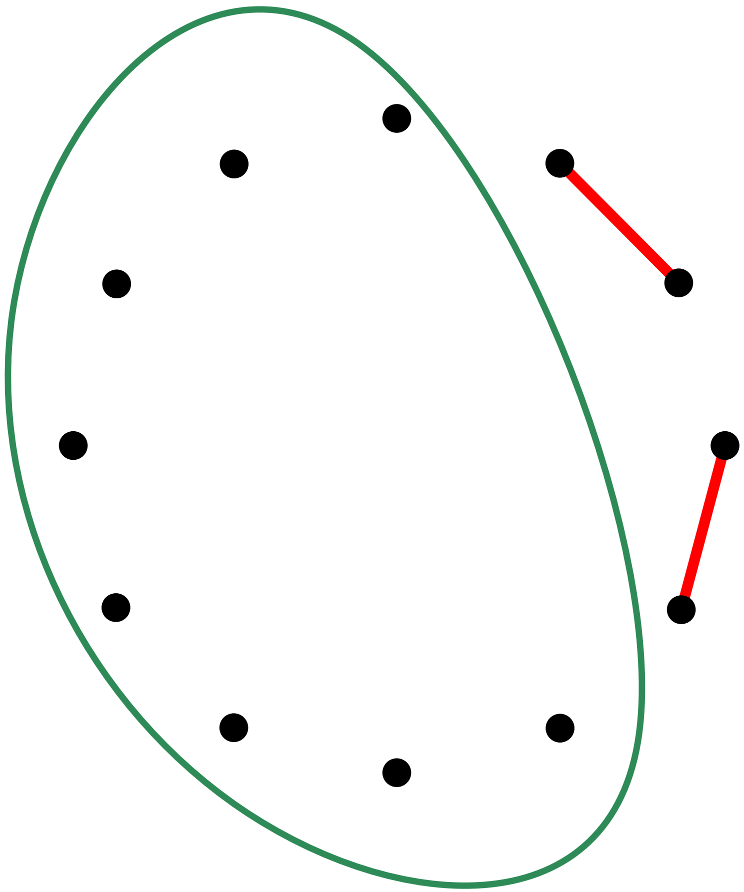

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An open problem:

Characterize isolated matchings for point sets in **general** position (for odd k).

