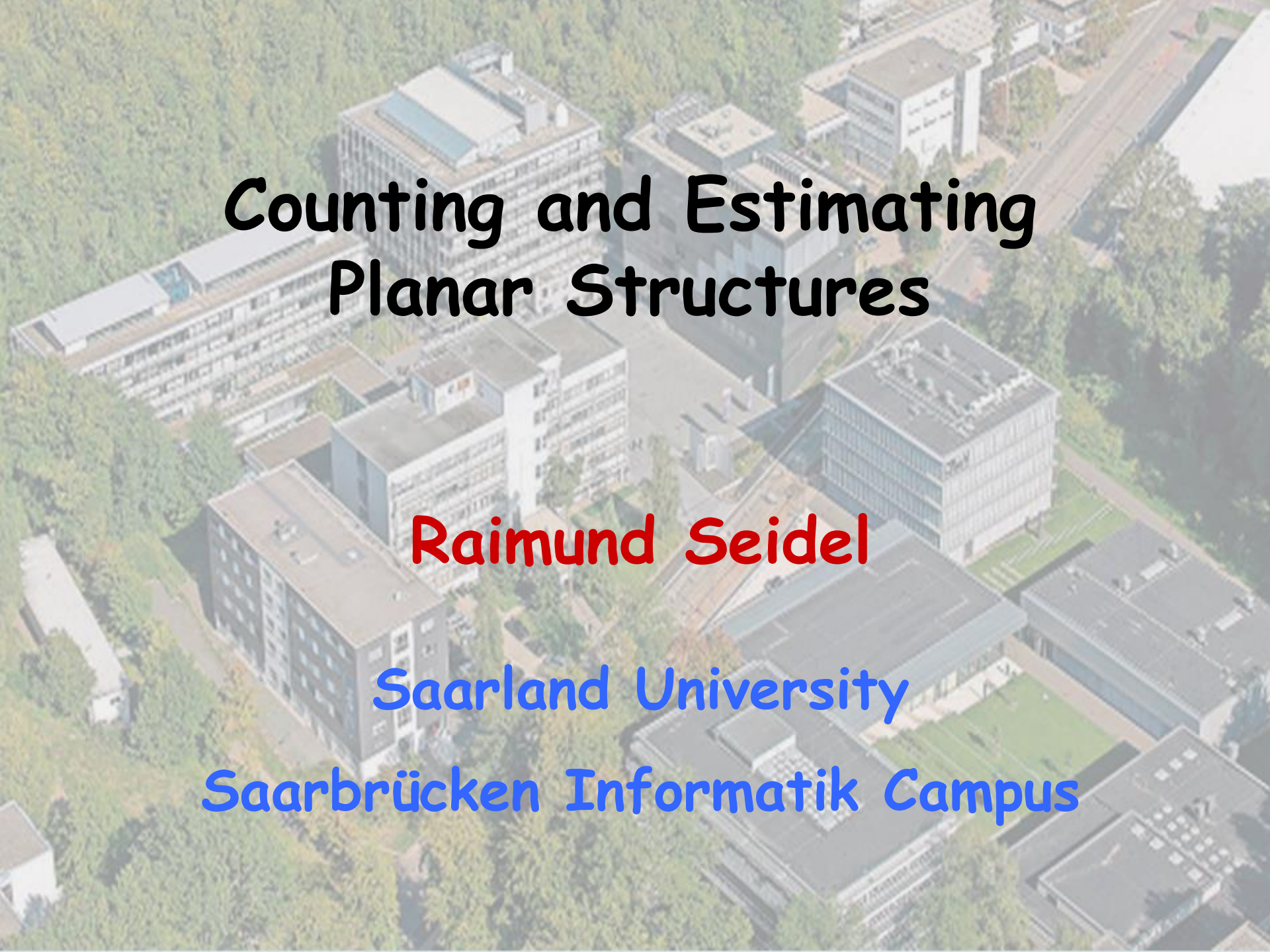
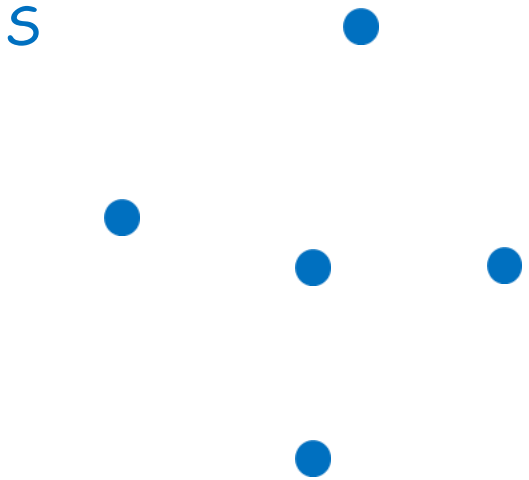


Counting and Estimating Planar Structures

Raimund Seidel

Saarland University

Saarbrücken Informatik Campus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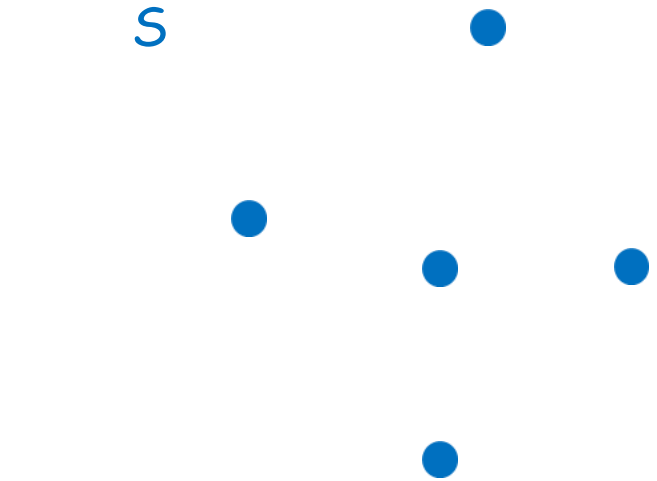
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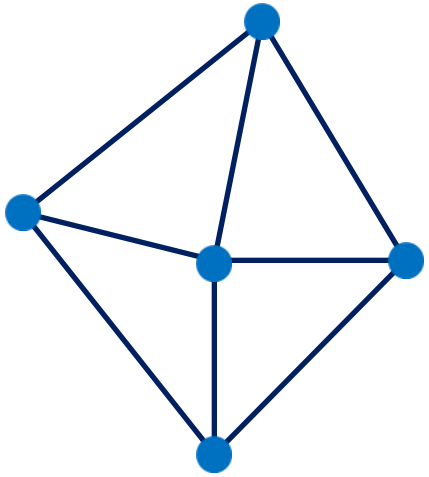

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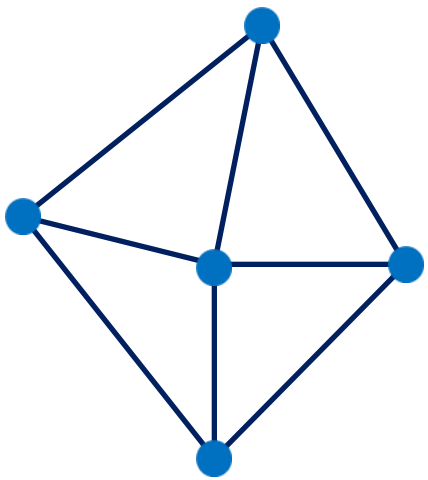

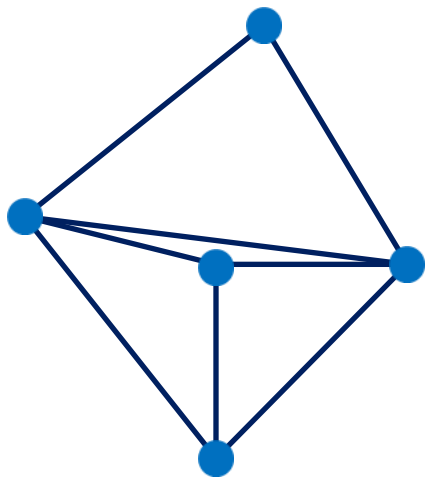

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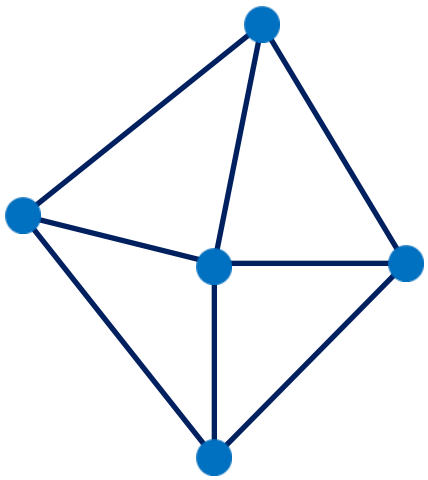
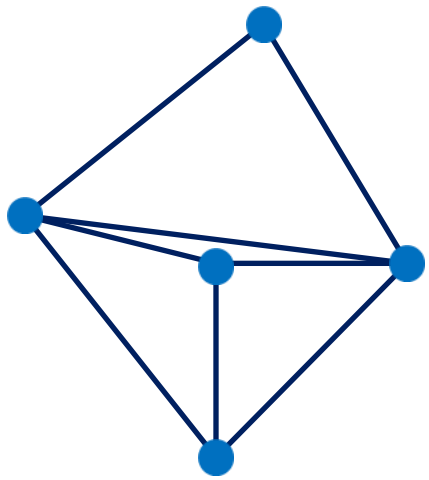
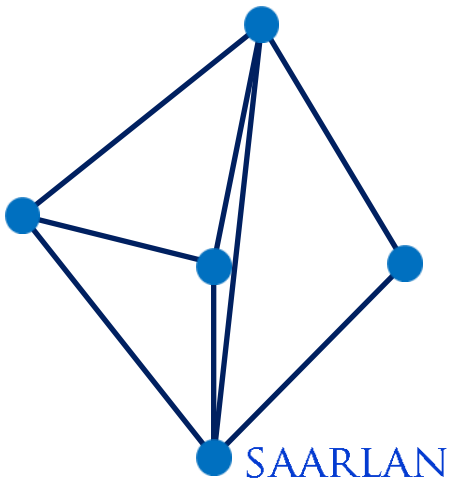


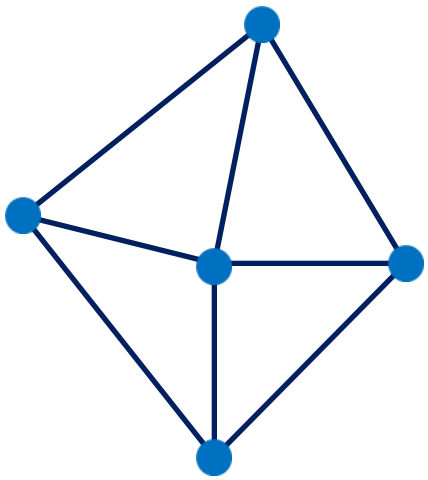


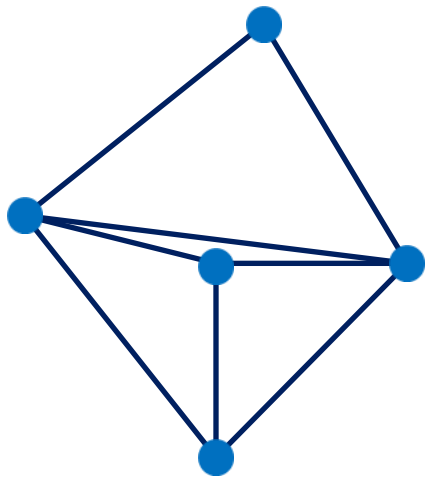


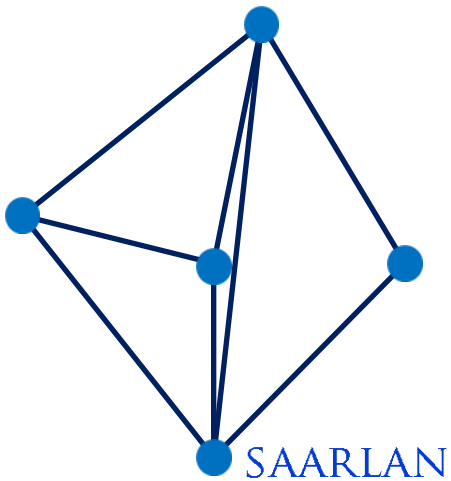
S

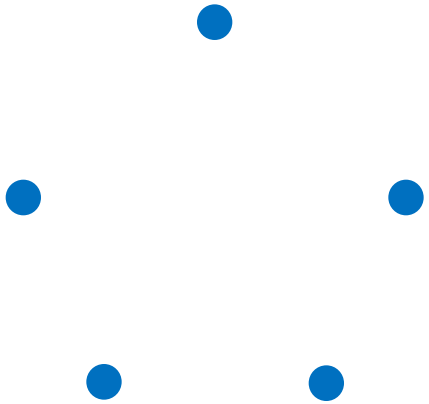
$tr(S)$ = # triangulations of S

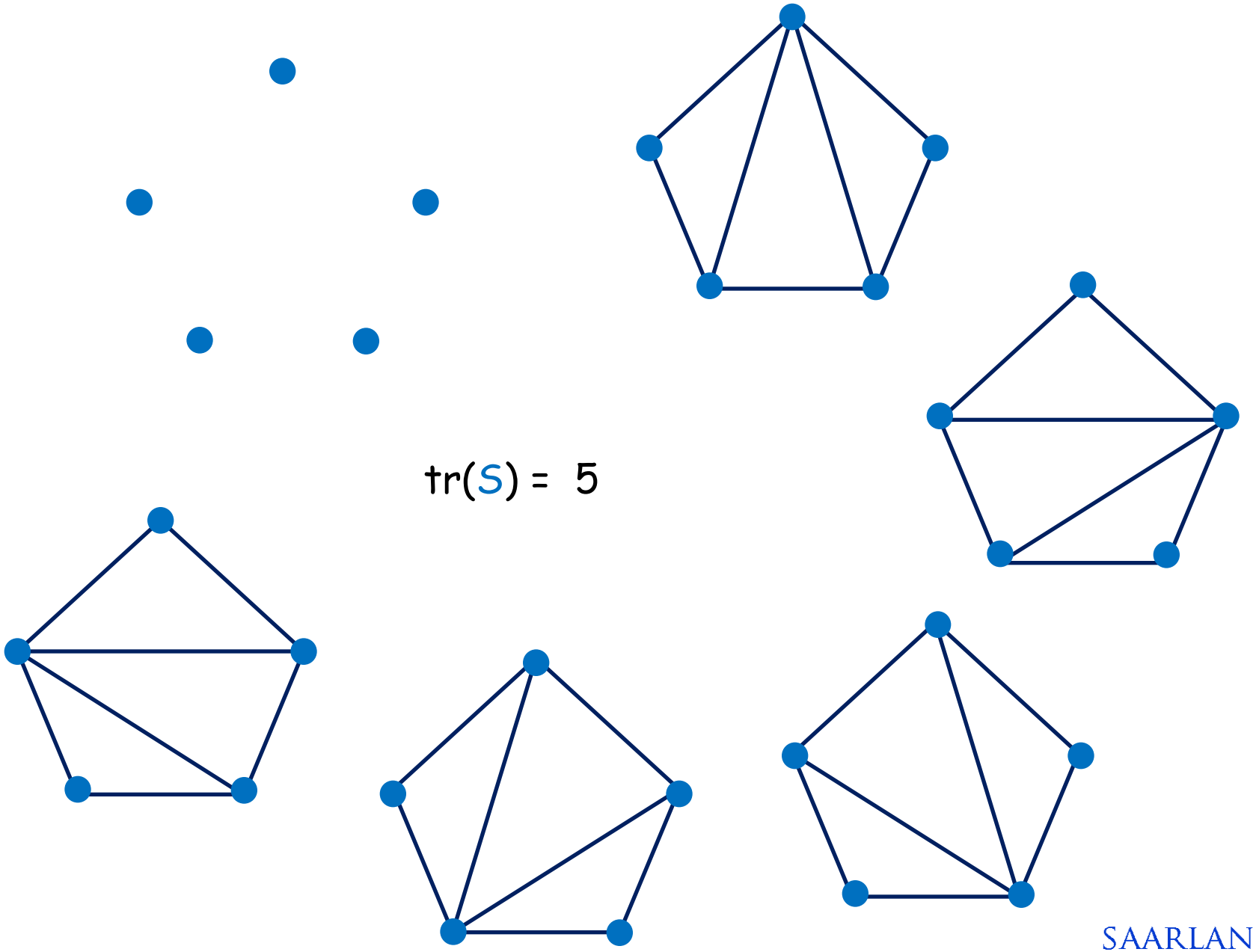
$tr(S) = 3$



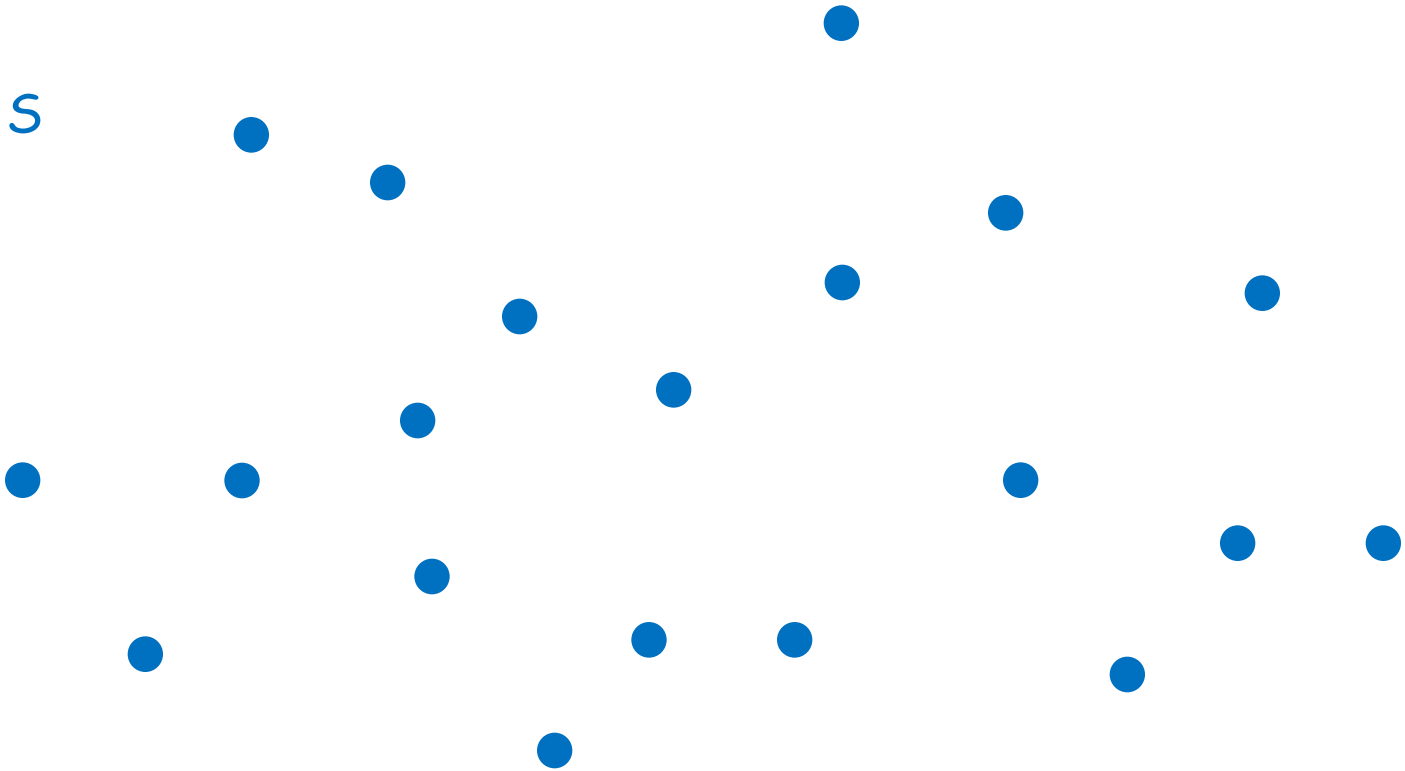








tr(S) = ?



$tr(S)$ = # triangulations of S

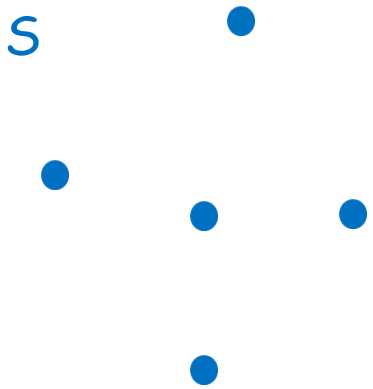
$Tr(n) = \max\{ tr(S) : |S|=n \}$

$tr(n) = \min\{ tr(S) : |S|=n \}$

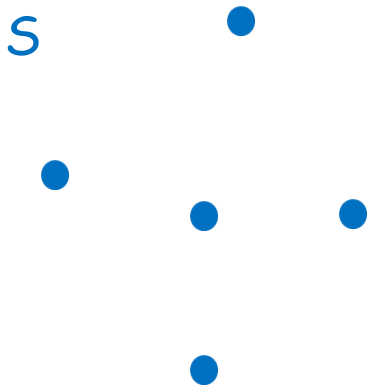
Questions:

- Bounds for $Tr(n)$
- Bounds for $tr(n)$
- Computing $tr(S)$
- Approximating/Estimating $tr(S)$
- Enumerating all triangulations of S
- random sampling a triangulation of S

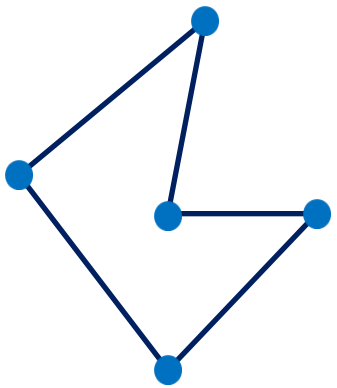
hc(S) = # **non-crossing** straight-edge hamiltonian cycles on S



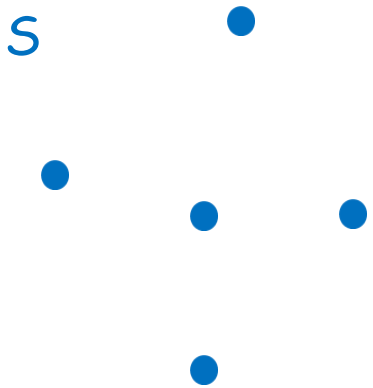
hc(S) = # **non-crossing** straight-edge hamiltonian cycles on S



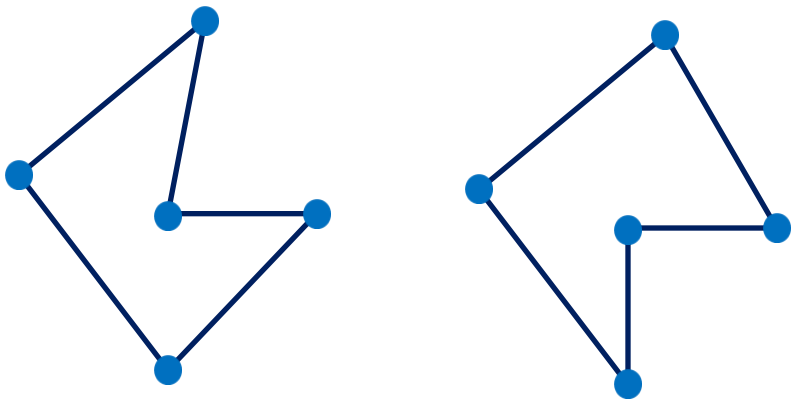
hc(S) = ?



