Straight Skeletons in 3-space

Franz Aurenhammer **Gernot Walzl**

Institute for Theoretical Computer Science
Graz University of Technology

2014-02-20

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Straight skeletons in the plane

Straight Skeletons of Simple Polygons

were introduced by Aichholzer, Aurenhammer, Alberts, Gärtner in 1995. [AAAG95]

- Defined by *shrinking process*
- Each edge moves inwards at the same speed

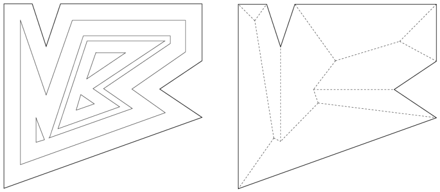

Figure: Straight skeleton [AAAG95]

Events

- edge event
- split event

Complexity

- n faces
- $n-2$ nodes
- $2n-3$ arcs

Notice

- Straight skeleton is unique for a given polygon
- Its structure may be interpreted as plane tree
- *Bisector graph* is **not** unique

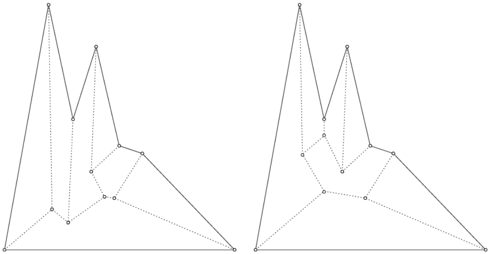

Figure: Bisector graph [AAAG95]

Construction

- Straight skeleton is interpreted as 2-dim. projection of a 3-dim. roof model
- A horizontal plane Π moves upwards
- Events are stored in a priority queue
- Priority reflects the height of Π

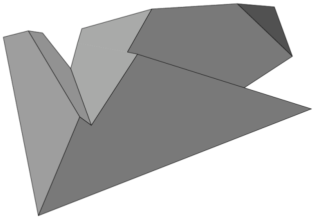

Figure: Roof model [AAAG95]

Complexity bounds

- The straight skeleton of a convex polygon is equal to its medial axis. It can be constructed in $\Theta(n)$ time. [AGSS87]
- Lower bound for polygons with holes: $\Omega(n \log n)$

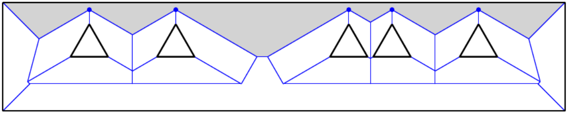

Figure: Lower bound for polygons with holes [Hub11]

Algorithm	Time	Space
[AAAG95]	$O(nr \log n)$	$O(n)$
[AA98]	$O(n^3 \log n)$ (pract. $O(n \log n)$)	$O(n)$
[VY13]	$O(n^{4/3+\varepsilon})$	$O(n)$

Straight skeletons in 3-space

Straight Skeletons of 3-dim. Polyhedra

- Defined by a *shrinking process*
- Each face moves inwards at the same speed

First work was done by Barequet, Eppstein, Goodrich and Vaxman in 2008. [BEGV08]

Contents of their work

- Orthogonal polyhedra formed by unions of cubical voxels
- Polyhedra with axis-parallel edges and faces
- Ambiguities of the straight skeleton for non-axis-aligned polyhedra

Skeleton computation of orthogonal polyhedra

A robust implementation was done by Martinez et al. [MVPG11]

Contents

- Compute Voronoi diagram with L_∞ distance
- Medial axis with L_∞ distance of orthogonal polyhedra is equal to the straight skeleton

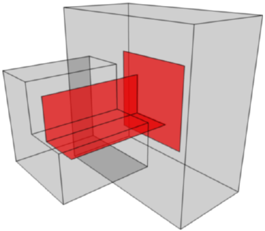

Figure: Example of their method [MVPG11]