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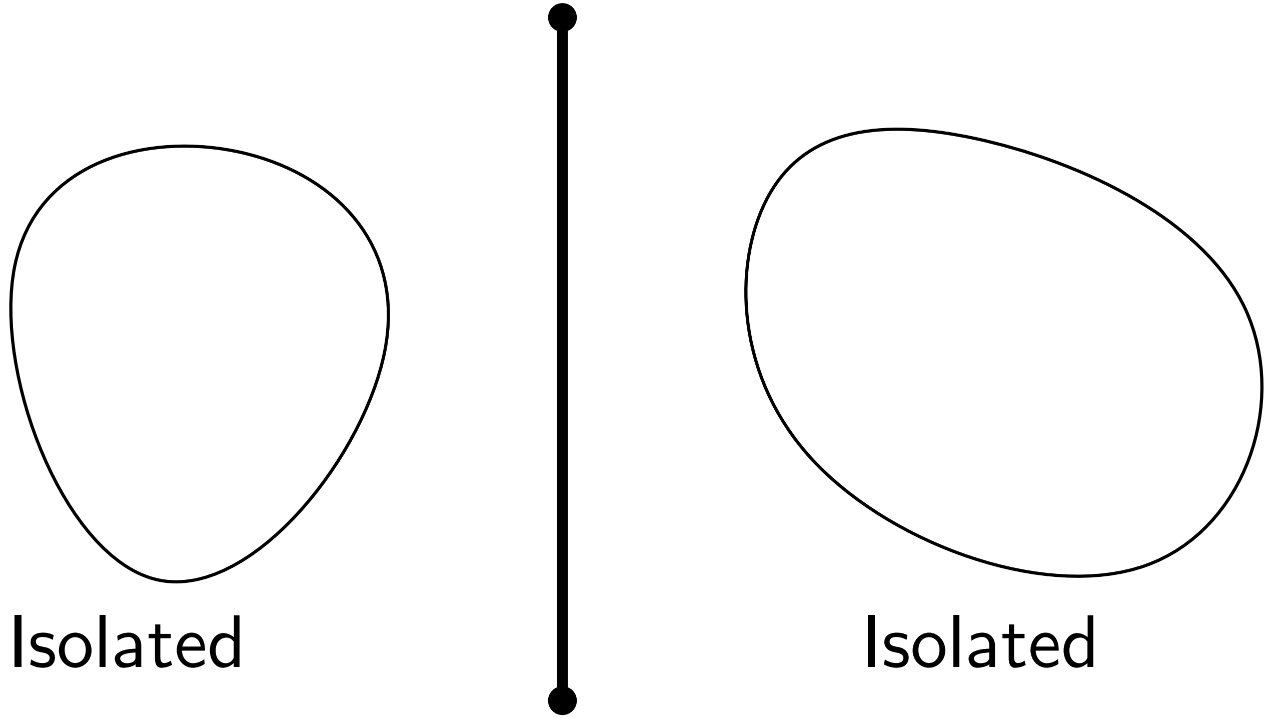
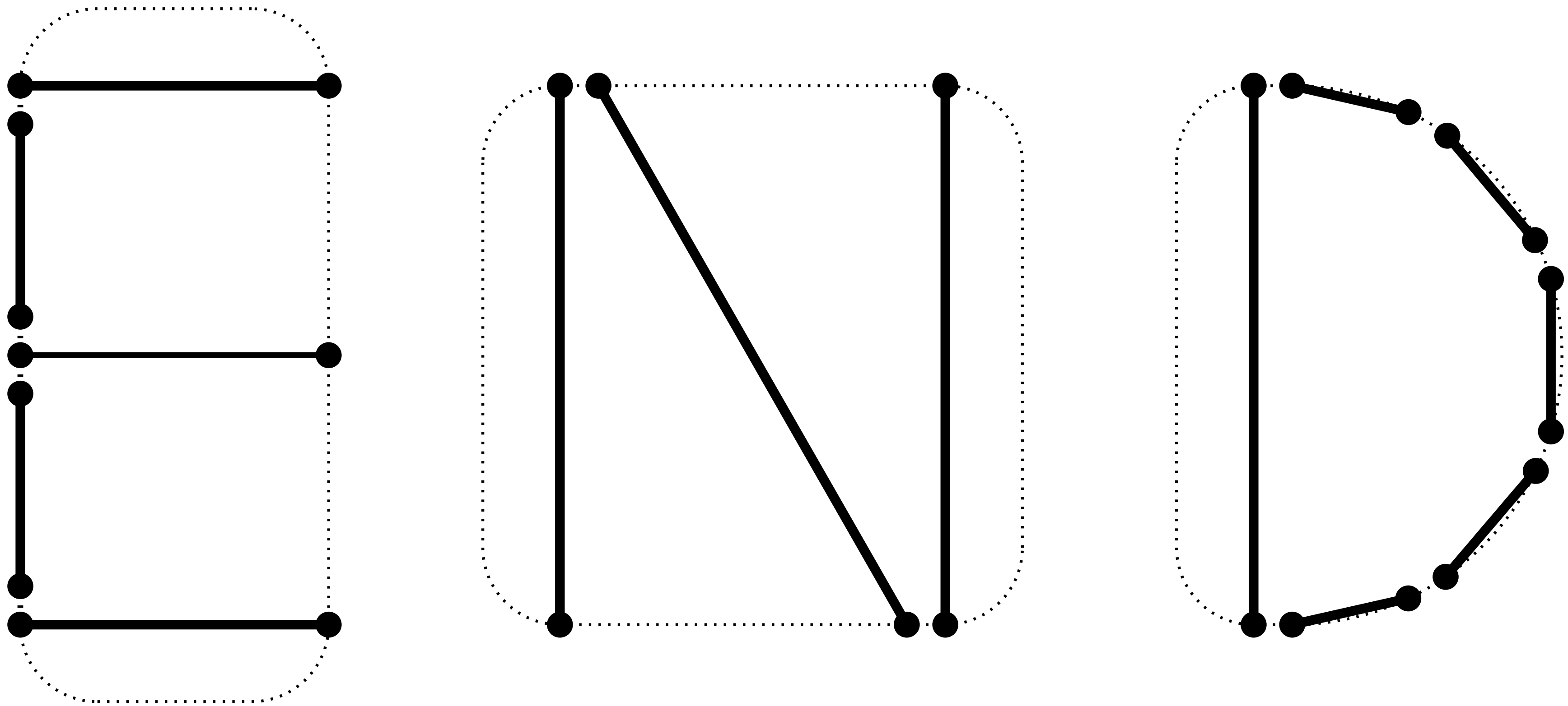
A sufficient (but not necessary) condition:

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