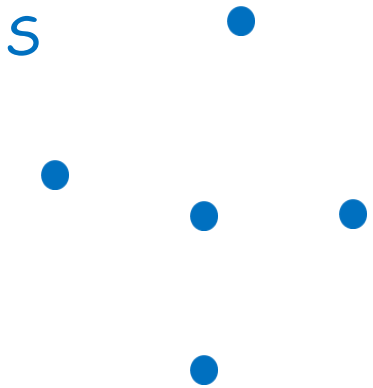
hc(S) = # **non-crossing** straight-edge hamiltonian cycles on S



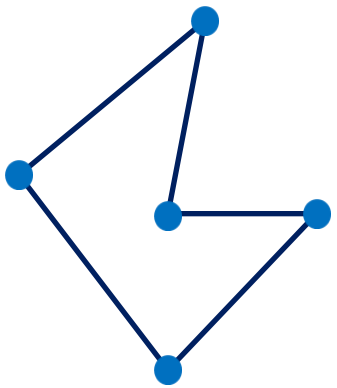
hc(S) = ?

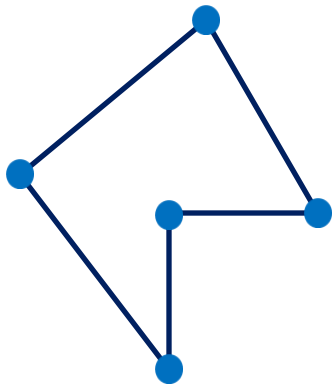


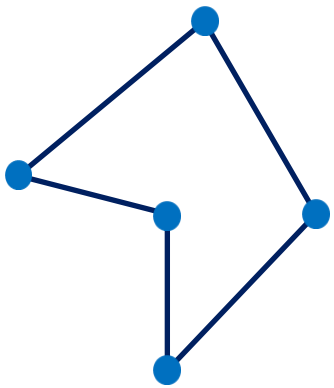
hc(S) = # **non-crossing** straight-edge hamiltonian cycles on S



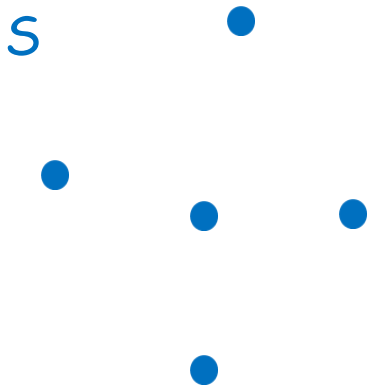
hc(S) = ?



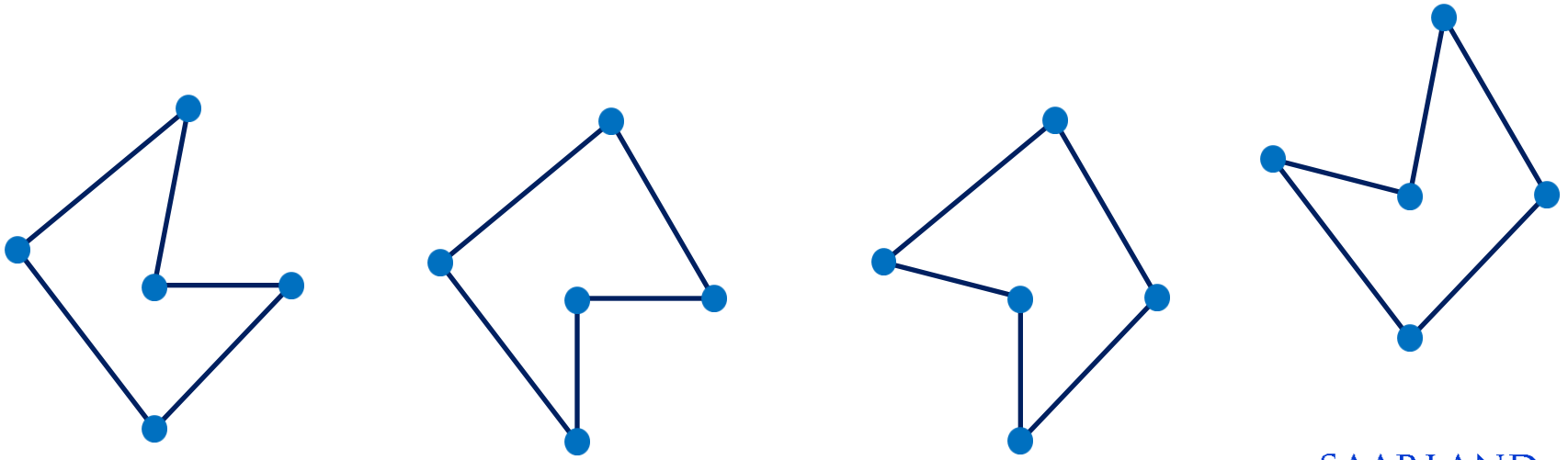




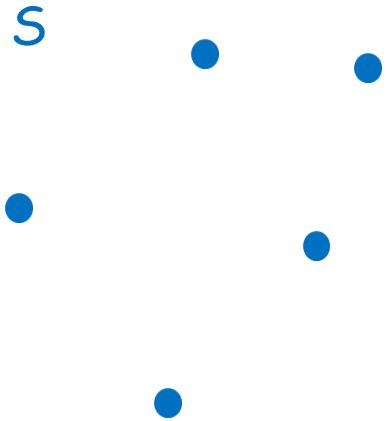
hc(S) = # **non-crossing** straight-edge hamiltonian cycles on S



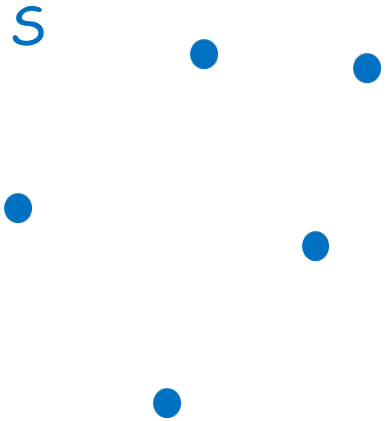
hc(S) = 4



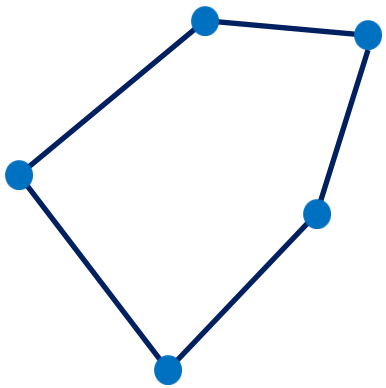
hc(S) = # **non-crossing** straight-edge hamiltonian cycles on S



hc(S) = # **non-crossing** straight-edge hamiltonian cycles on S



hc(S) = 1



hc(S) = # non-crossing straight-edge hamiltonian cycles on S

„plane“

hc(S) = # plane hamiltonian cycles on S

$Hc(n) = \max\{ hc(S) : |S|=n \}$

$hc(n) = \min\{ hc(S) : |S|=n \}$

$hc(S)$ = # plane hamiltonian cycles on S

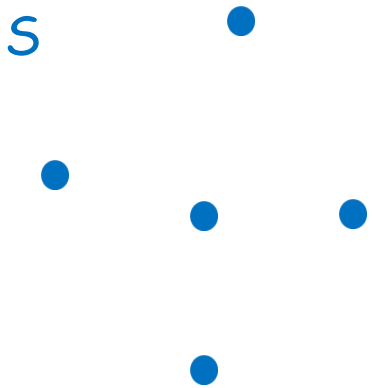
$Hc(n) = \max\{ hc(S) : |S|=n \}$

$hc(n) = \min\{ hc(S) : |S|=n \}$

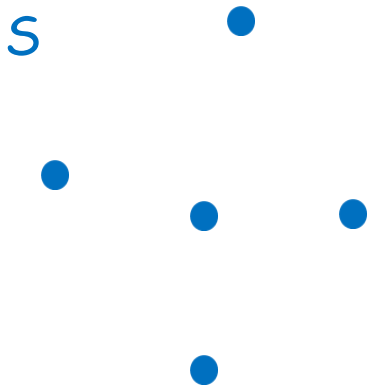
Questions:

- Bounds for $Hc(n)$
- Bounds for $hc(n)$
- Computing $hc(S)$
- Approximating/Estimating $hc(S)$
- Enumerating plane hamiltonian cycles on S
- random sampling a plane hamiltonian cycle on S

pm(S) = # plane maximum matchings on S

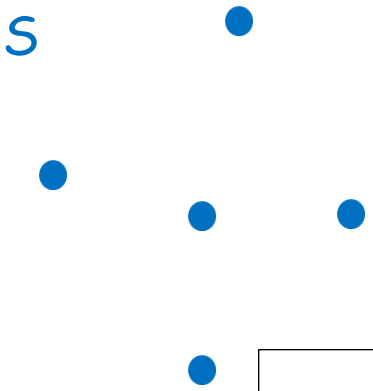


pm(S) = # plane maximum matchings on S

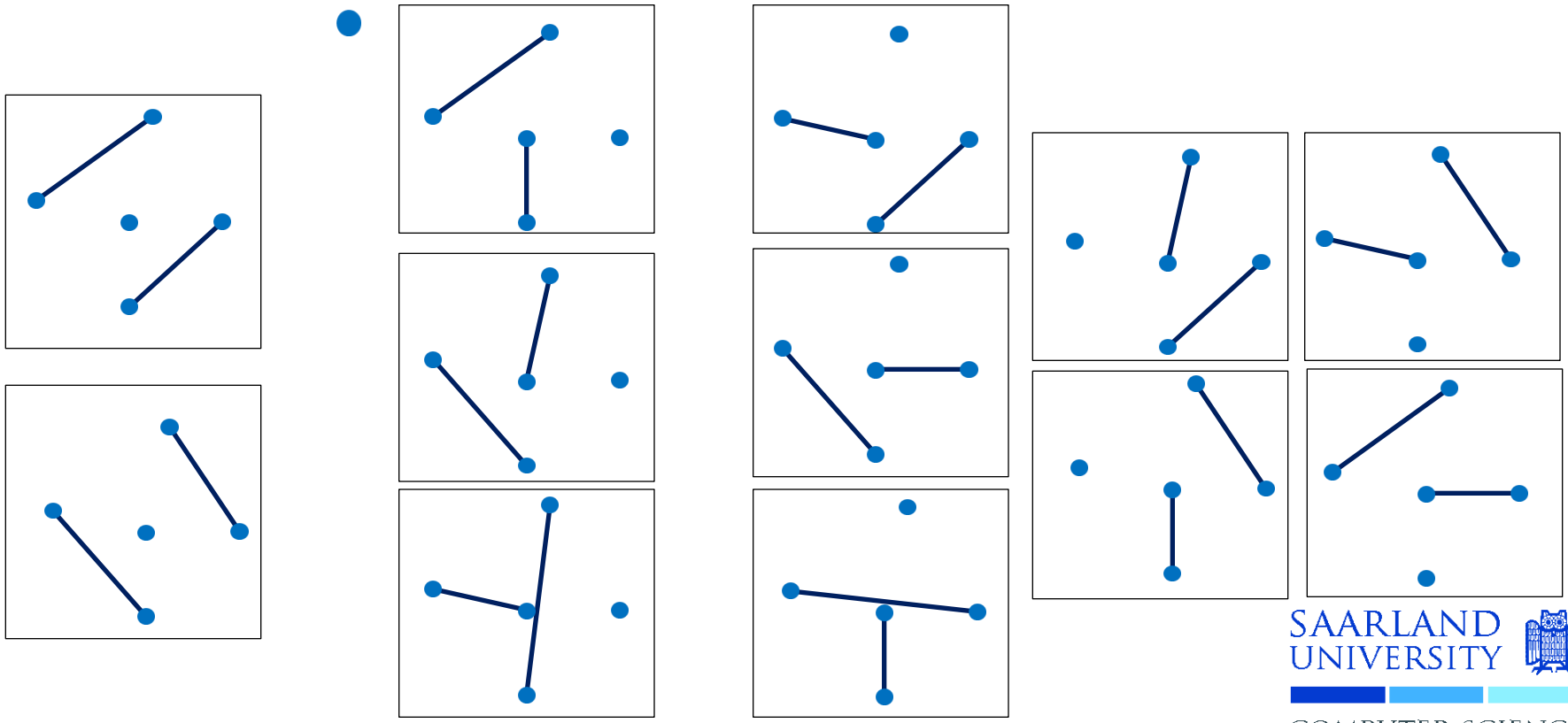


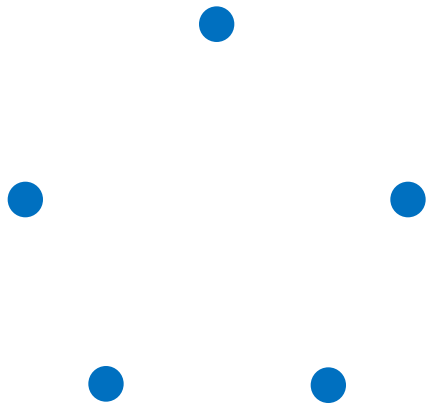
pm(S) = 12

pm(S) = # plane maximum matchings on S

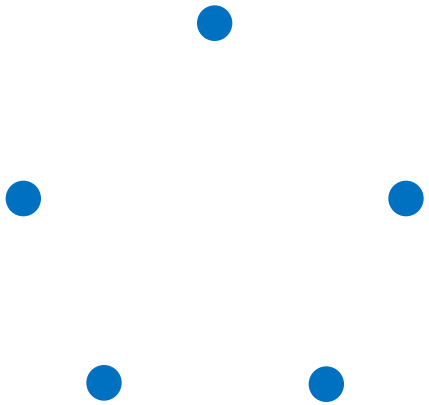


pm(S) = 12

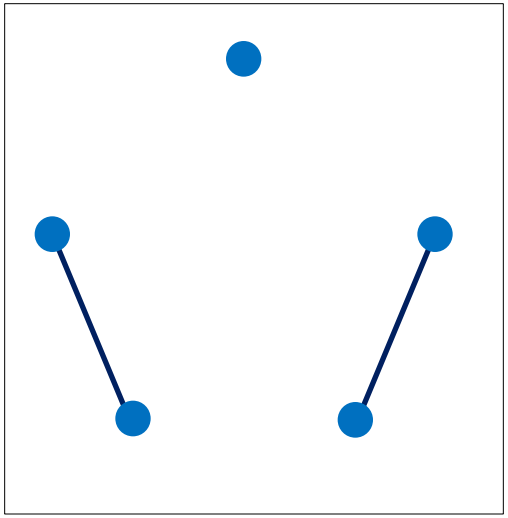


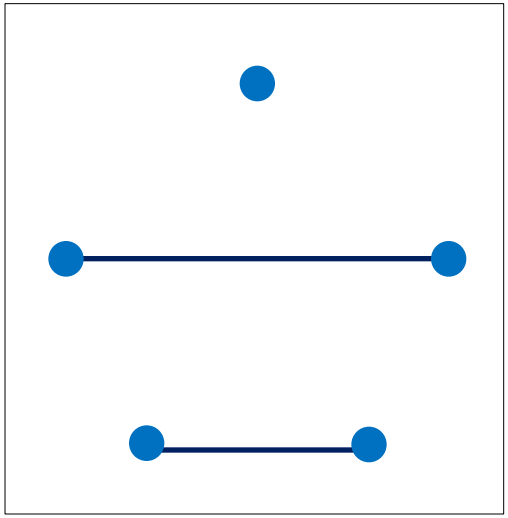


$pm(S) = ?$



pm(S) = 10





$pm(S)$ = # plane maximum matchings on S

$$Pm(n) = \max\{ pm(S) : |S|=n \}$$

$$pm(n) = \min\{ pm(S) : |S|=n \}$$

Questions:

- Bounds for $Pm(n)$
- Bounds for $pm(n)$
- Computing $pm(S)$
- Approximating/Estimating $pm(S)$
- Enumerating plane maximum matchings on S
- random sampling a plane maximum matching on S

Other classes of "plane structures" on S:

- plane spanning trees
- plane cycle covers
- plane graphs
- plane convex covers
- ...

$xx(S)$ = # plane xxxxx on S

$Xx(n) = \max\{ xx(S) : |S|=n \}$

$xx(n) = \min\{ xx(S) : |S|=n \}$

Questions:

- Bounds for $Xx(n)$
- Bounds for $xx(n)$
- Computing $xx(S)$
- Approximating/Estimating $xx(S)$
- Enumerating plane xxxxx on S
- random sampling a plane xxxxx on S

Inspite of decades of research each problem is either

trivial

or

not answered satisfactorily.

Triangulations

tr(S) = # triangulations of S

Tr(n) = max{ tr(S) : $|S|=n$ }

tr(n) = min{ tr(S) : $|S|=n$ }

Questions:

- Bounds for Tr(n)
- Bounds for tr(n)
- Computing tr(S)
- Approximating/Estimating tr(S)
- Enumerating all triangulations of S
- random sampling a triangulation of S

$\Omega(2.43^n) \leq \mathrm{tr}(S) \leq O(30^n)$ $\quad\quad$ $n=|S|$

Aichholzer, Hackl, Vogtenhuber, Huemer, Hurtado, Krasser, McCabe, Seidel, Sharir, Sheffer, Noy, Flajolet, …
Ajtai, Chvátal, Newborn, Szemerédi, Smith, Seidel, Santos, Sharir, Welzl, Sheffer, Dumitrescu, Schulz, Toth, …

$$\Omega(8.65^n) \leq \mathrm{Tr}(n) \leq O(30^n)$$

$$\Omega(2.43^n) \leq \mathrm{tr}(n) \leq O(3.464^n)$$

McCabe, Seidel, EuroCG2003

tr(h,m) = min{ tr(S) | S has h extreme points and m "interior" points }

Theorem 1:

$tr(h,m) \geq \lambda \cdot (30/11)^h \cdot (11/5)^m$

$\geq \lambda \cdot 2.7272^h \cdot 2.2^m$ $\qquad \lambda \geq 0.0414$

Theorem 2: For every fixed h:

$tr(h,m) \geq \Omega(2.63^m)$

Computing tr(S)

enumeration via reverse search $\Omega(\mathrm{tr}(S)) \geq \Omega(2.43^n)$
(Avis, Fukuda, Bespamyatnikh)

divide-and-conquer using „triangulation paths" ???
(Aichholzer)

dynamic programming ???
(Ray, Seidel)

sweep using „triangulation paths" ???
(Alvarez, Bringmann, Ray)

dynamic programming using „onion structure" $O(3.1414^n)$
(Alvarez, Bringmann, Curticapean, Ray)

via reduction to path counting in DAGs $O(2^n n^2)$

(Alvarez, S.)