We are interested in straight skeletons of general polyhedra

Complexity bounds of straight skeletons

Description	Bound	Reference
convex polyhedra	$\Omega(n^2)$ $O(n^2)$	well known (medial axis) well known (medial axis)
general polyhedra	$\Omega(n^2\alpha^2(n))$ $O(n^4)$	[BEGV08] trivial

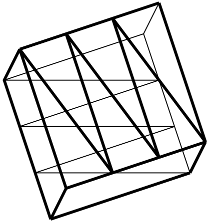

Figure: Lower bound for convex polyhedra

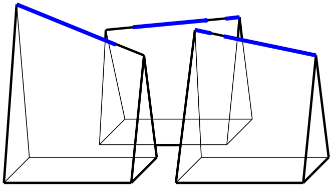

Figure: Lower bound for general polyhedra (set of prisms)

Our work

Straight skeletons for general polyhedra

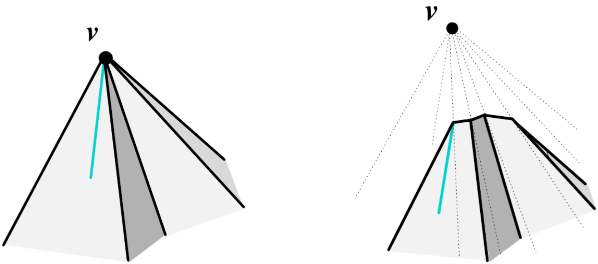

Figure: Example of an initial offset of vertex v (deg. 6)

Initial offset

- At the very first moment, each vertex (deg. ≥ 4) gets decomposed into vertices of degree 3.
- If we know how to split these vertices, we know how to handle all occurring events.

Splitting convex vertices

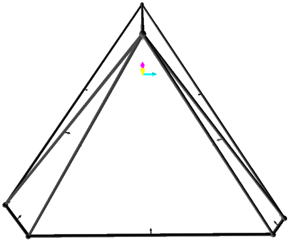

Figure: Convex vertex (deg. 5)

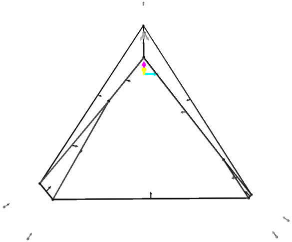

Figure: Split vertex

- For each incident face of the given vertex:
  - Calculate the offset plane of adjacent faces (3 planes)
  - Determine the distance between the intersection point of the offset planes and the vertex
- Split the vertex using the farthest distance

Combinatorial vertex splitter

Vertices of degree 3 span an unrooted binary tree.

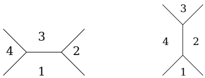

Figure: Combinations for deg. 4

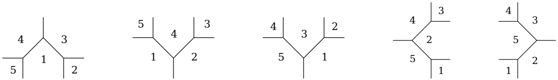

Figure: Combinations for deg. 5

Number of possible unrooted binary trees

Catalan number

$$C_n := \frac{1}{n+1}\binom{2n}{n} = O(c^n), c > 1$$

Check for self-intersections

- All facets need to be self-intersection free.
- No edges are allowed to intersect any other facet.

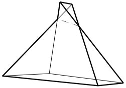

Figure: Invalid surface: self-intersecting facets

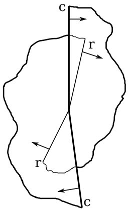

Figure: 2 tilted facets

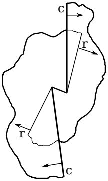

Figure: Valid solution

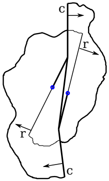

Figure: Self-intersecting polyhedron

Saddle points

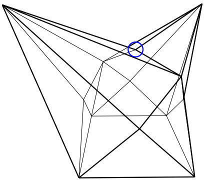

Figure: Front view

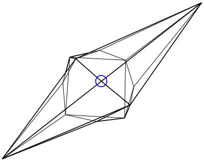

Figure: Top view

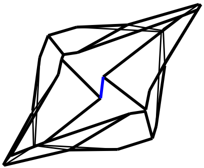

Figure: Convex solution

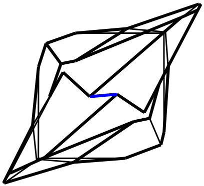

Figure: Reflex solution

- This simple example shows that offsetting a polyhedron is not unique.