Today: $O(2^n n^2)$ algorithm

via reduction to path counting in DAGs

"aggregative" …. substantially faster than enumeration

$\mathrm{tr}(S) \geq \Omega(2.43^n)$

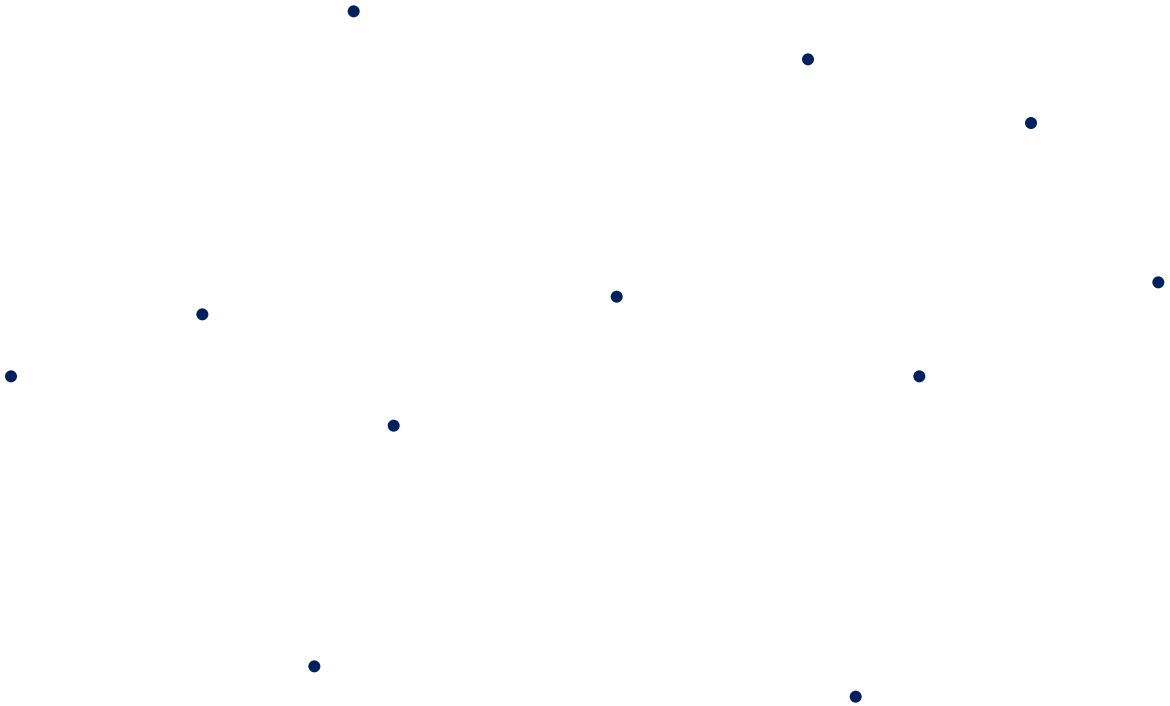
Today: $O(2^n n^2)$ algorithm

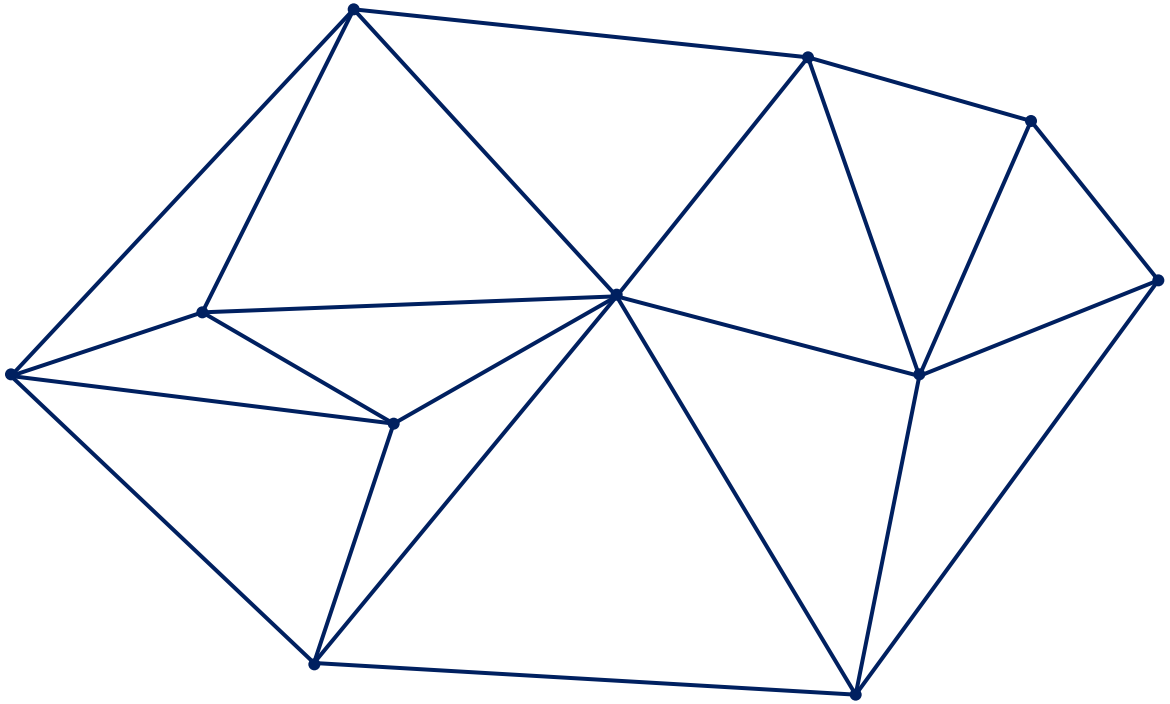
via reduction to path counting in DAGs

"aggregative" …. substantially faster than enumeration

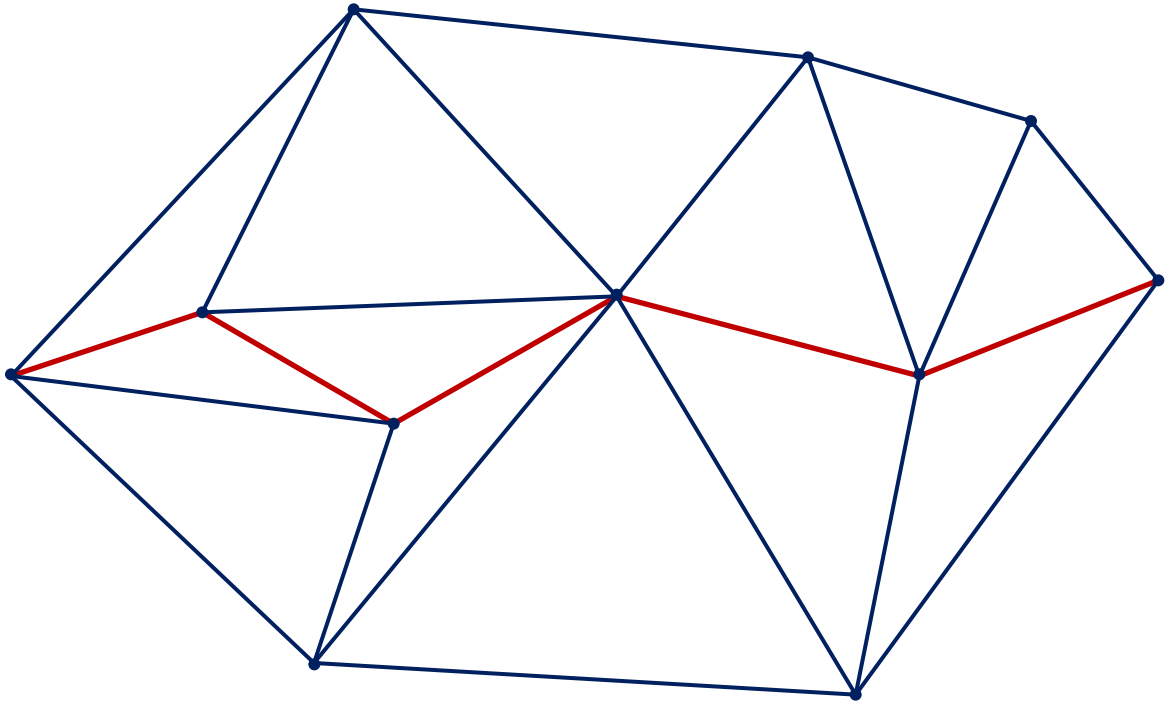
$\mathrm{tr}(S) \geq \Omega(2.43^n)$

Exponential space usage: $O(2^n n)$



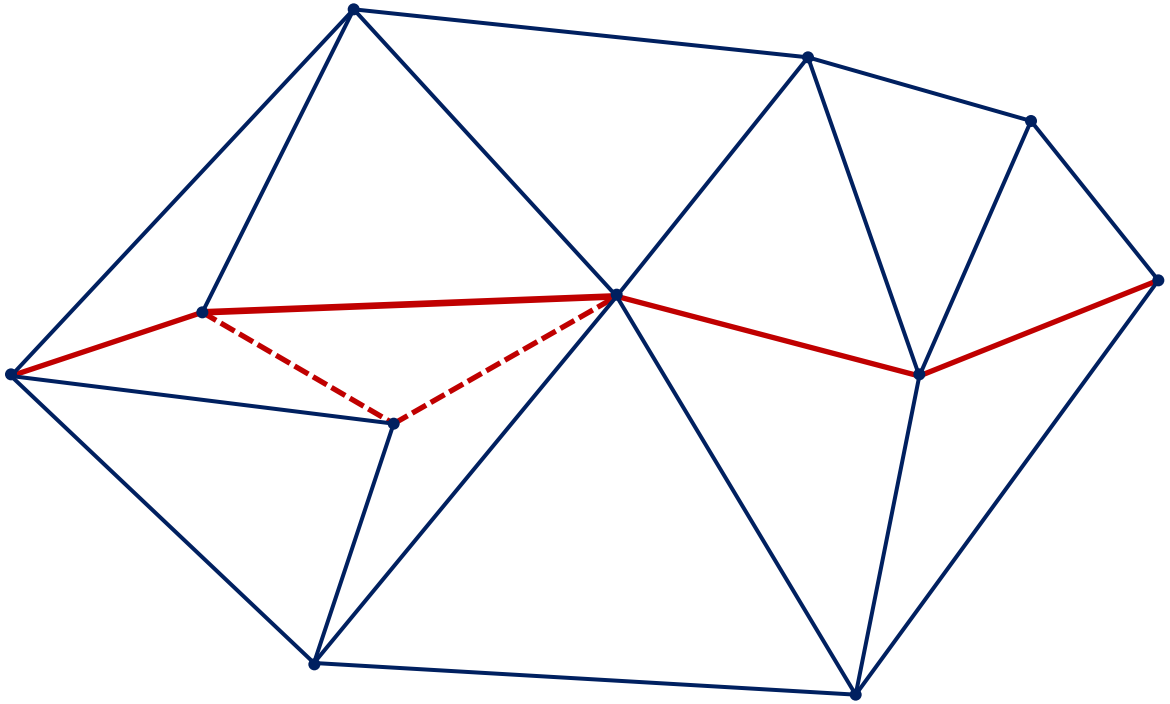


monotone chain



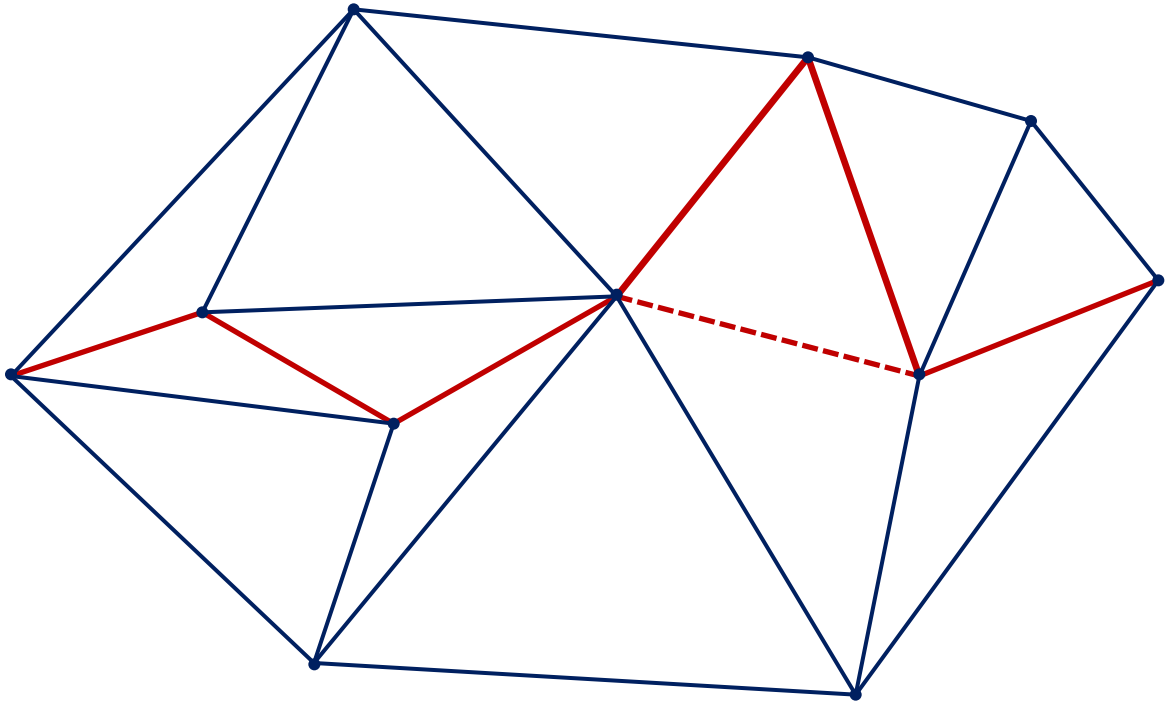
advance

(losing a vertex)



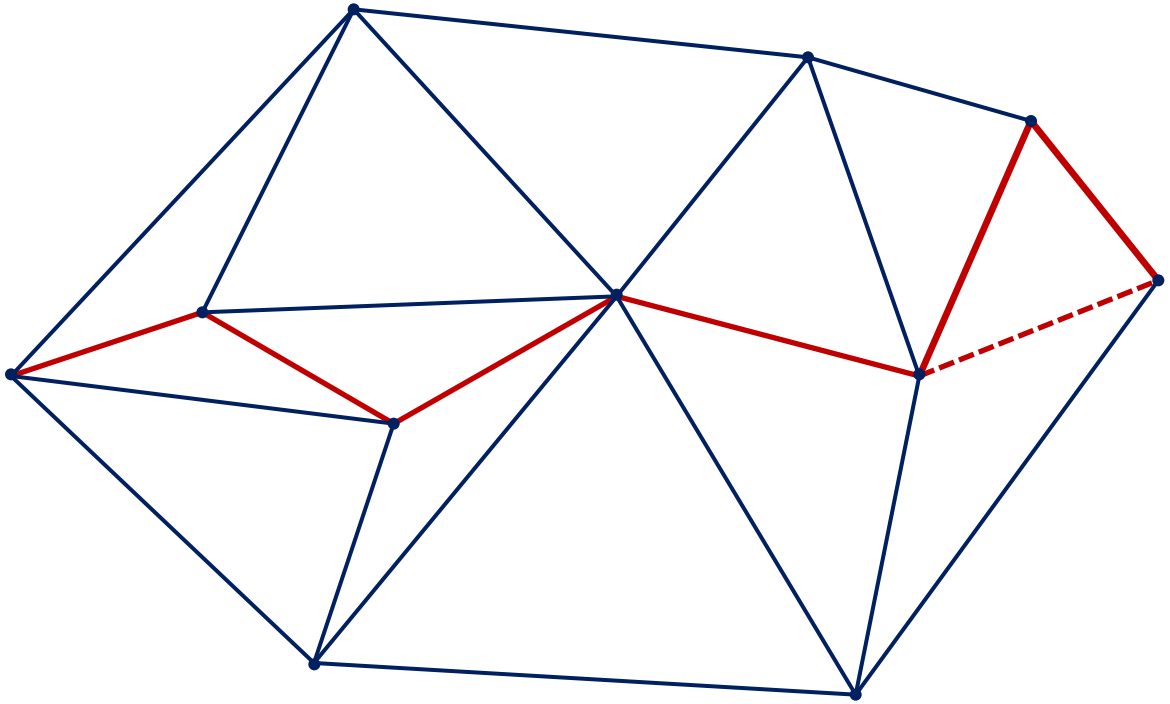
advance

(winning a vertex)



advance

(winning a vertex)



Claim:
For every monotone chain C in a triangulation there is an advance.

Claim:
For every monotone chain C in a triangulation there is an advance,

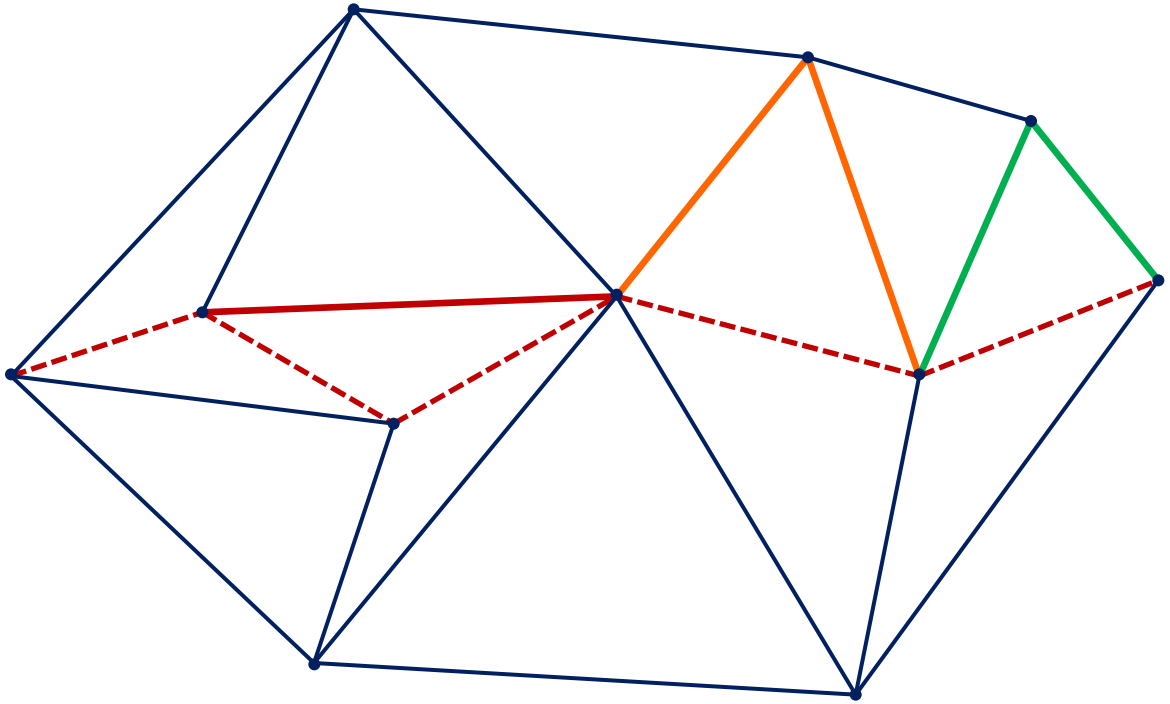
unless C is the chain formed by the upper hull.

Claim:
For every monotone chain C in a triangulation there is an advance,

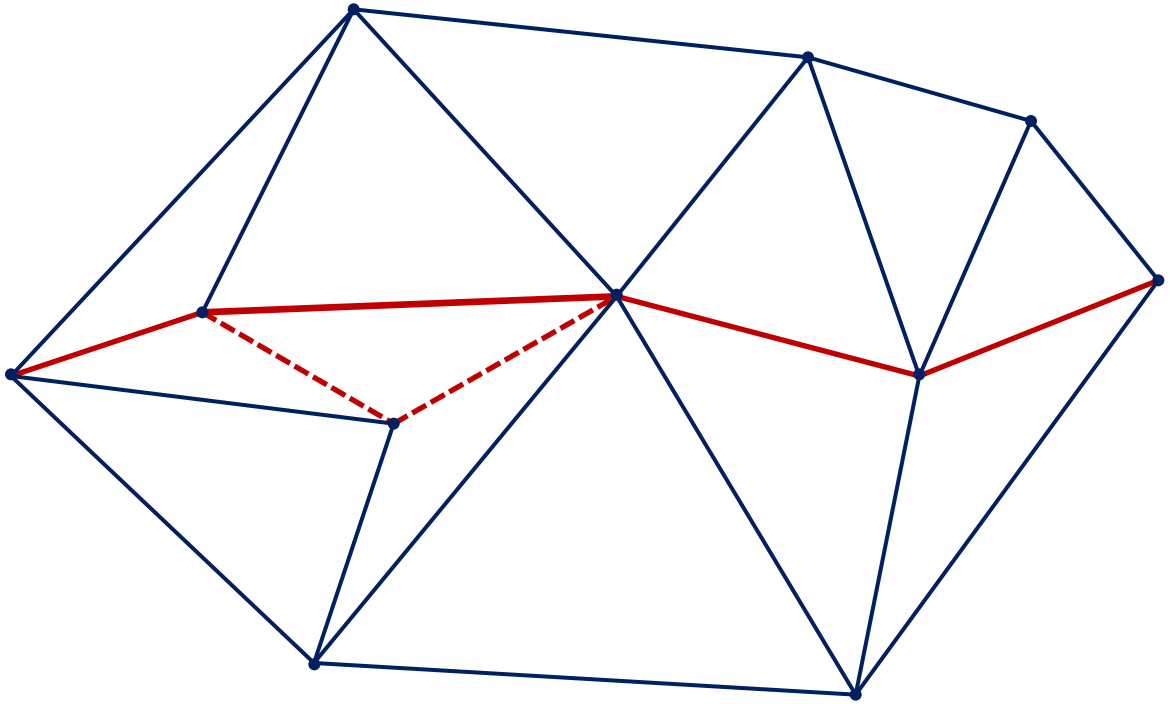
unless C is the chain formed by the upper hull.

Corollary:
For every monotone chain C in a triangulation there is a **leftmost** advance,

unless C is the chain formed by the upper hull.



leftmost advance

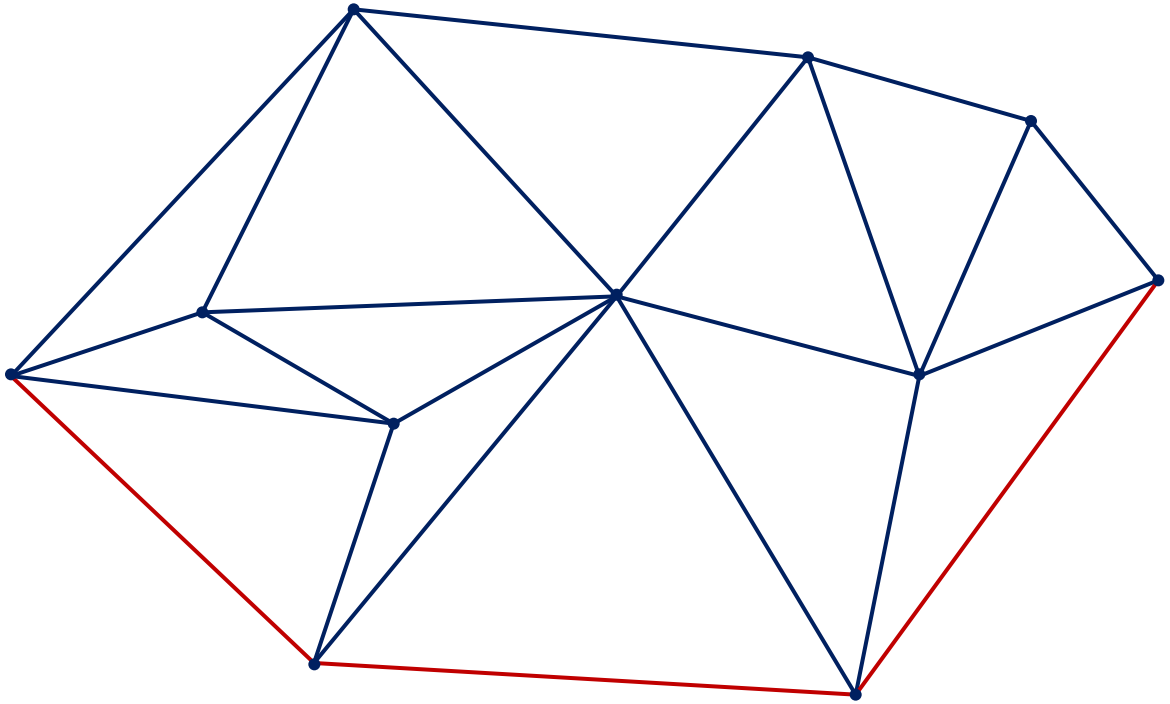


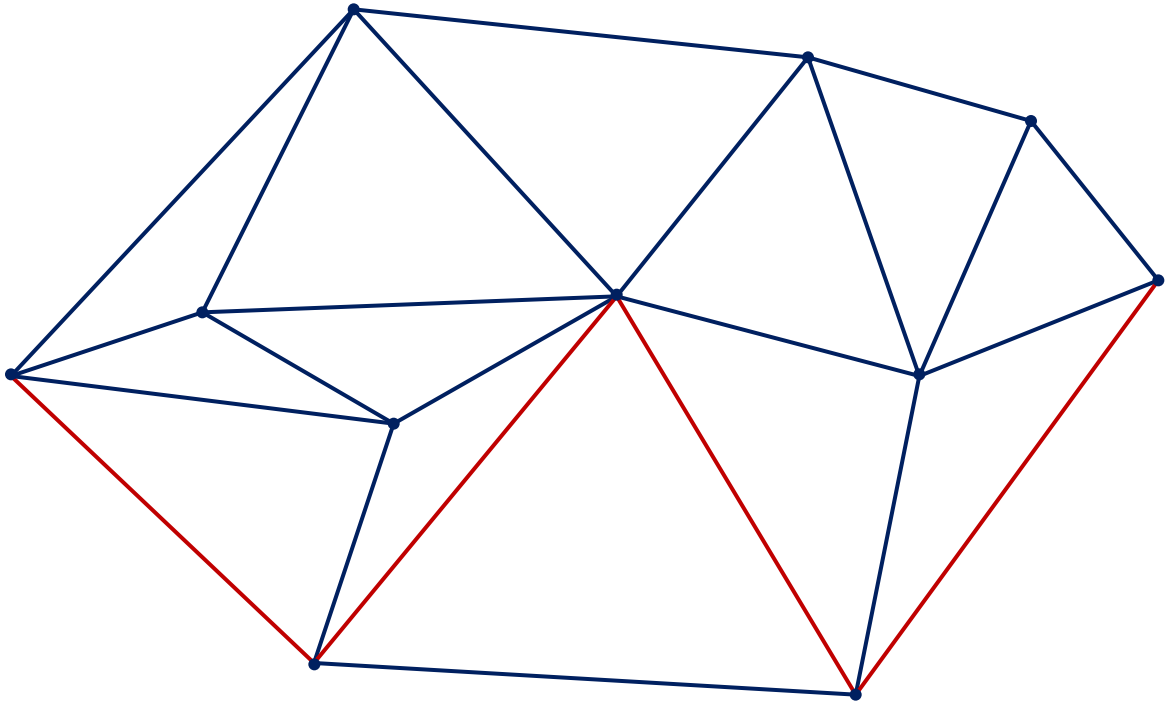
Leftmost advancing monotone chain sweep of a triangulation T

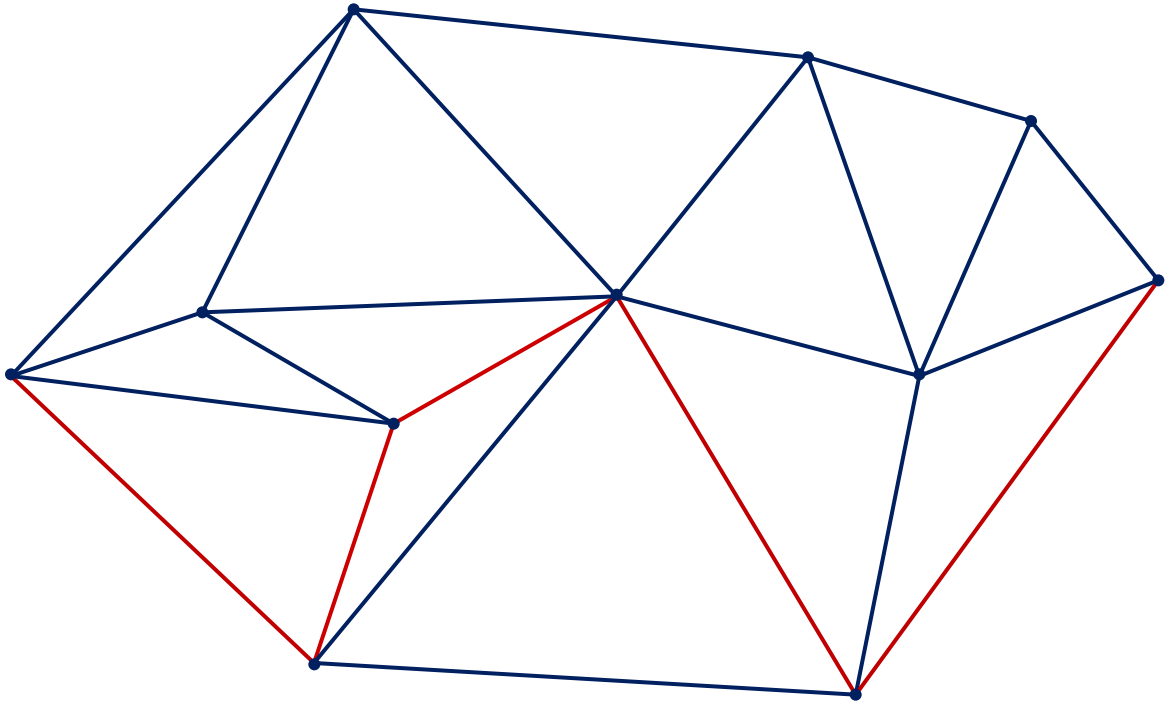
Sequence $C_0, C_1, \ldots, C_t$ of monotone chains of T such that

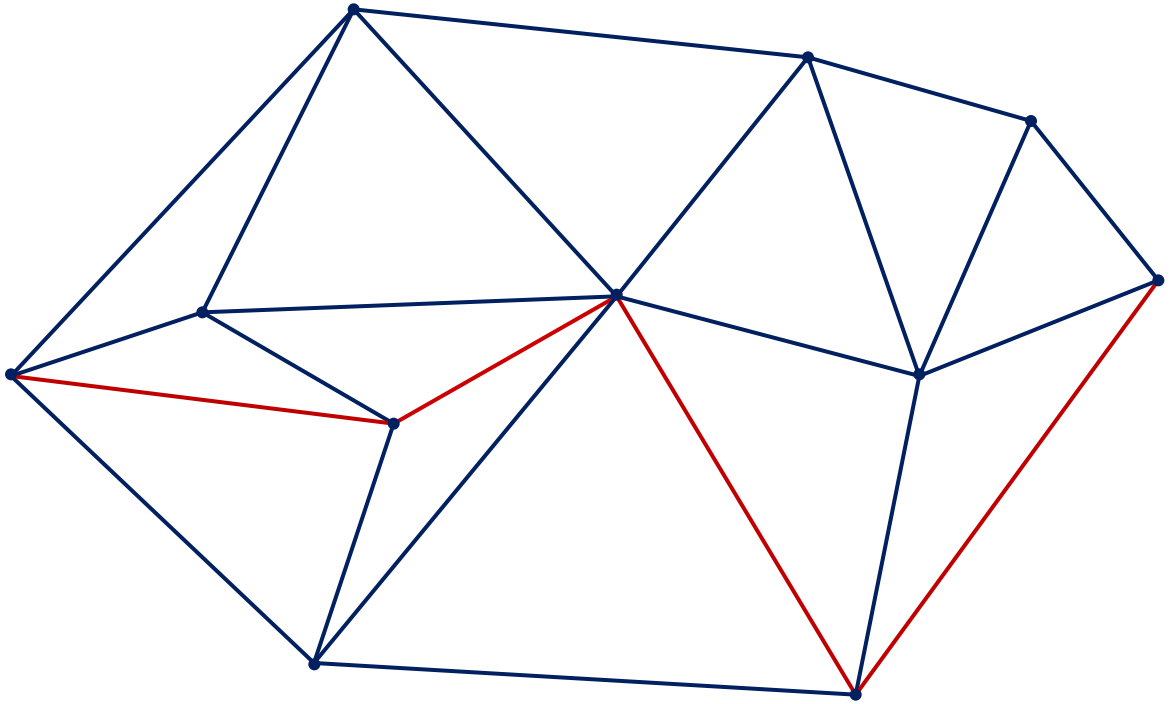
C_0 is the lower convex hull
C_t is the upper convex hull
C_{i+1} obtained from C_i by a leftmost advance

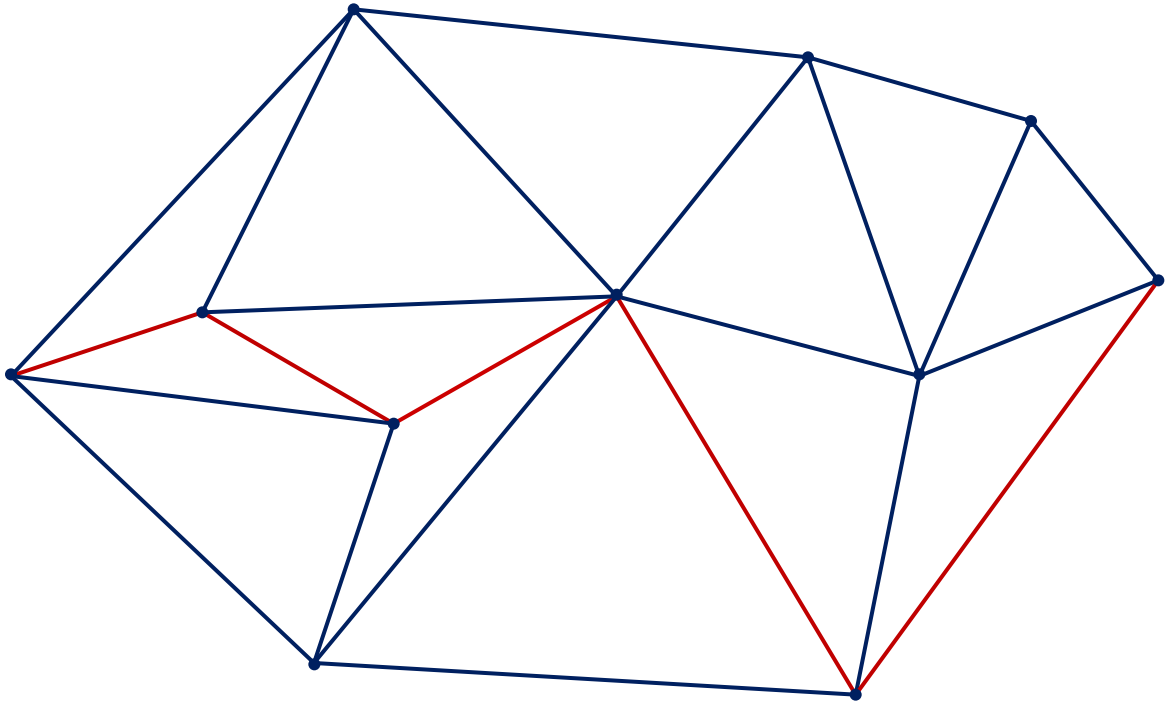
Claim:
Every triangulation has a unique leftmost advancing sweep.