Existence of a solution

Proof

- Consider any unrooted binary tree to split the vertex
- There will be parts that grow (increase the volume) and parts that shrink
- There is always a path that can be used to cut the growing parts
- This path is where the inside "wraps" to the outside

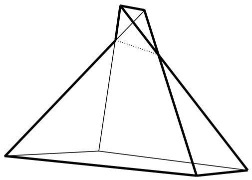

Figure: Path to cut

Implementation

- Implemented in C++ (approx. 35k LOC)
- Our geometry kernel has double precision only
  CGAL's kernel[1] can optionally be linked
- Resulting skeleton is stored using a relational database (SQLite)
- Interactive animation is done using OpenGL
- Software rendering for PostScript output
- Automated testing

- Live demo

[1] http://www.cgal.org/

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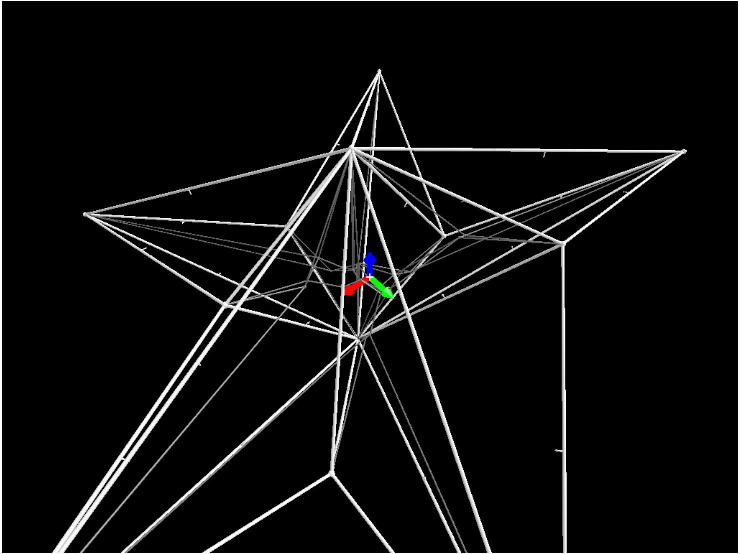
Live demo



Conclusion

- The offset of a polyhedron is not unique
- There is always a solution
- The vertex splitter shows how to handle occurring events
- Our work solved open problems [BEGV08]
- Implementation shows the results

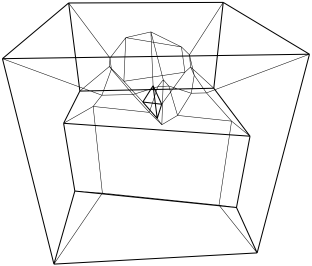


Figure: Wedge on tabletop

Future work

Mesh generation based on *straight skeletons*

Idea is based on
"Skeleton-based Modeling Operations on Solids" [STG+97]

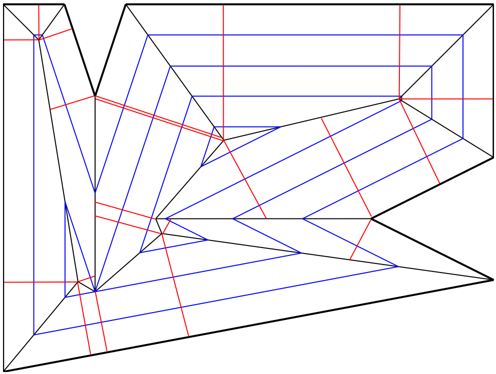


Figure: Mesh generation

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