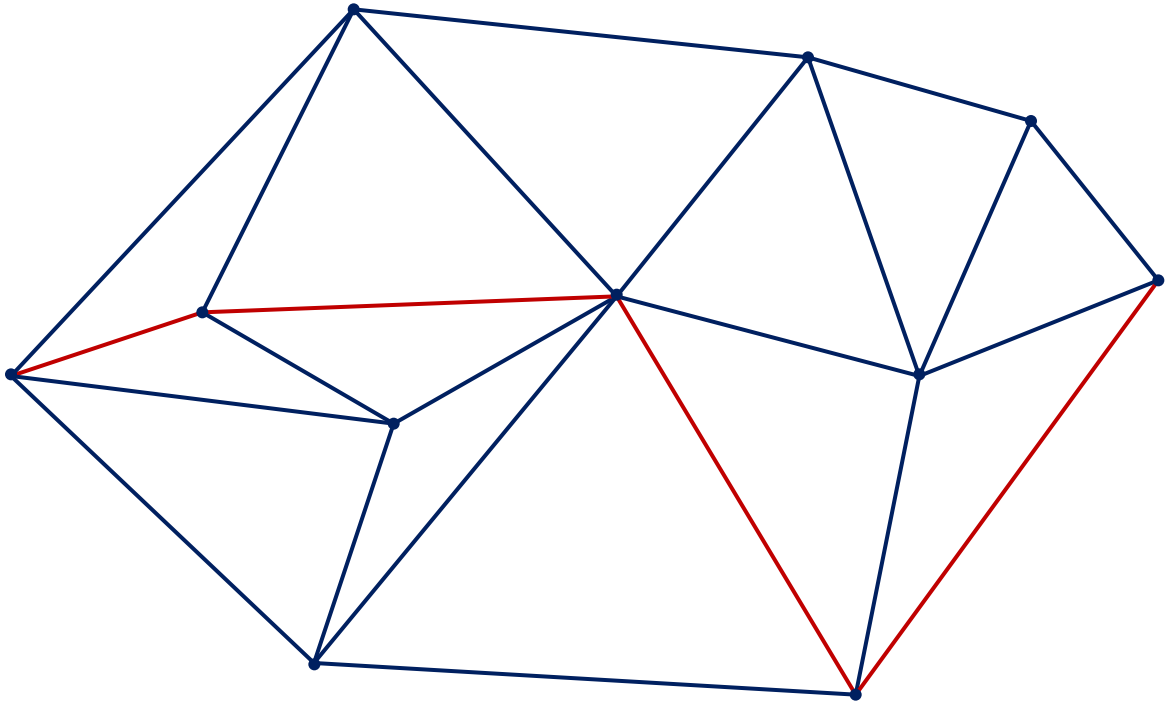


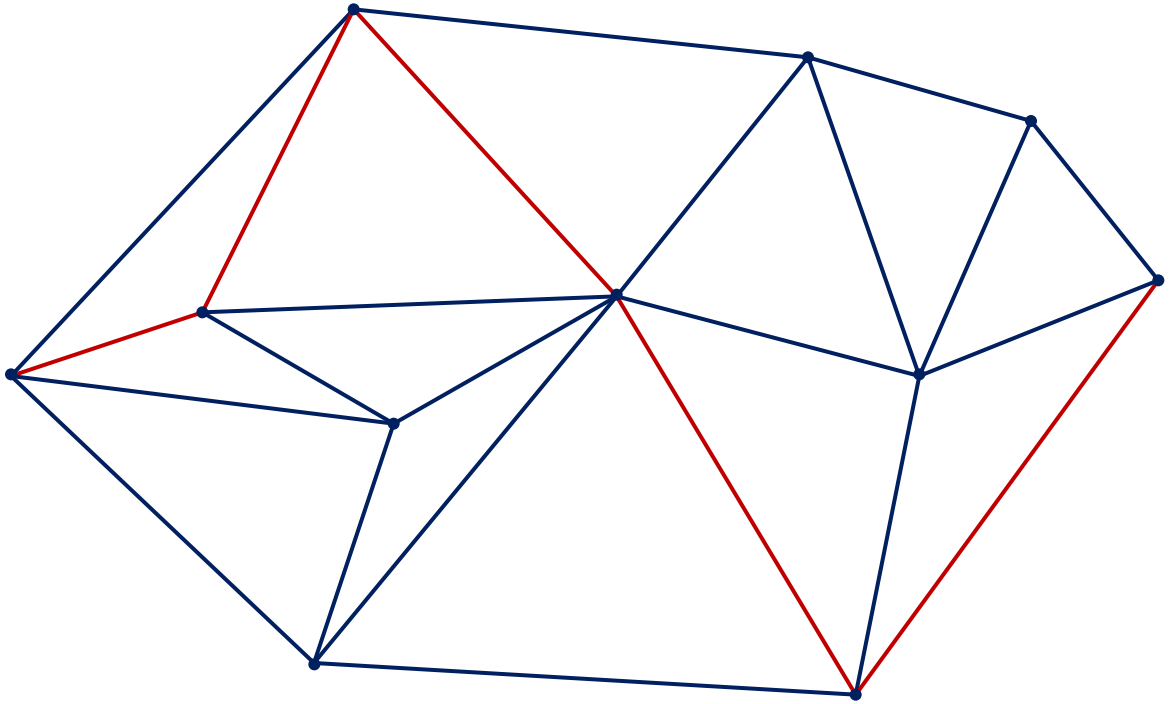


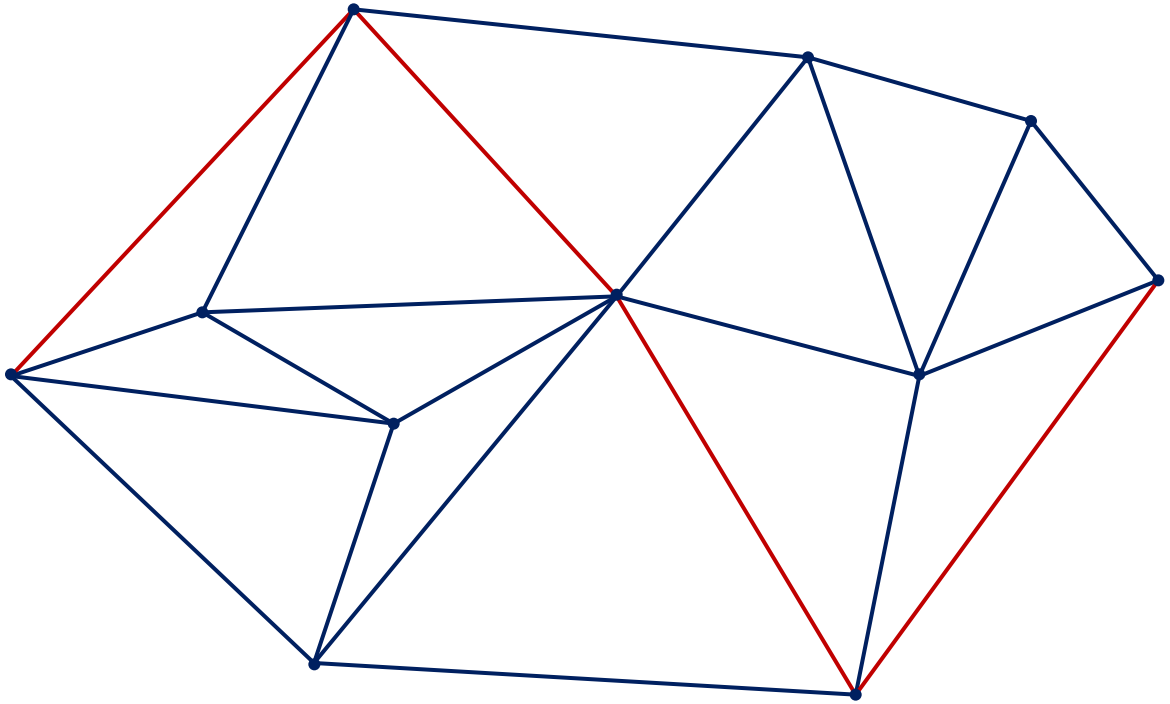


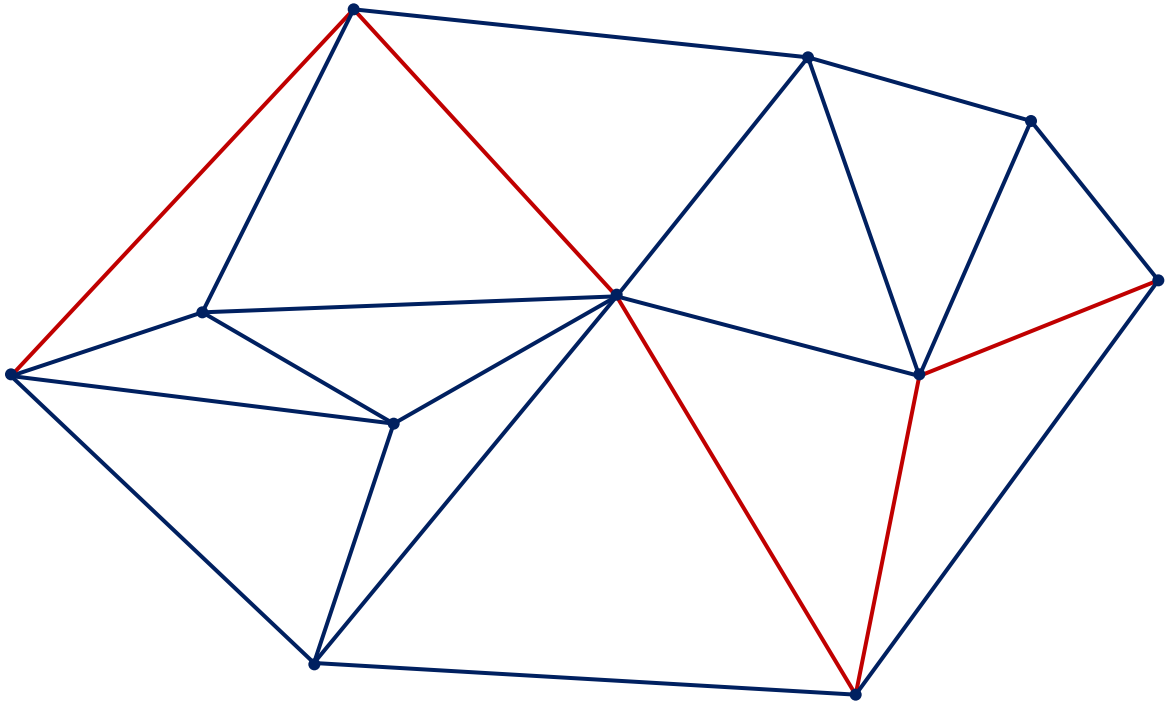


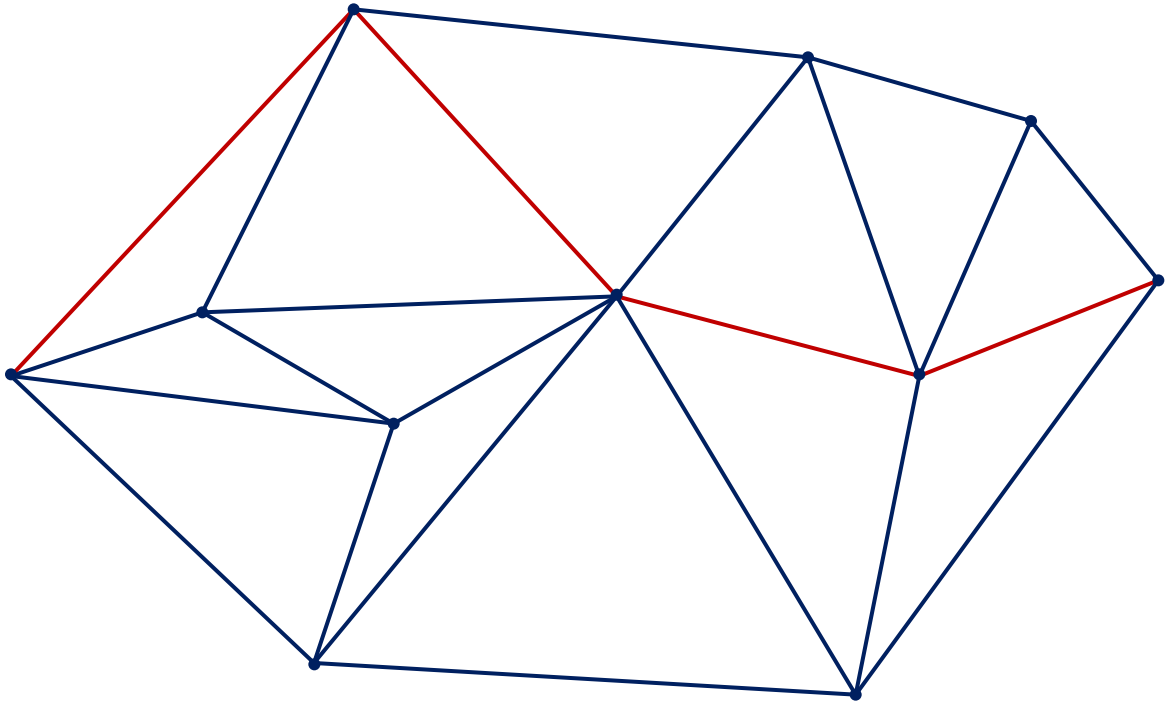


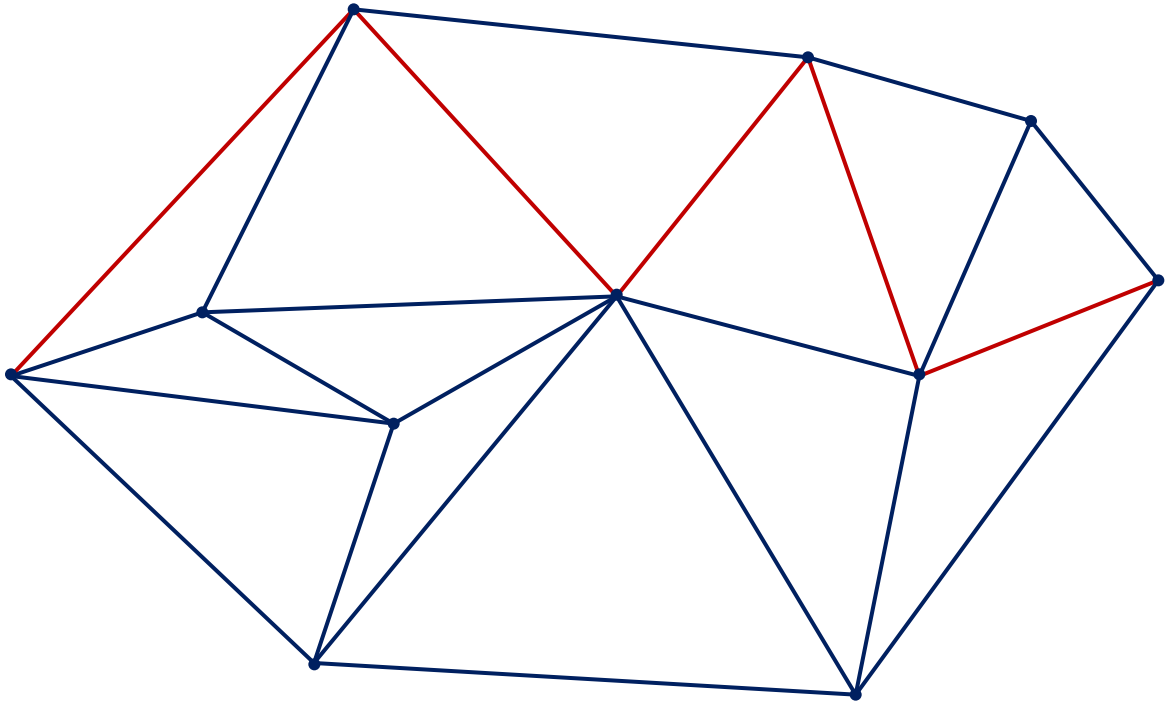


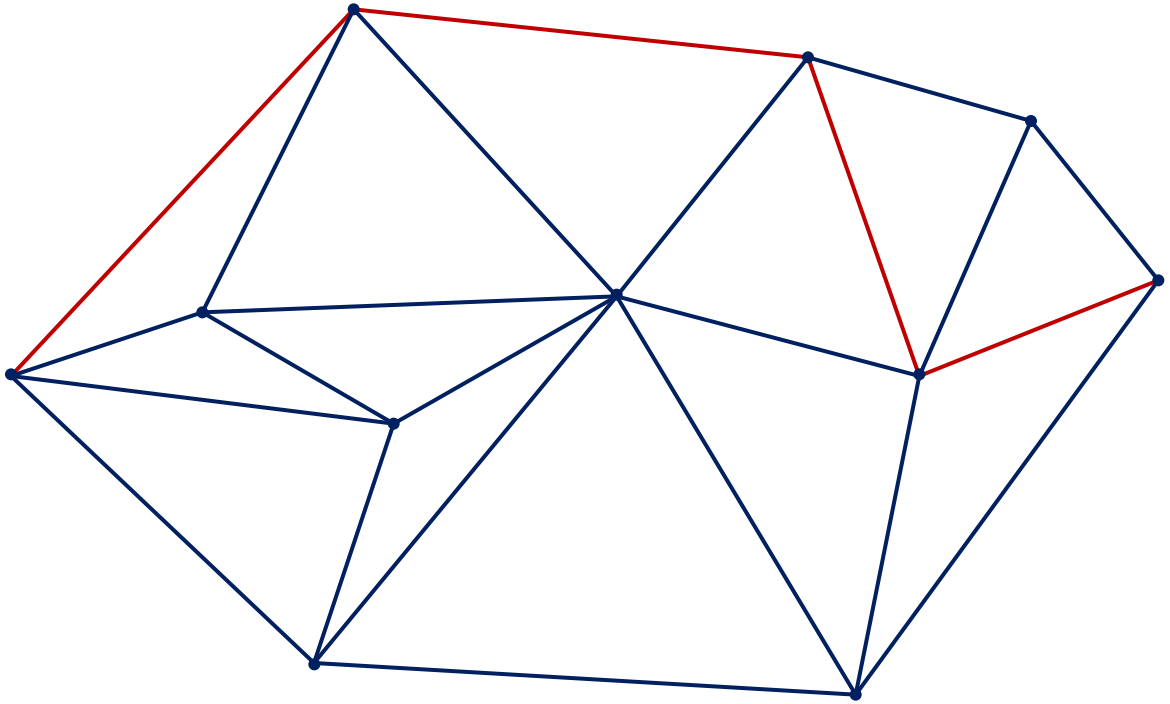


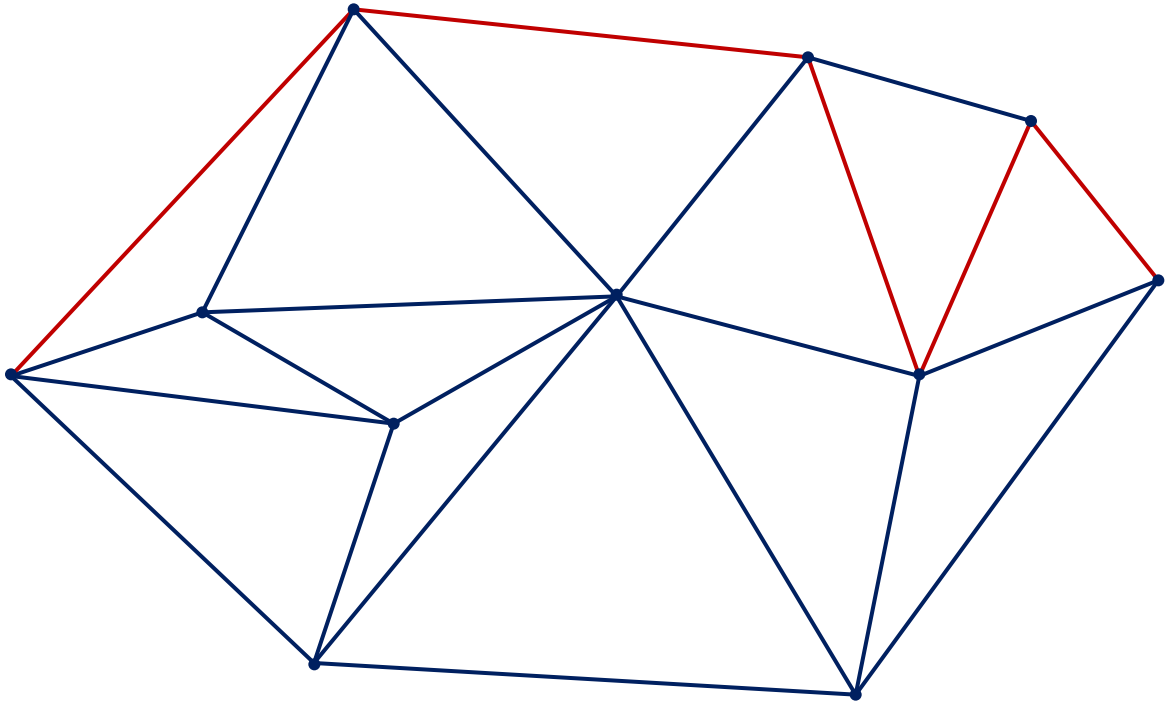


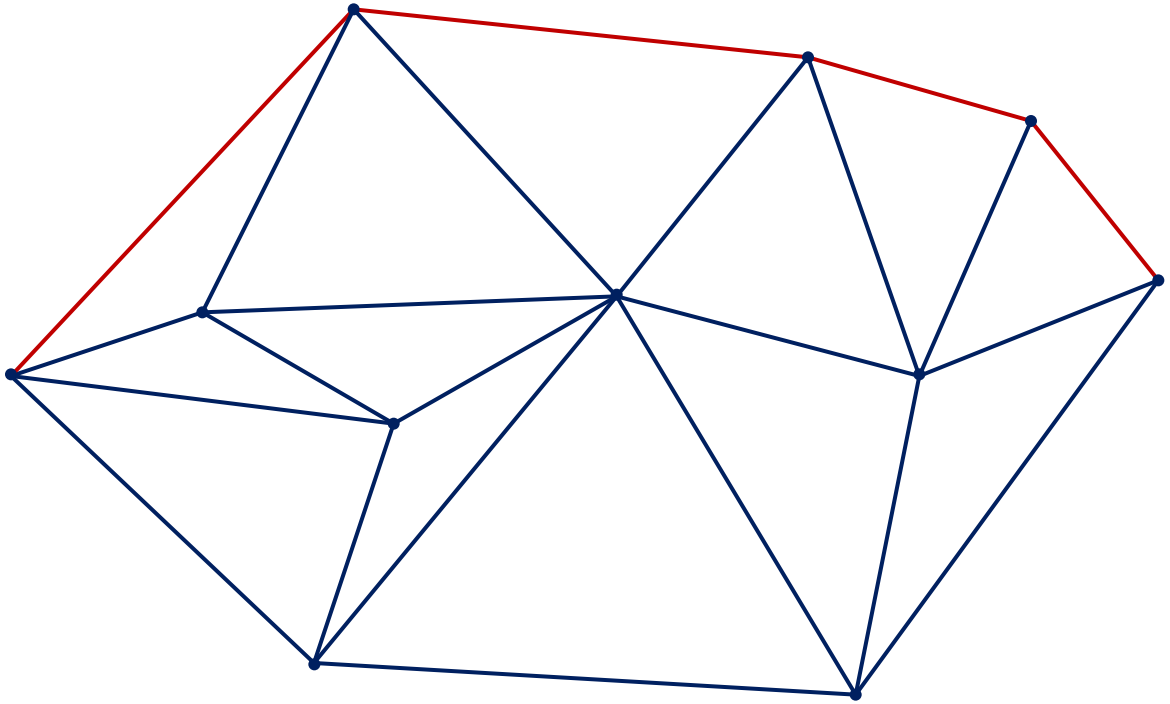












Leftmost advancing monotone chain sweep of a triangulation T

Sequence $C_0, C_1, \ldots, C_t$ of monotone chains of T such that

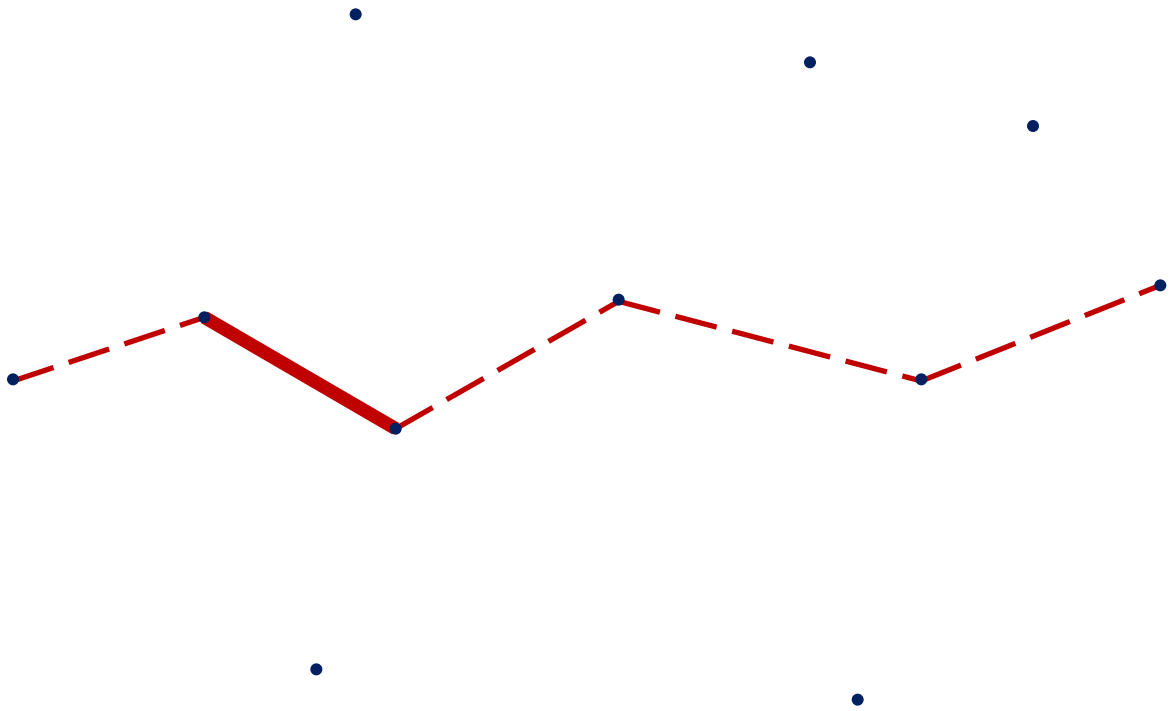
C_0 is the lower convex hull
C_t is the upper convex hull
C_{i+1} obtained from C_i by a leftmost advance

Claim:
Every triangulation has a unique leftmost advancing sweep.


S

S

marked monotone chain of S



S

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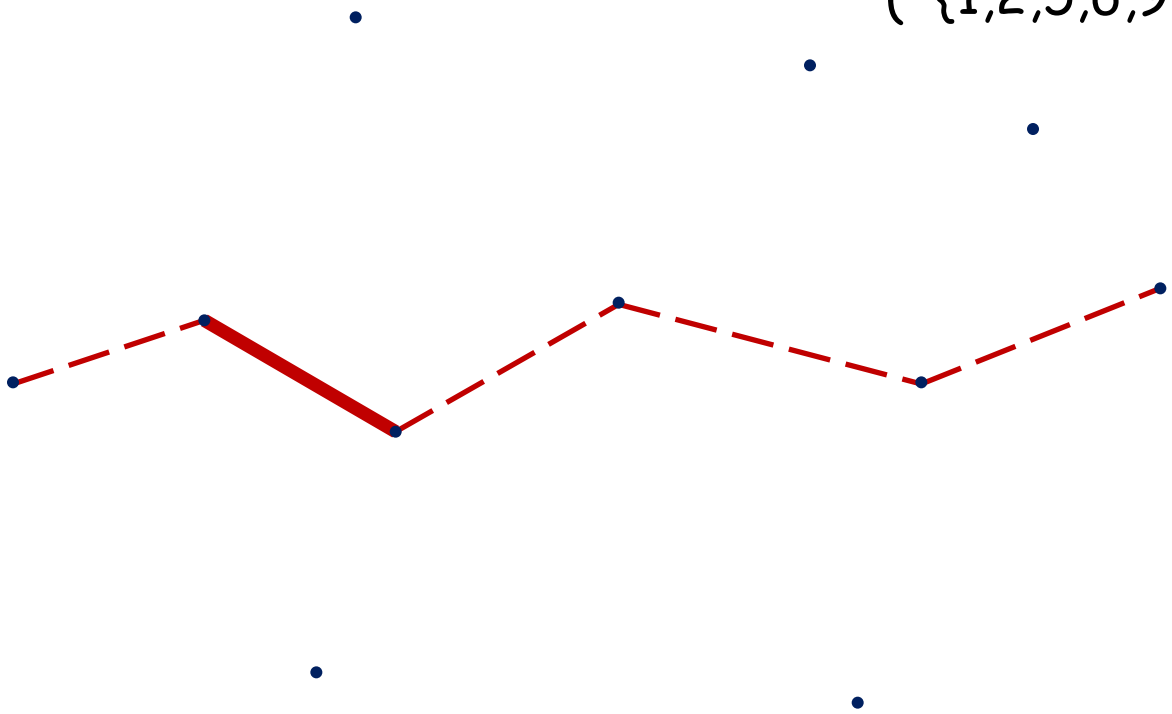
({1,2,5,6,9,11} , 2)



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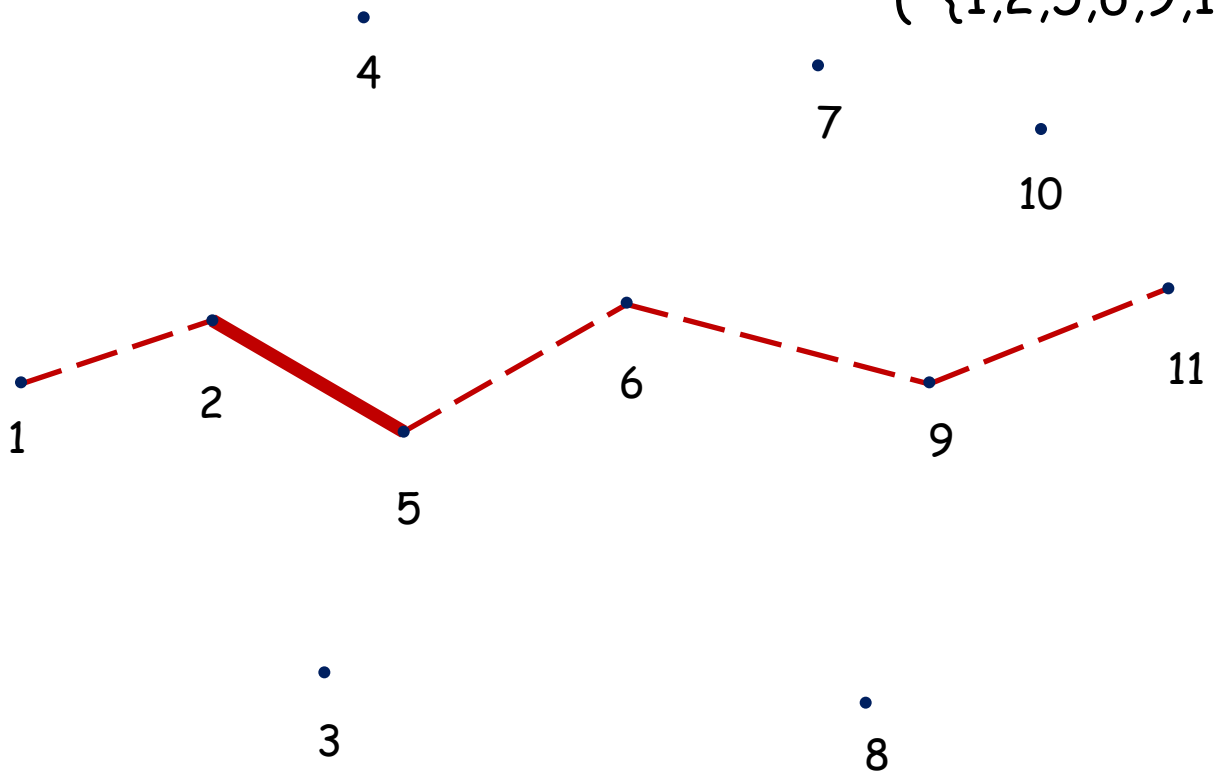

S

marked monotone chain of S

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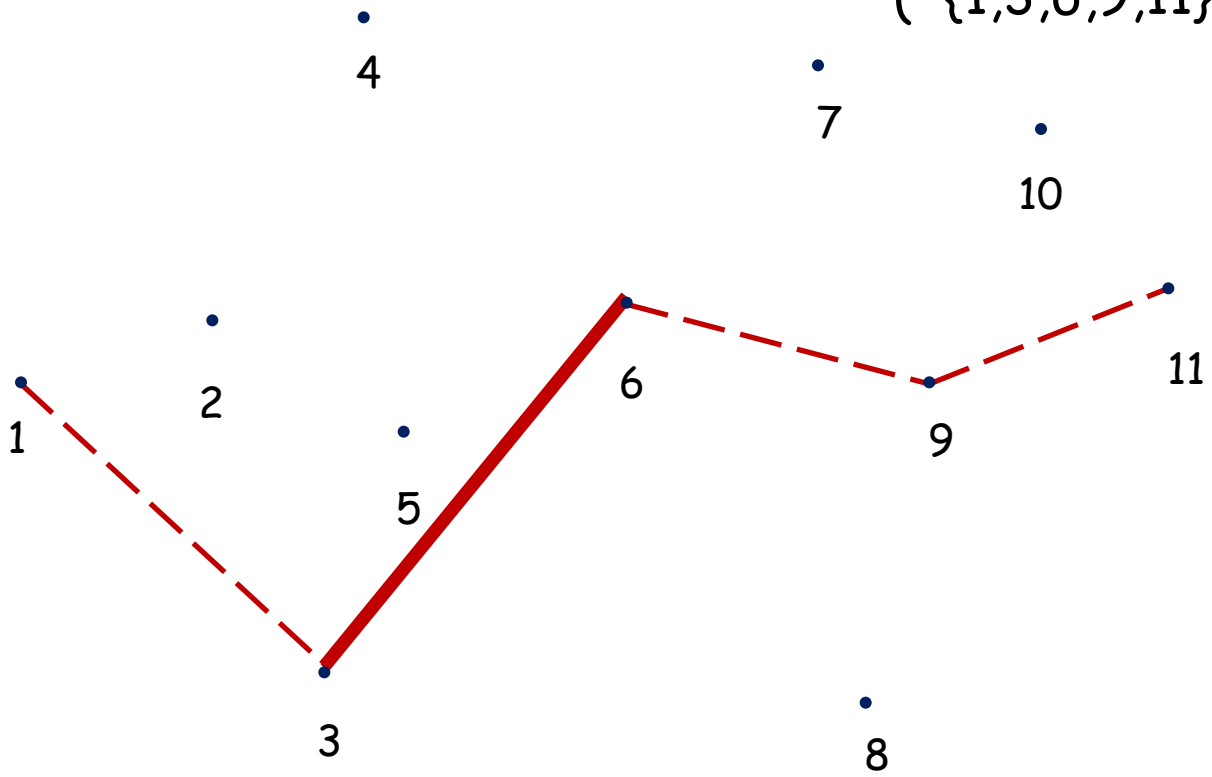
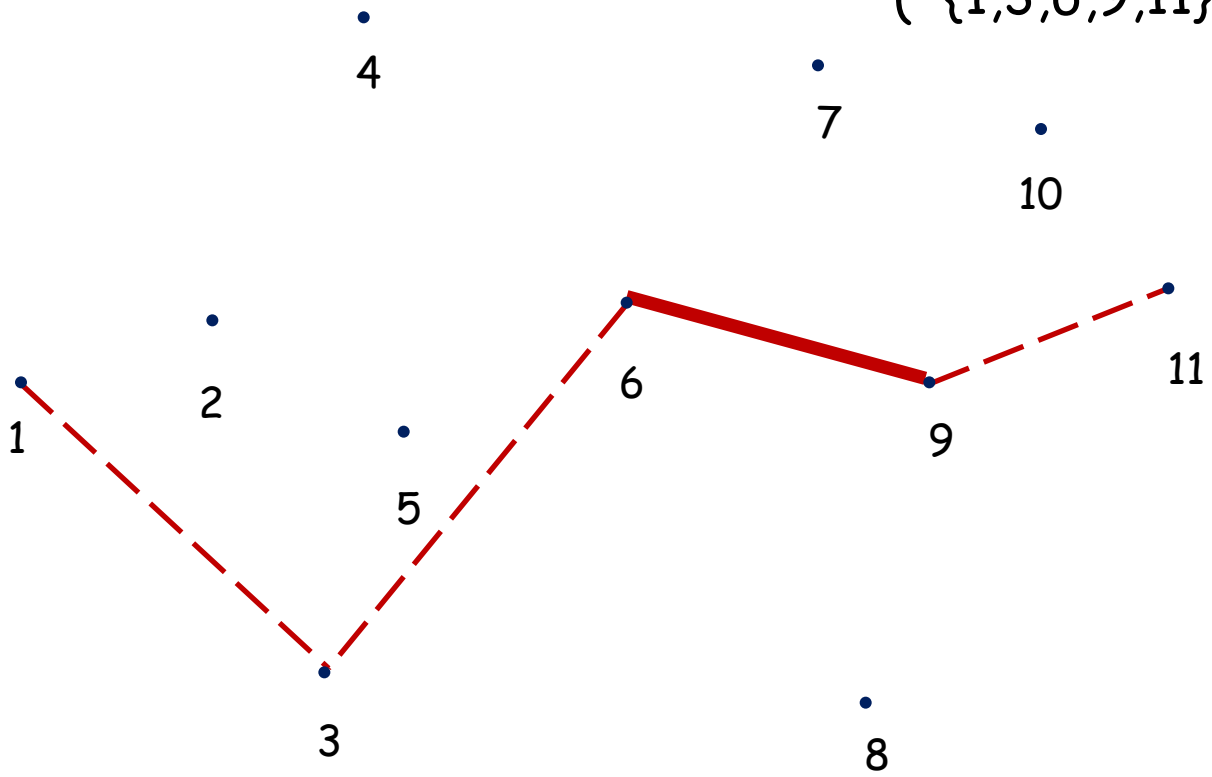


S

marked monotone chain of S

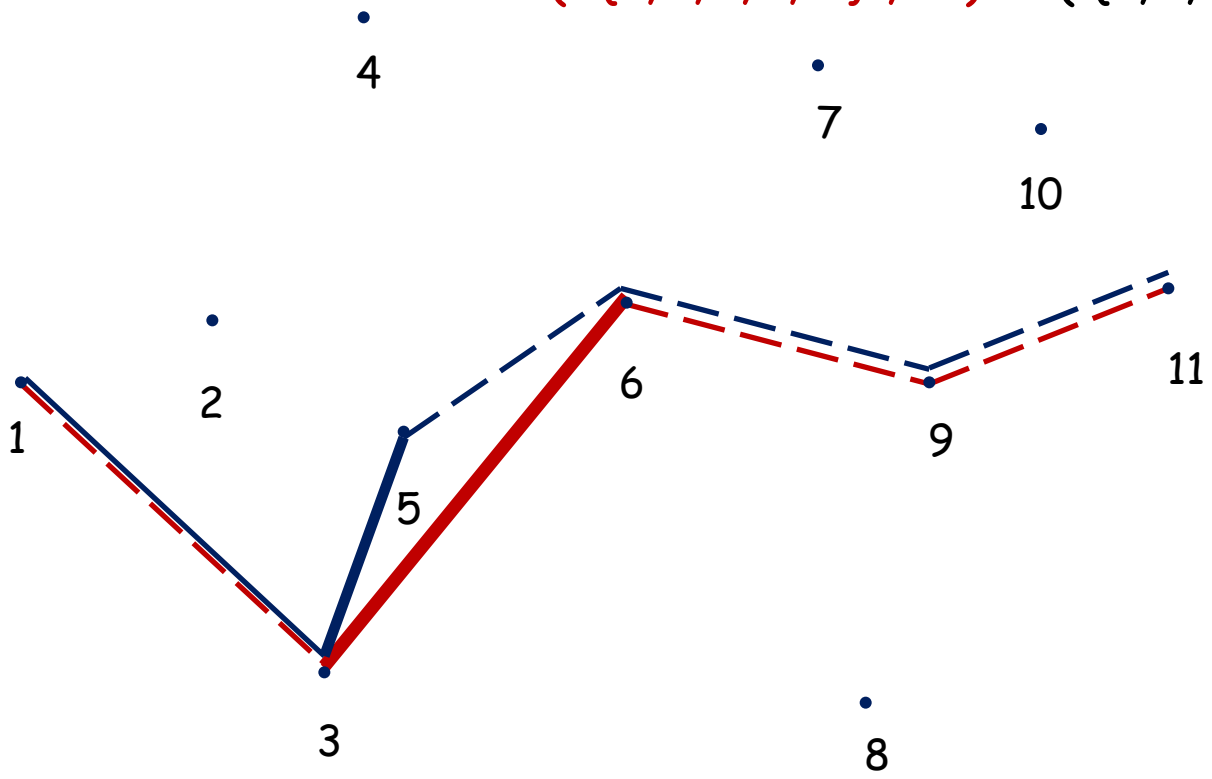
({1,3,6,9,11} , 3)



Successor relation $\prec$ on marked chains

marked monotone chain of S

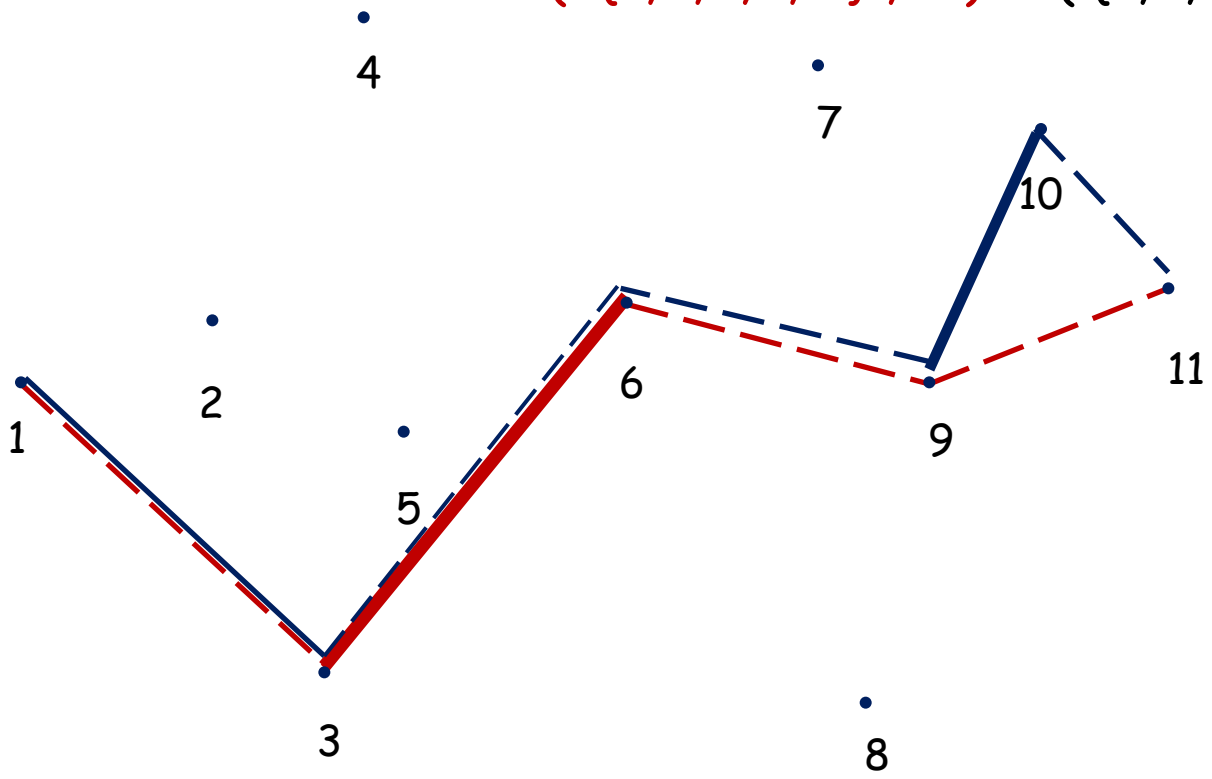
$(\{1,3,6,9,11\} , 2) \prec (\{1,3,5,8,9,11\} , 2)$



Successor relation $\prec$ on marked chains

marked monotone chain of S

({1,3,6,9,11} , 2) $\prec$ ({1,3,6,9,10,11} , 4)



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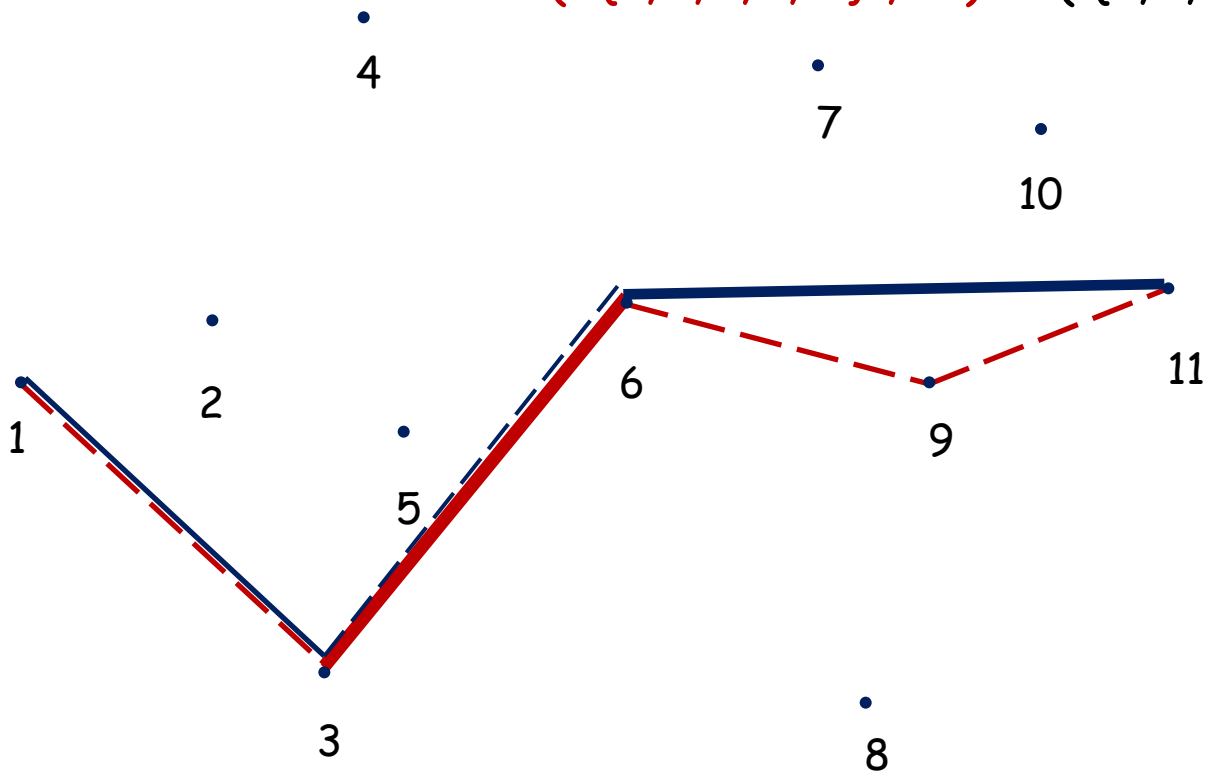

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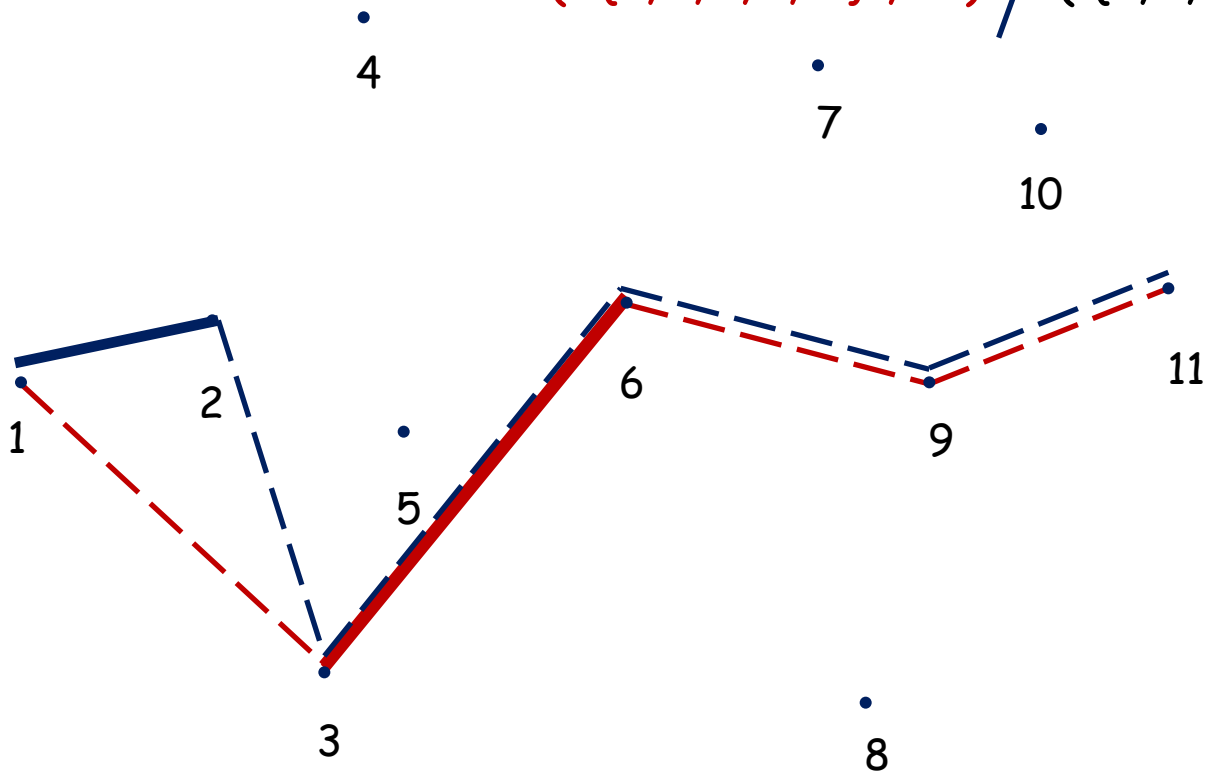
Successor relation $\prec$ on marked chains

marked monotone chain of S

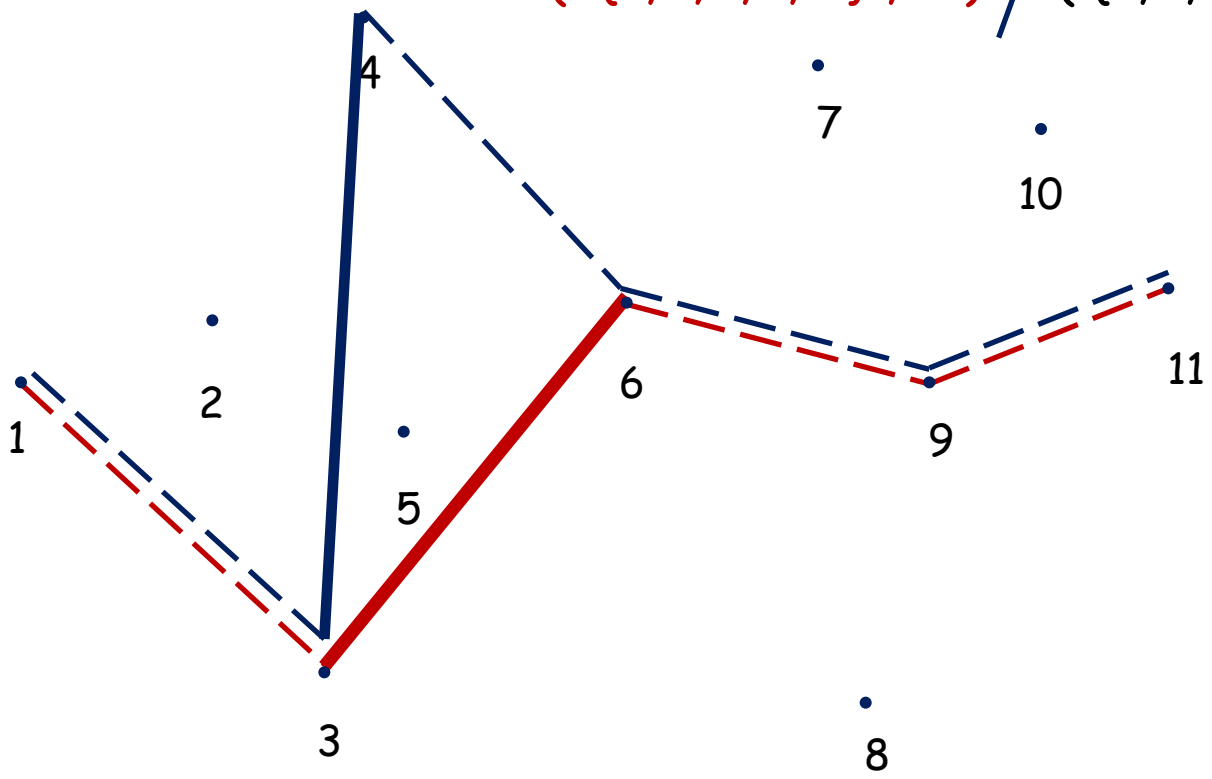
({1,3,6,9,11} , 2) $\not\prec$ ({1,2,3,6,9,11} , 2)



Successor relation $\prec$ on marked chains

marked monotone chain of S

({1,3,6,9,11} , 2) $\not\prec$ ({1,3,4,6,9,11} , 2)



Successor relation $\prec$ on marked chains

Intuition:

(C, k) is to mean that C arose in a leftmost advancing sweep, by an advance that created the k-th edge of C

the next advance must not happen to the left of the k-th edge (otherwise C would not have arisen through a leftmost advance!)

$(C, k) \prec (C', k')$ means that in a leftmost advancing sweep of some triangulation C' could have been created by an advance from C

$G(S)$ Marked monotone chain graph of S

directed acyclic graph
nodes are the marked monotone chains of S
arc from (C, k) to (C', k') iff $(C, k) \prec (C', k')$

$L \subset S$ set of points of S on the lower hull
$U \subset S$ set of points of S on the upper hull

Theorem:
The paths in $G(S)$ from $(L, 1)$ to (U, k), for any k, correspond 1 to 1 with the triangulations of S.

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Theorem:
The paths in $G(S)$ from $(L, 1)$ to TOP, correspond 1 to 1 with the triangulations of S,
where TOP is an additional vertex of $G(S)$ with arcs leading into it from (U, k) for every k.

Theorem:
The paths in $G(S)$ from $(L, 1)$ to TOP, correspond 1 to 1 with the triangulations of S.

Corollary:
The number of triangulations of S can be determined by computing the number of paths in $G(S)$ from $(L, 1)$ to TOP.

Since $G(S)$ is a DAG this can be done
in time linear in the size of $G(S)$, which is $O(2^n n^2)$
and in space linear in the number of vertices of $G(S)$,
which is $O(n2^n)$.

Implementation

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can handle random point sets up to size 50, which was out of reach before

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Very space intensive!!

- 128 Gbyte main memory machine
- do not store entire $G(S)$
- run algorithm several times on the same input,
  - count modulo different primes, and use Chinese Remaindering

Easy Generalizations

Count triangulations of S that are constrained to contain certain edges.

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Truly uniform random generation of triangulations of a given set S.

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Truly uniform random generation of triangulations of a given set S.

Fast unbiased estimation of tr(S).

Truly uniform random generation of a triangulation of S

1. Preprocessing:
   Generate $G(S)$
   with each node v store
   $p(v)$ = #paths from source to v
   with each node v store the prefix sums
   of the sequence of $p(u_1),\ldots,p(u_k)$,
   where $u_1,\ldots,u_k$ are the predecessor of v in $G(S)$

   store $Z = p(TOP) = tr(S)$

2. Sampling:
   Generate a random number Y in $\{1,\ldots,Z\}$
   Using the info computed in 1. determine the Y'th source-sink path in $G(S)$ (in reverse order)

Truly uniform random generation of a triangulation of S

Theorem:
After preprocessing that takes time $O(2^n n^2)$ and uses space $O(2^n n^2)$,
triangulations of S can be generated truly uniformly at random in
time $O(n \log n)$ per sample.

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Why is this useful ???

Estimating the proportion of triangulations of S that are **regular**

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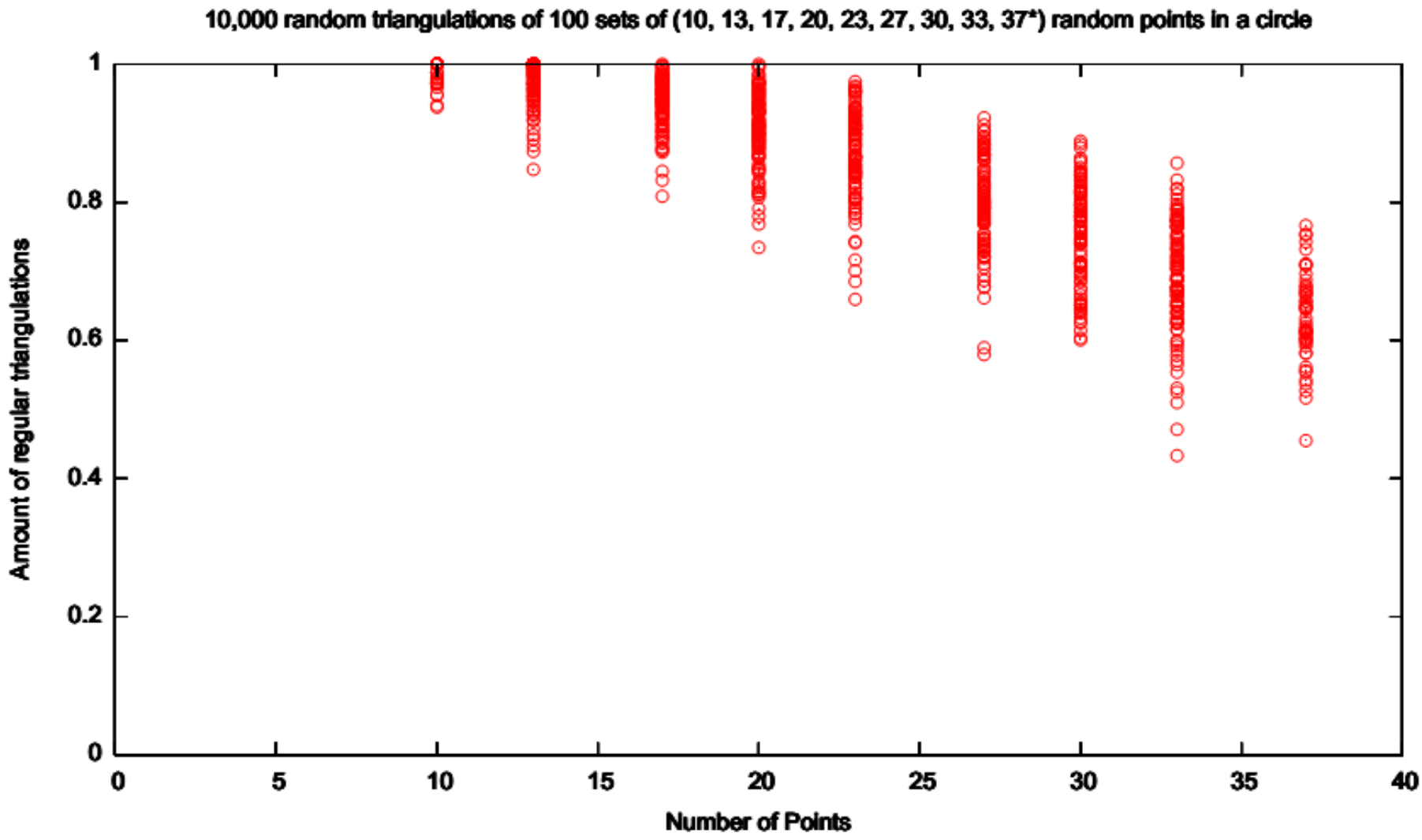
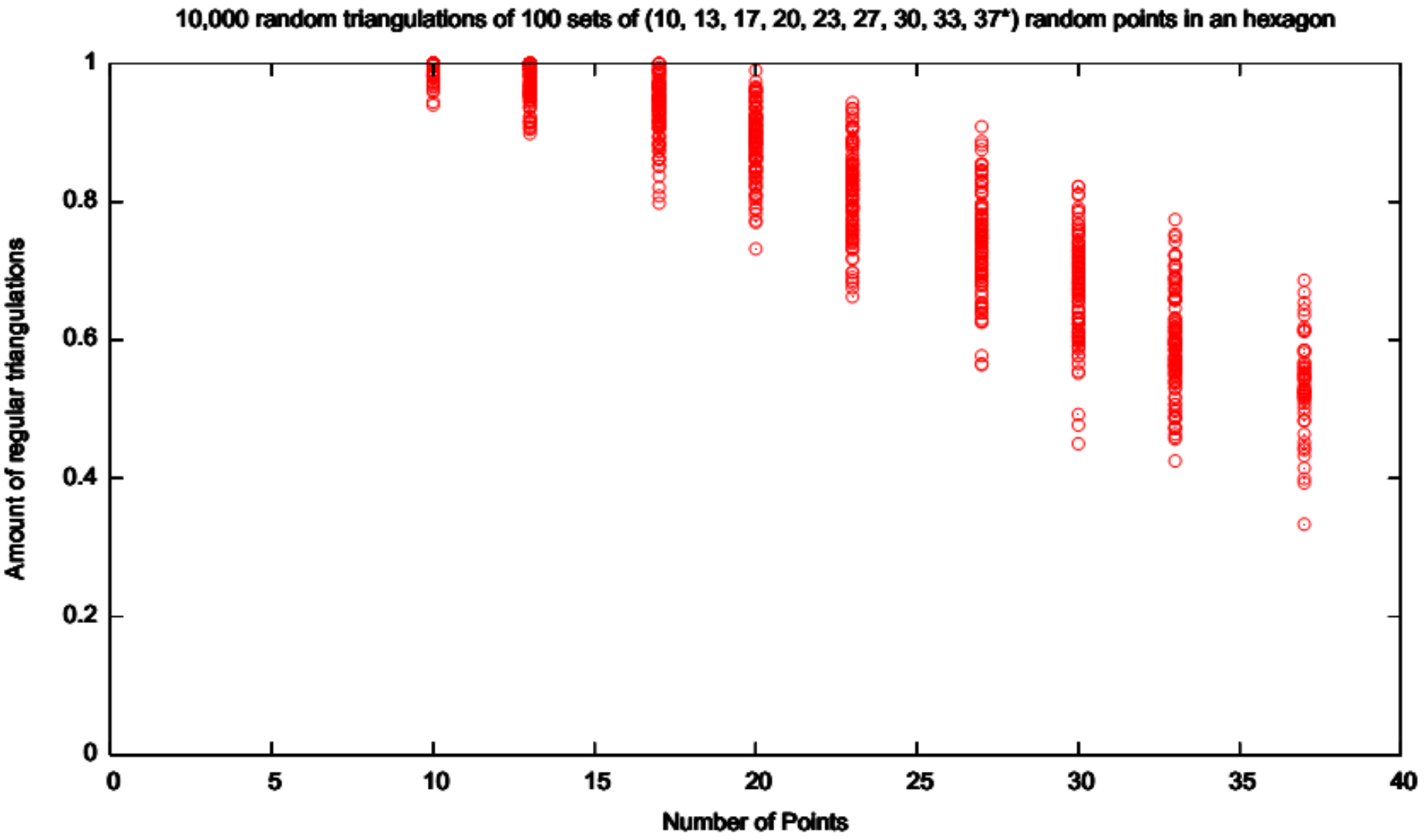
triangulation T of S is regular means

> S can be lifted to a set S′ in $\mathbb{R}^3$ so that T is the projection of the lower convex hull of S′

Regularity of a triangulation can be tested by checking a certain derived LP for feasibility.

Estimate the proportion of the triangulations of S that are regular by repeated random sampling and testing.





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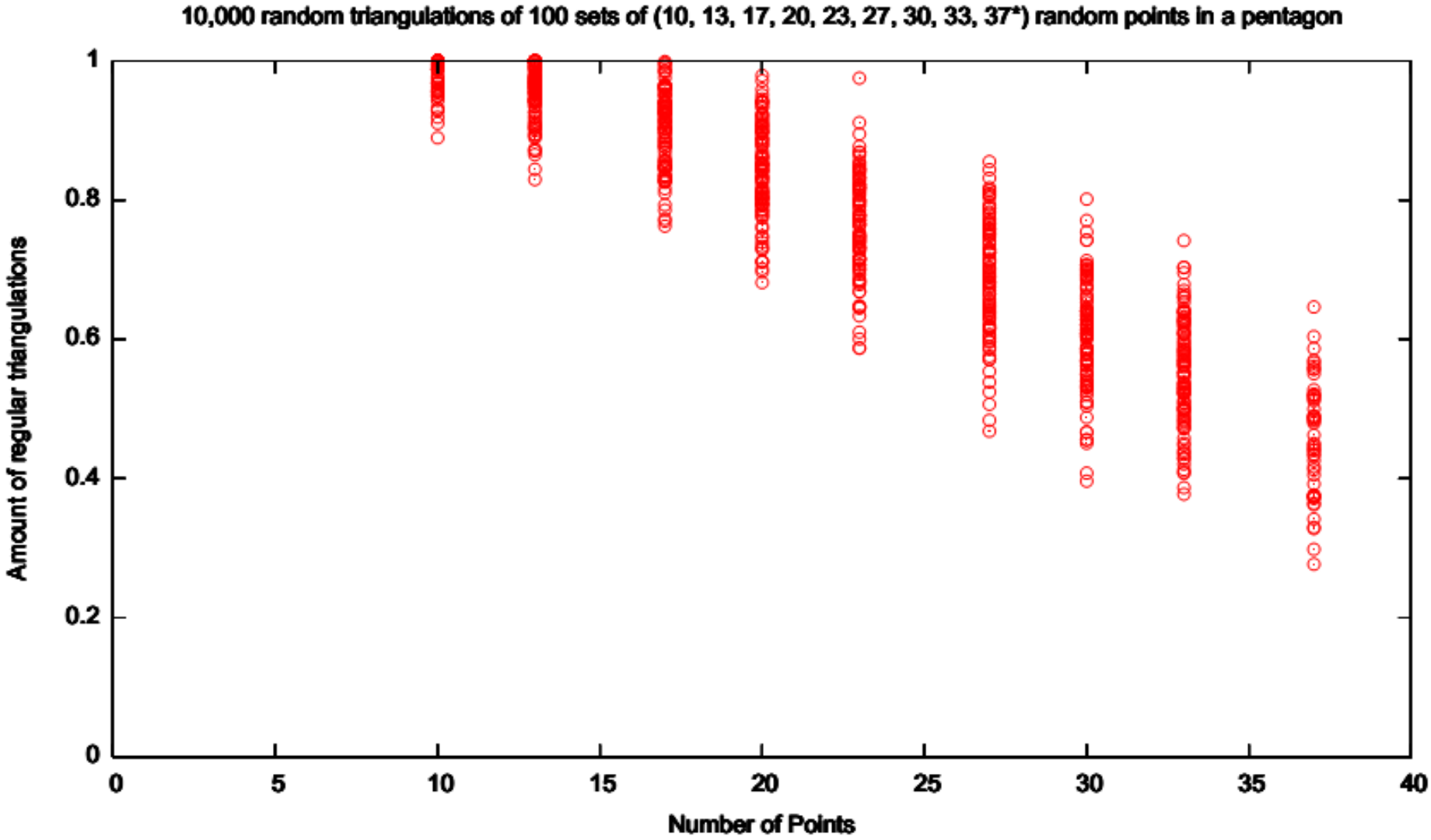

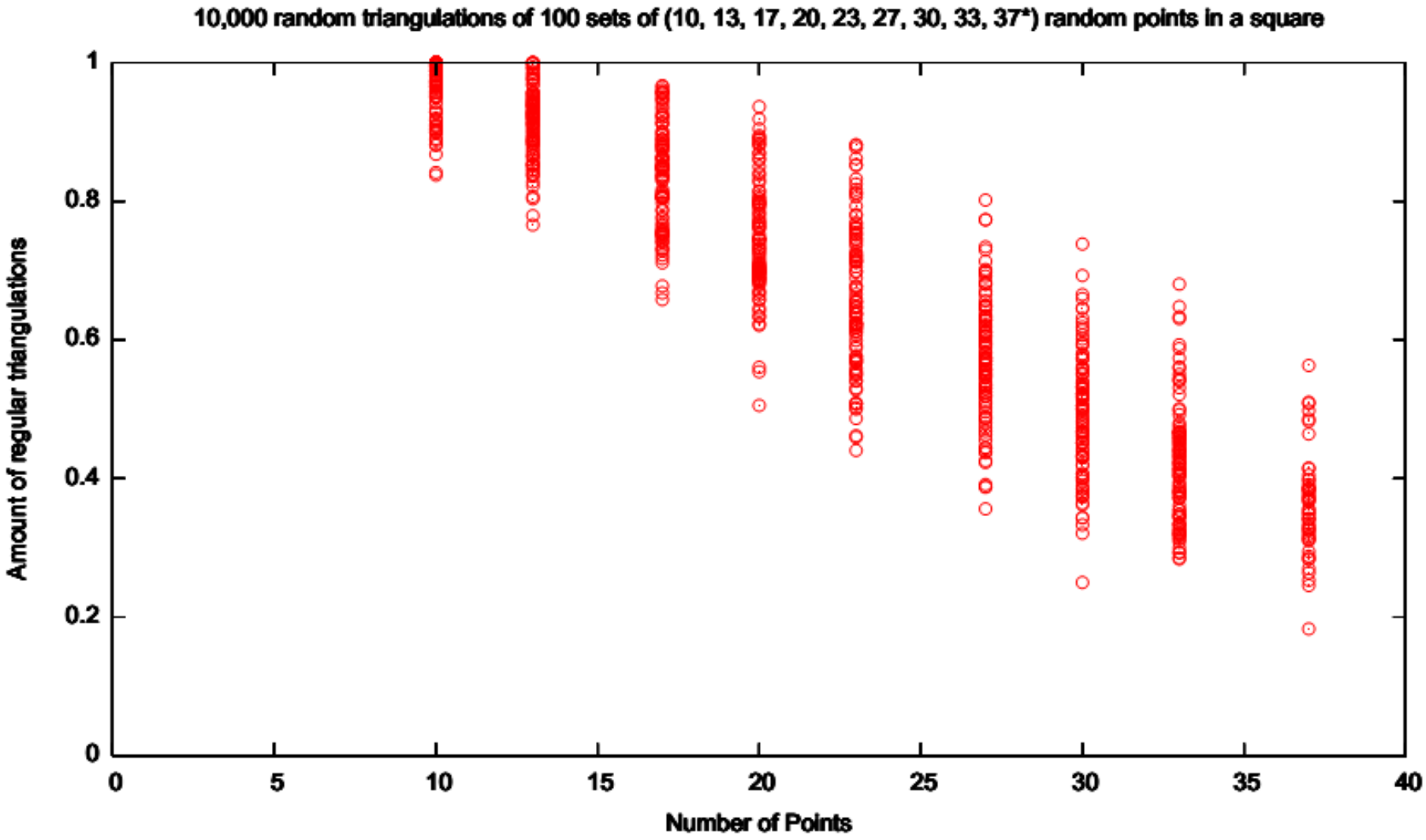

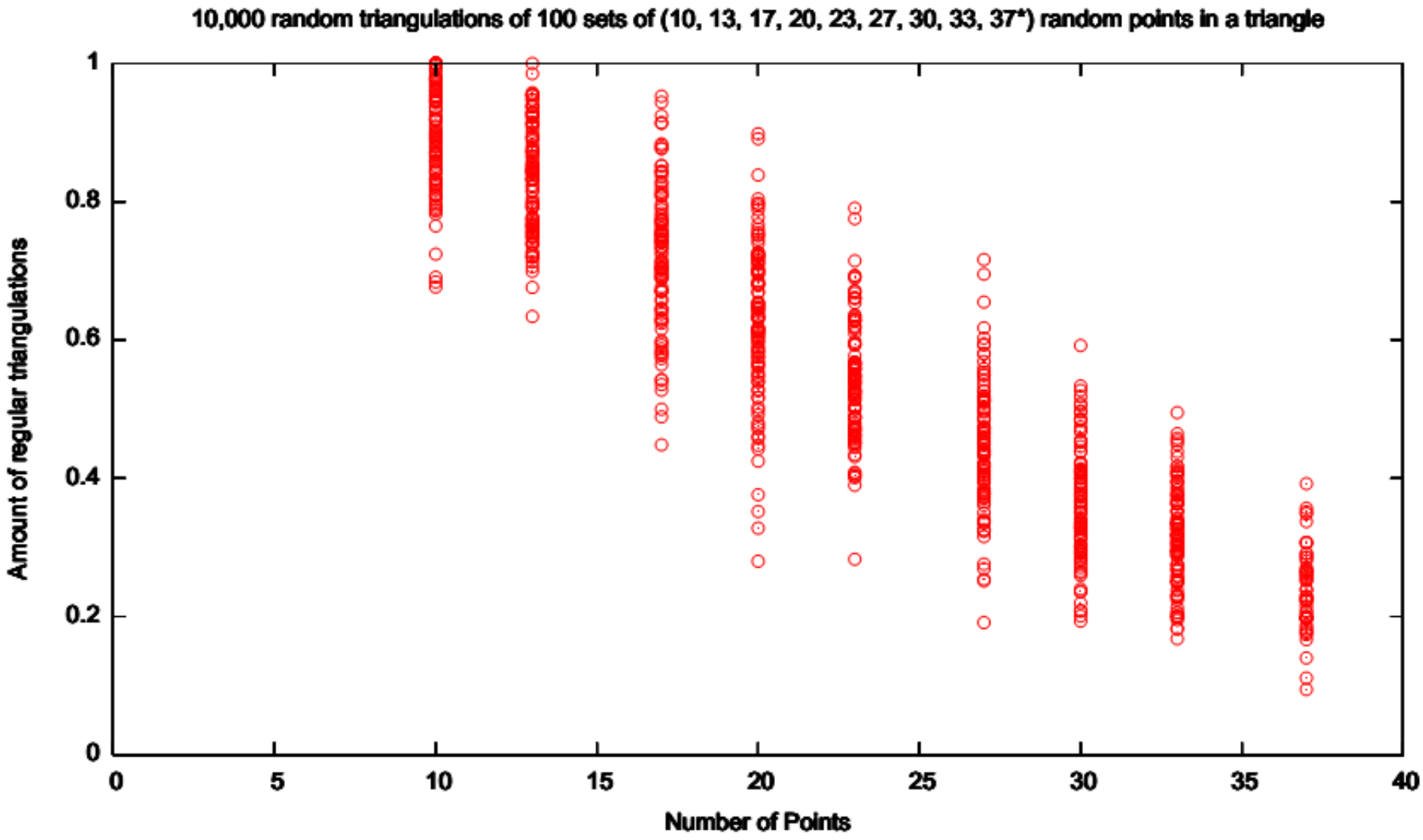

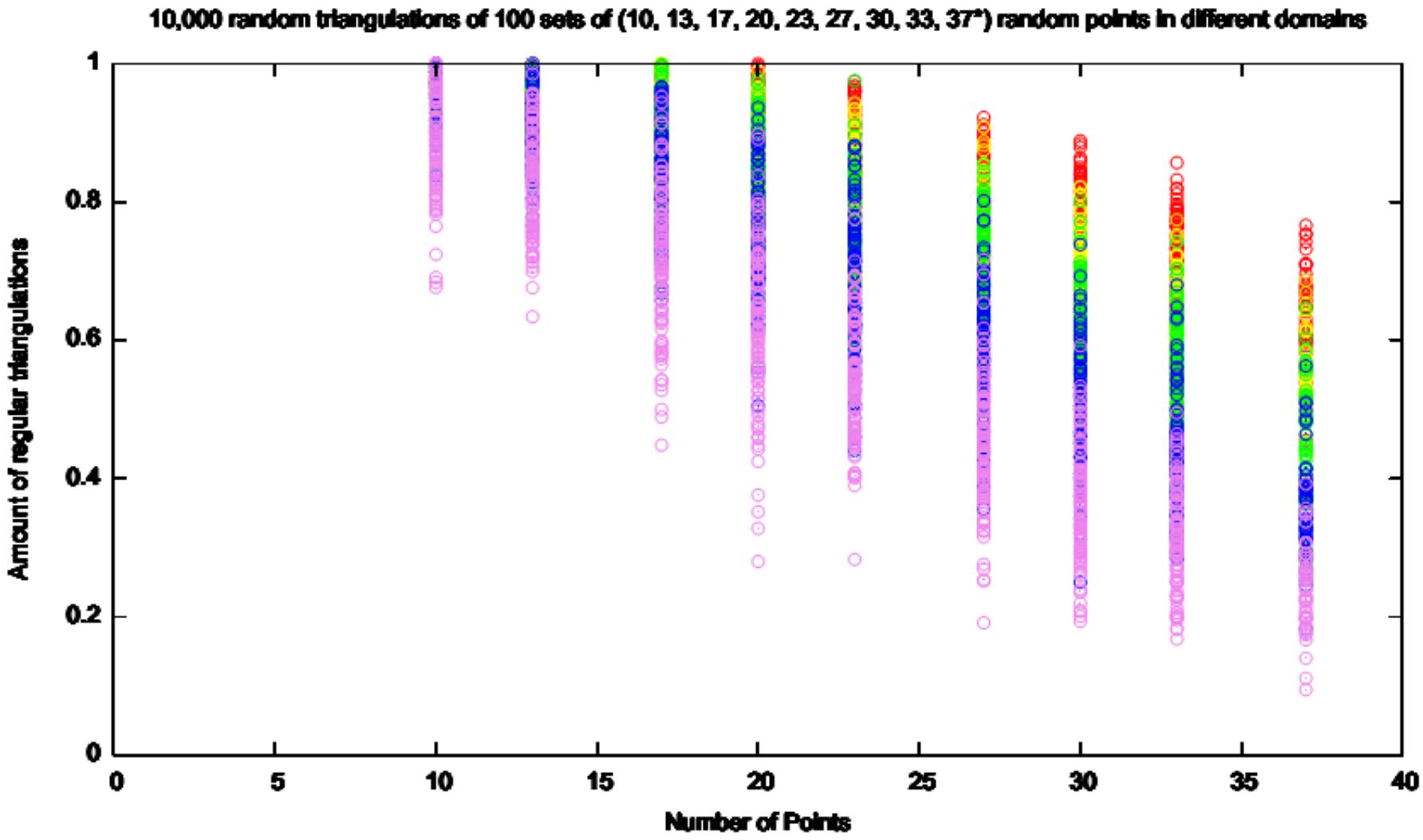

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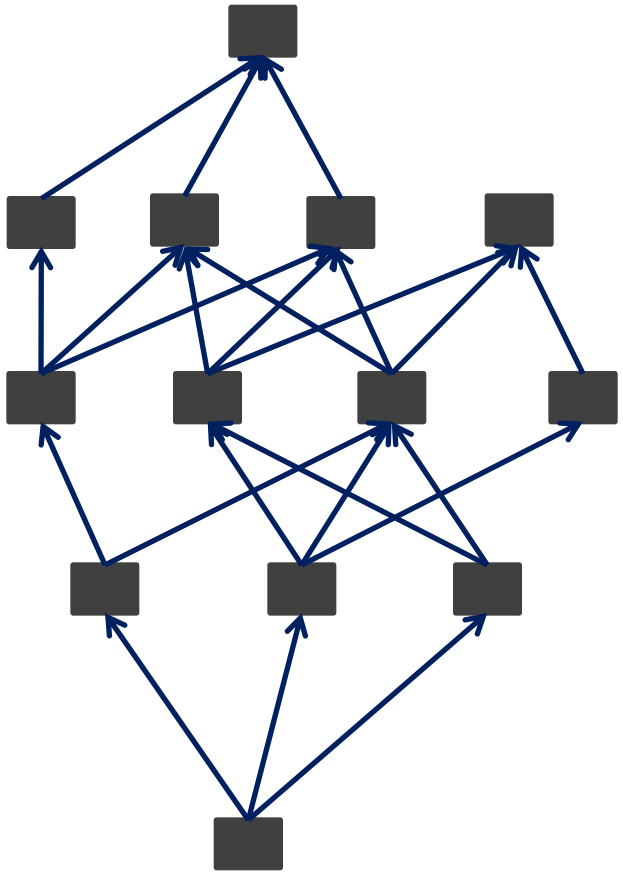
Estimating the number of source-sink paths in a graded DAG



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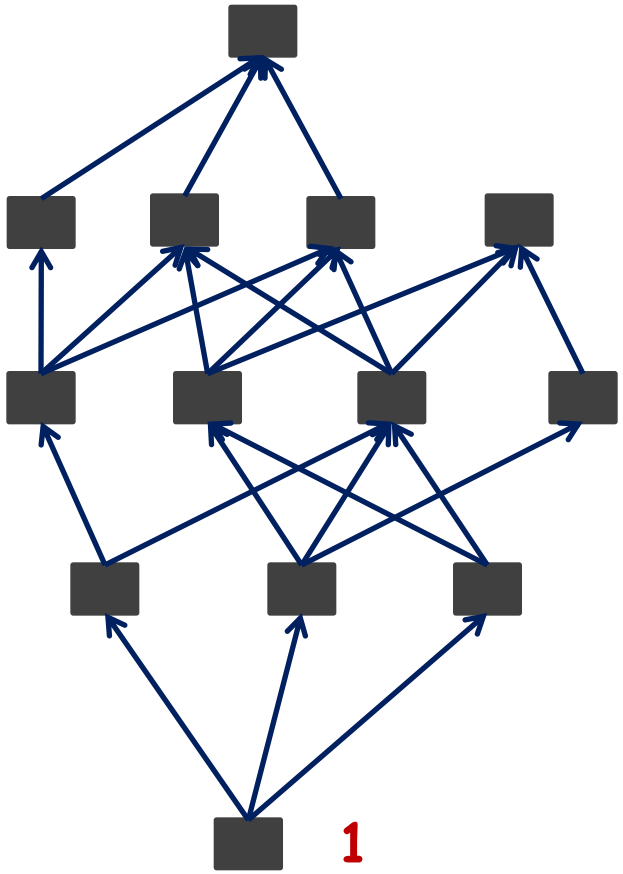

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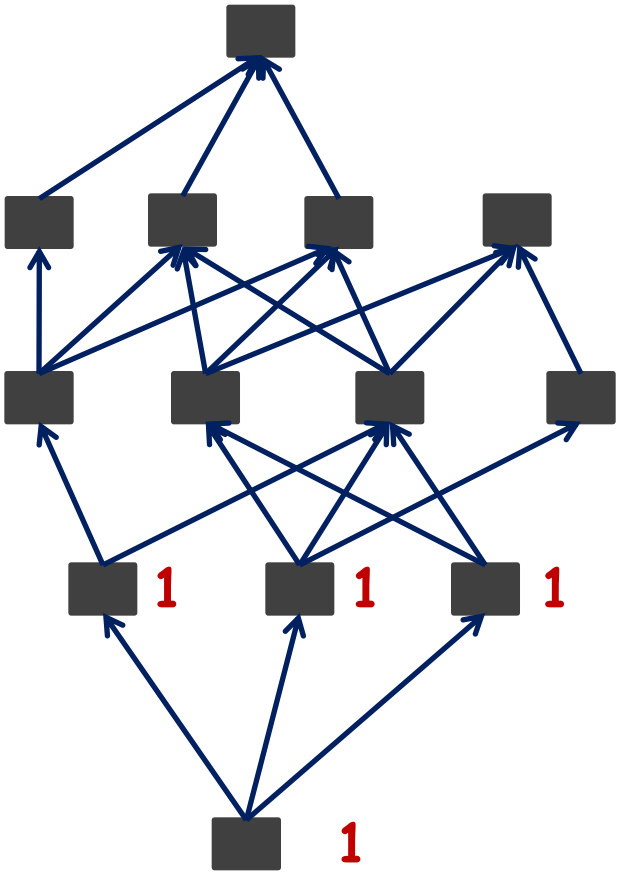

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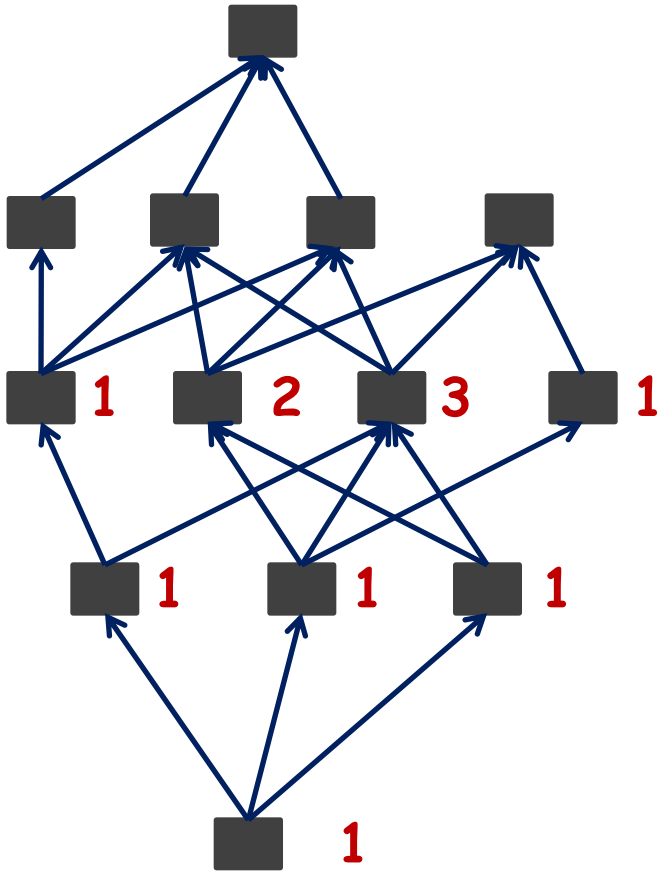

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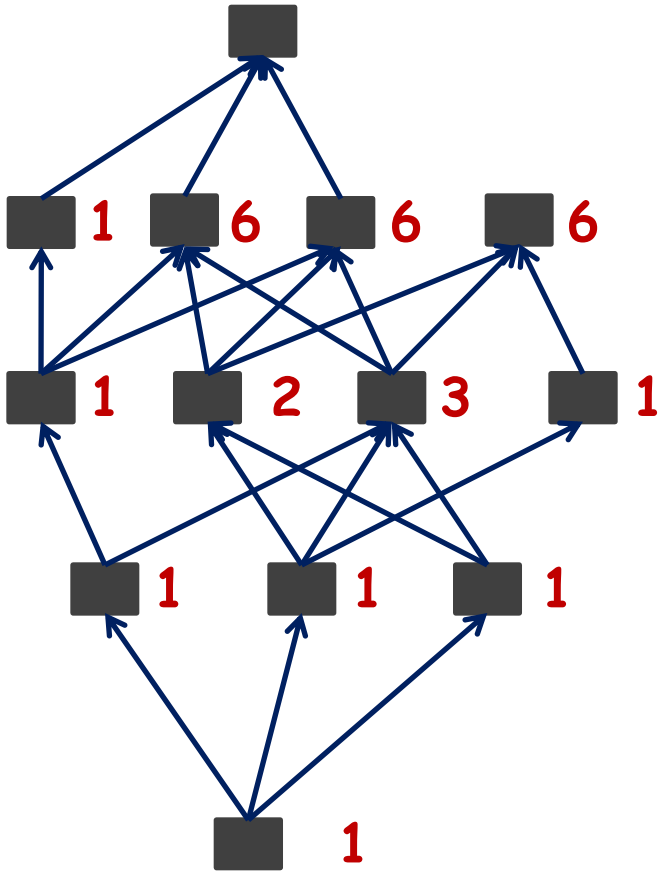

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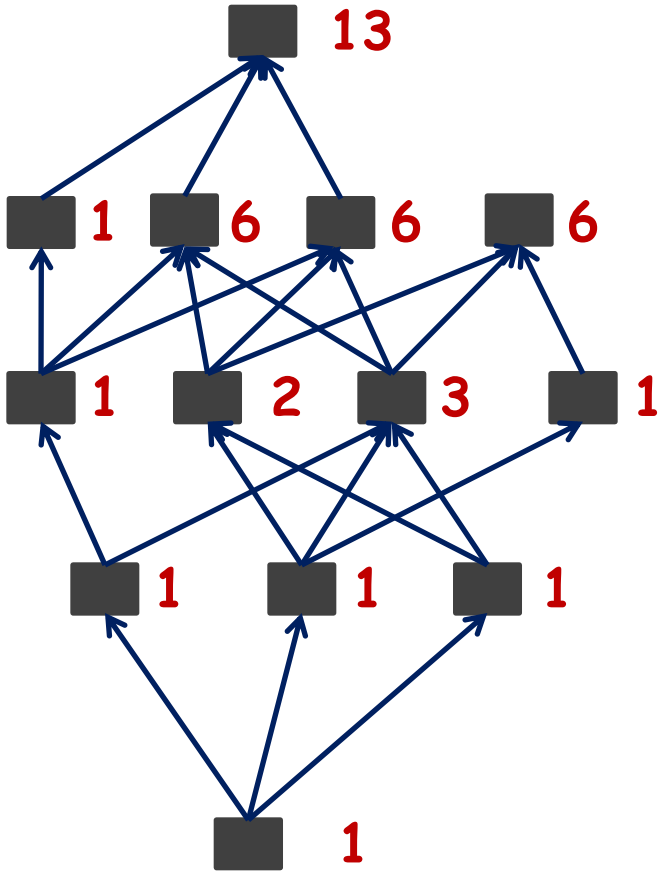

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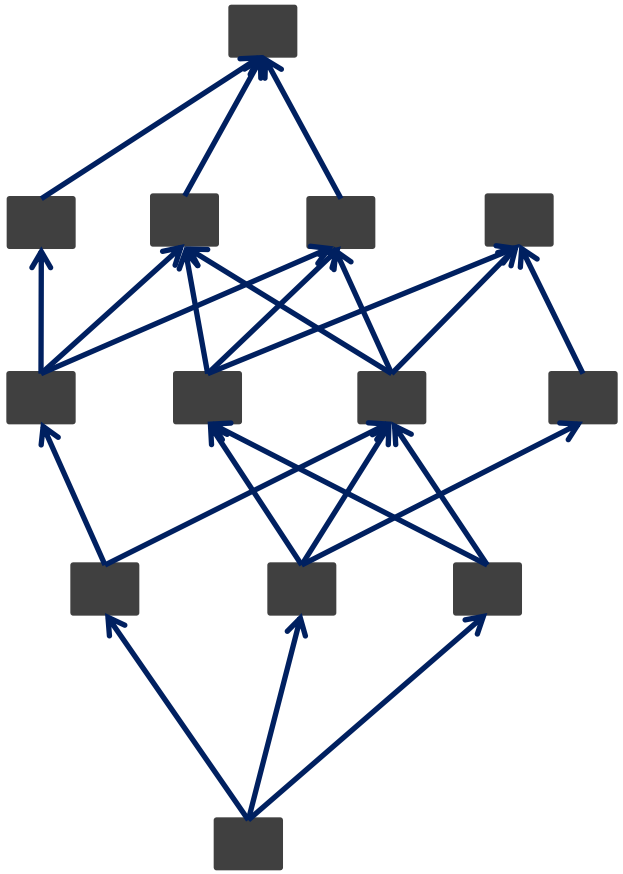

on level i keep each node independently with probability p_i

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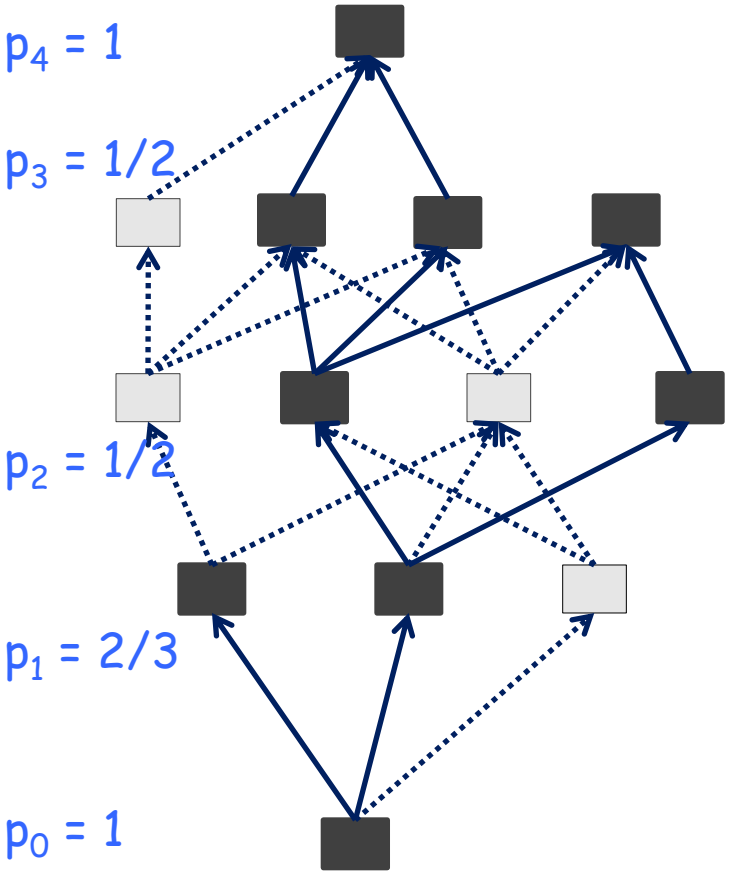

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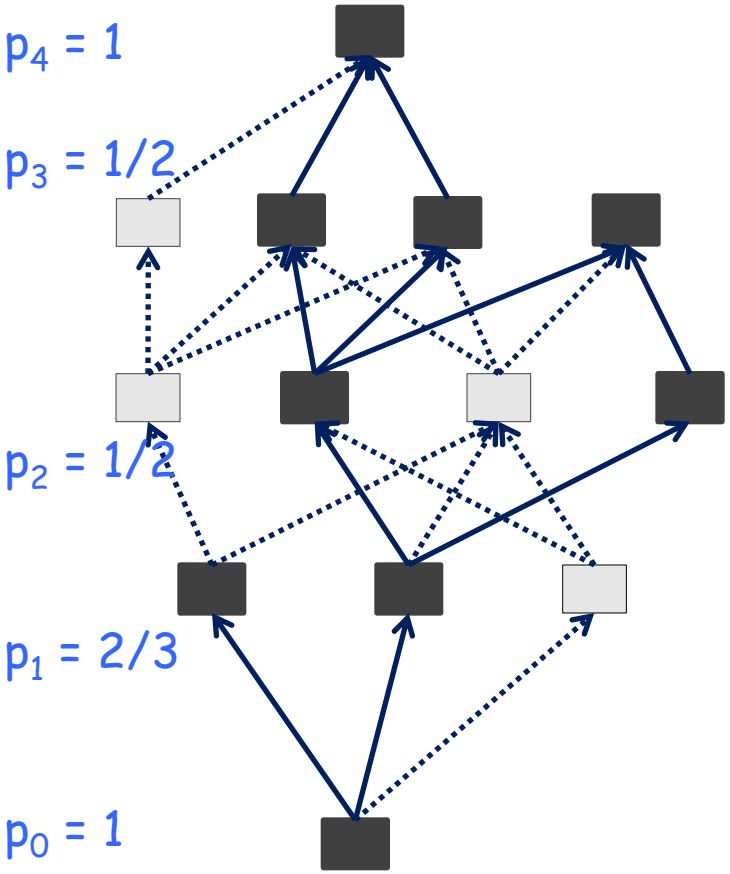


on level i keep each node independently with probability p_i

X = # surviving souce-sink paths

X_w = 1 iff path w survives, 0 ow.

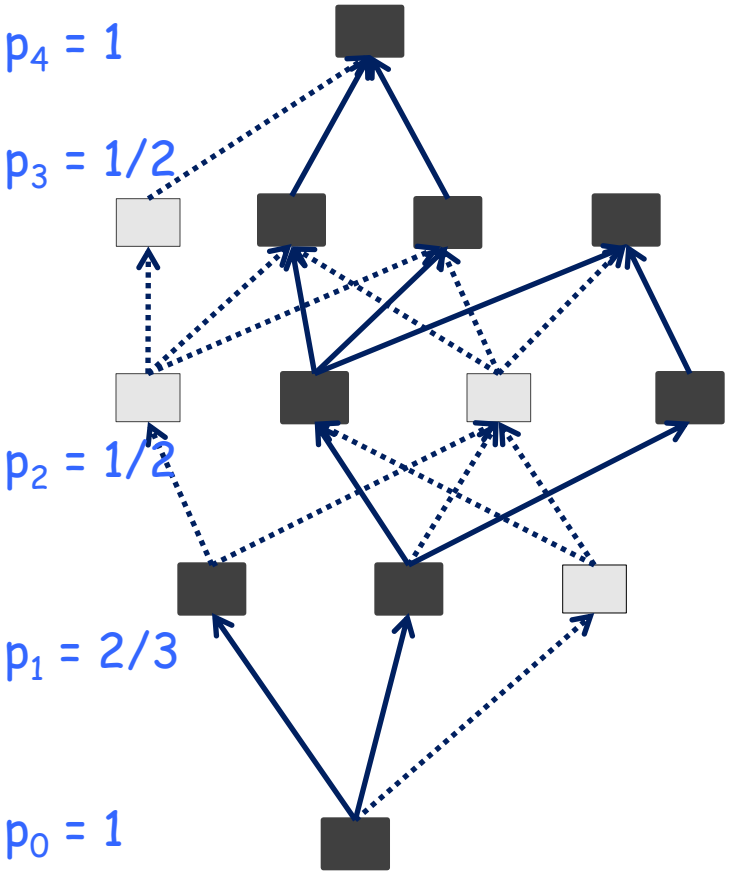
$X = \sum_w X_w$ $\qquad$ $Ex[X] = \sum_w Ex[X_w]$

$Ex[X_w]$ = Pr(path w survives) = $\prod_i p_i = P$

$Ex[X] = \sum_w P = Z\,P$

where Z total # of paths

Estimating the number of source-sink paths in a graded DAG



$Z = Ex[X] / P$

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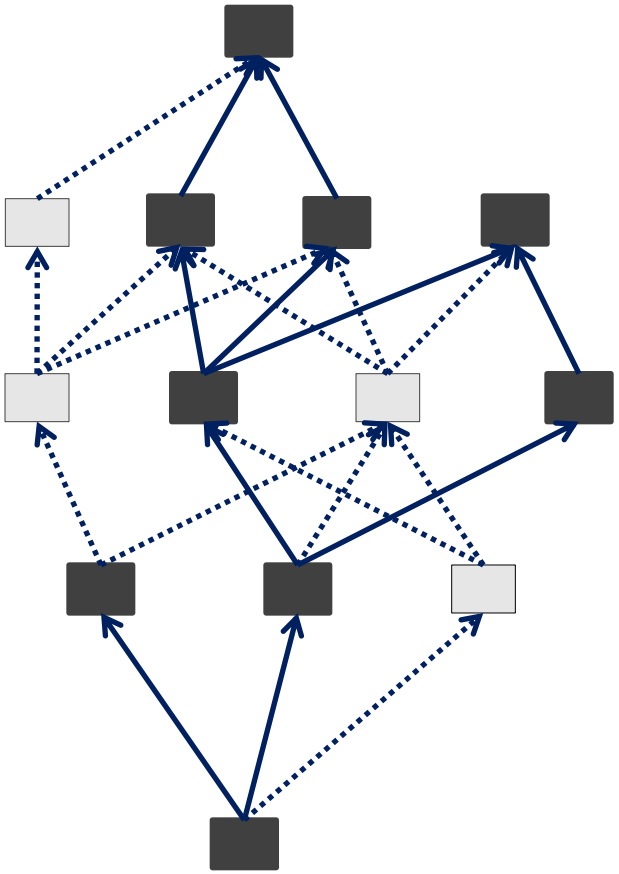

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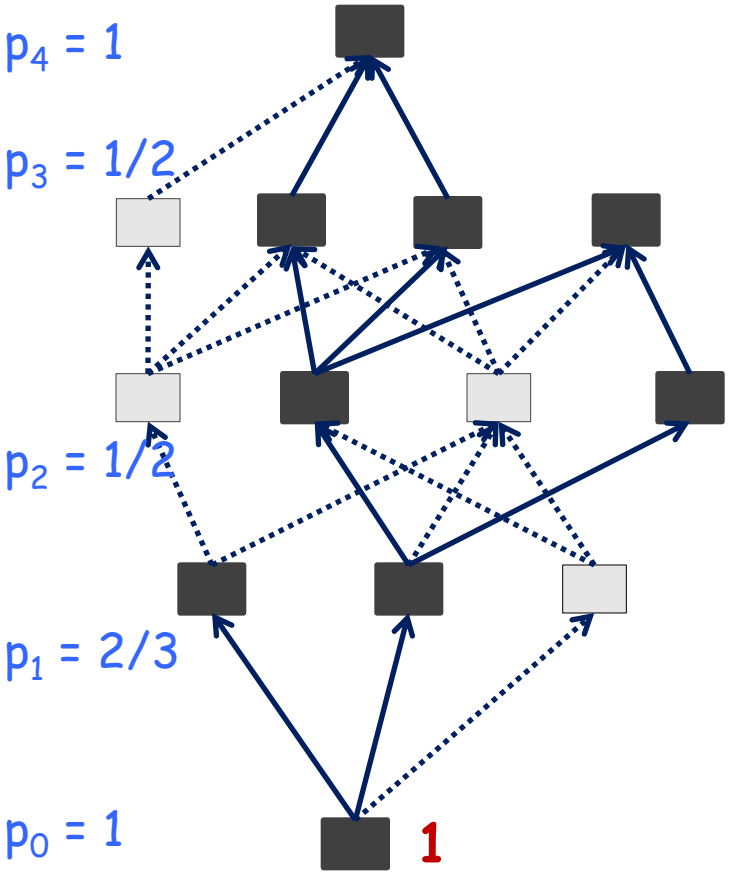

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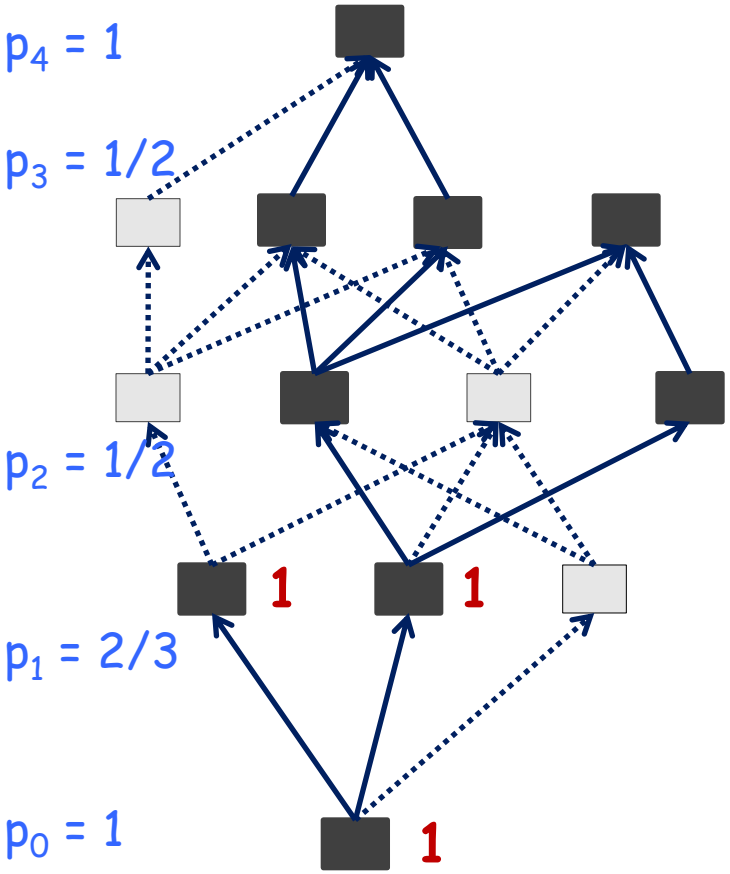

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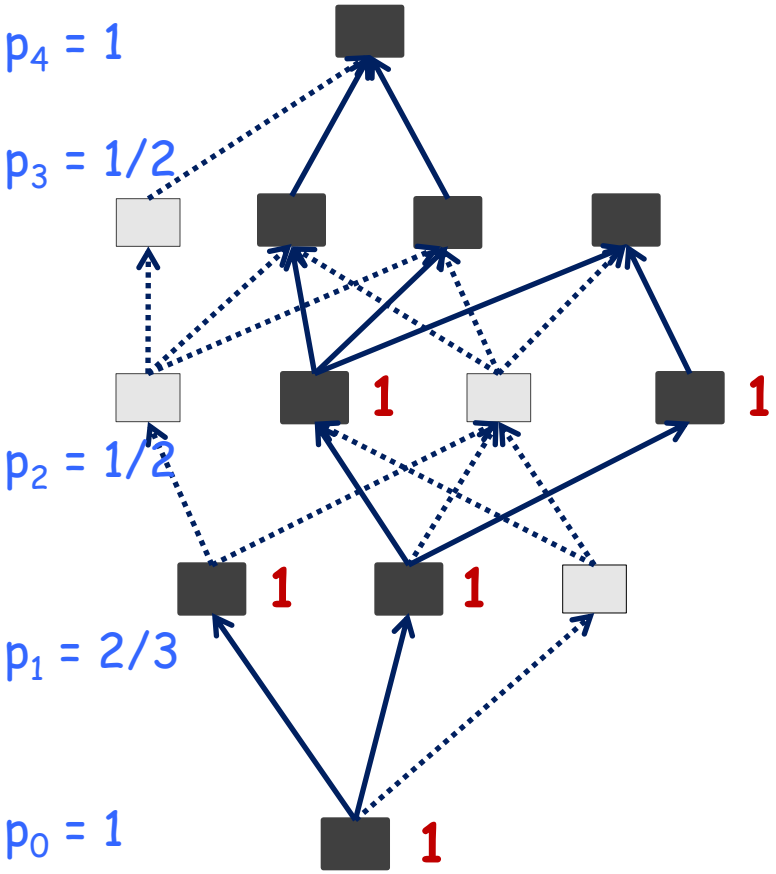

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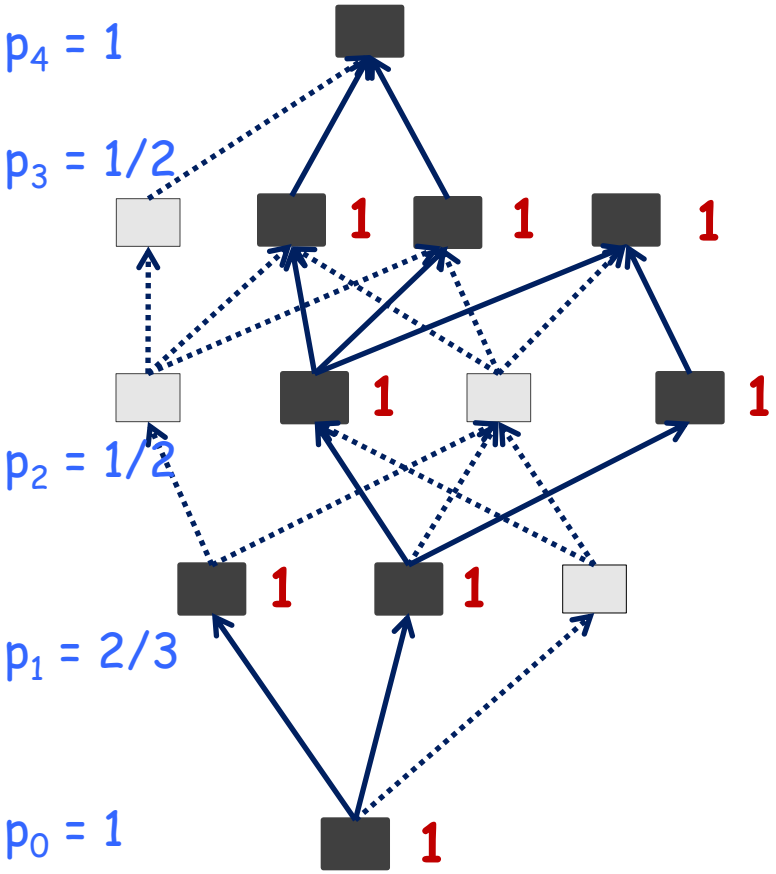
Estimating the number of source-sink paths in a graded DAG



$$Z = Ex[X] / P$$

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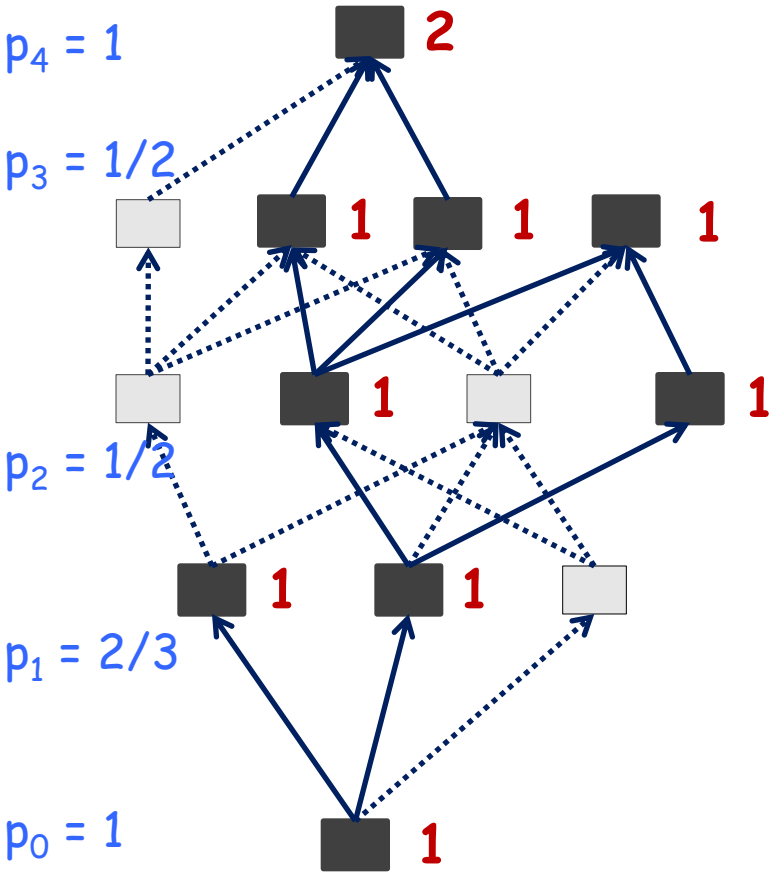
Estimating the number of source-sink paths in a graded DAG



$$Z = Ex[X] / P$$

$$P = \Pi_i p_i = 1/6$$

Estimate for Z: 2/(1/6) = 12

How good is this estimator Y = X/P?

Y is unbiased E[Y] = Z (the true number of paths)

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so far no good concentration results for Y

$$\Pr(|Y - Ex[Y]| < c\, Ex[Y]) < ???$$

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$$\Pr(|Y - Ex[Y]| < c\, Ex[Y]) < ???$$

$$\Pr(|Y - Ex[Y]| < c\, Ex[Y]) < M / c^2$$

path coupling coefficient $M = \left(\sum_{w,v} 1/p^{C(w,v)} \right) / Z^2 - 1$

$C(w,v)$ = # of common vertices of paths w and v

each $p_i = p$

50 random points

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27 hours computing time using
128 GB main memory

50 random points

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147019897942999105259582587551602 triangulations

1.470198… ee_32

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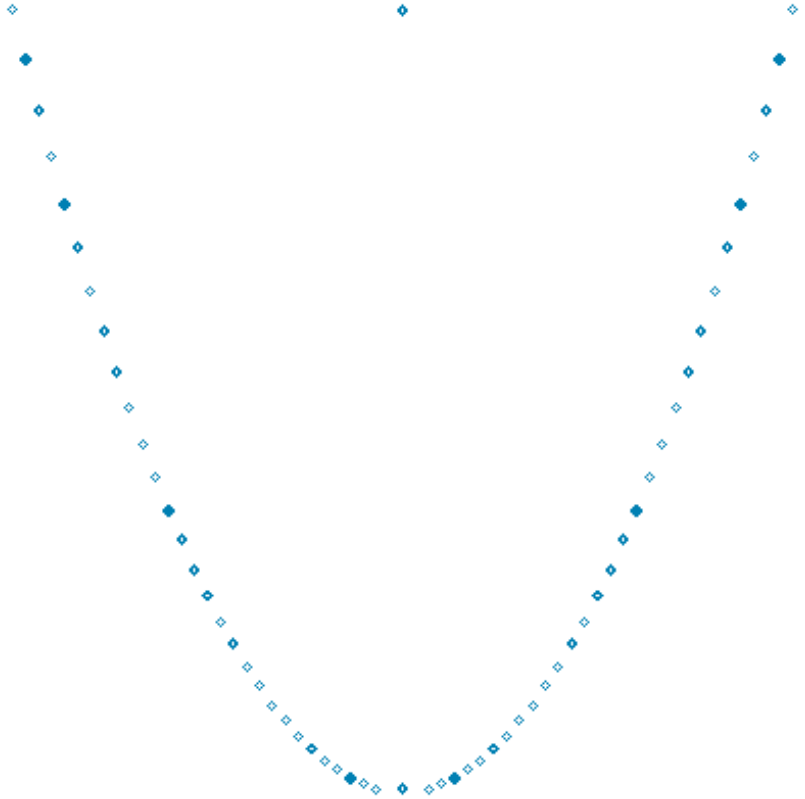
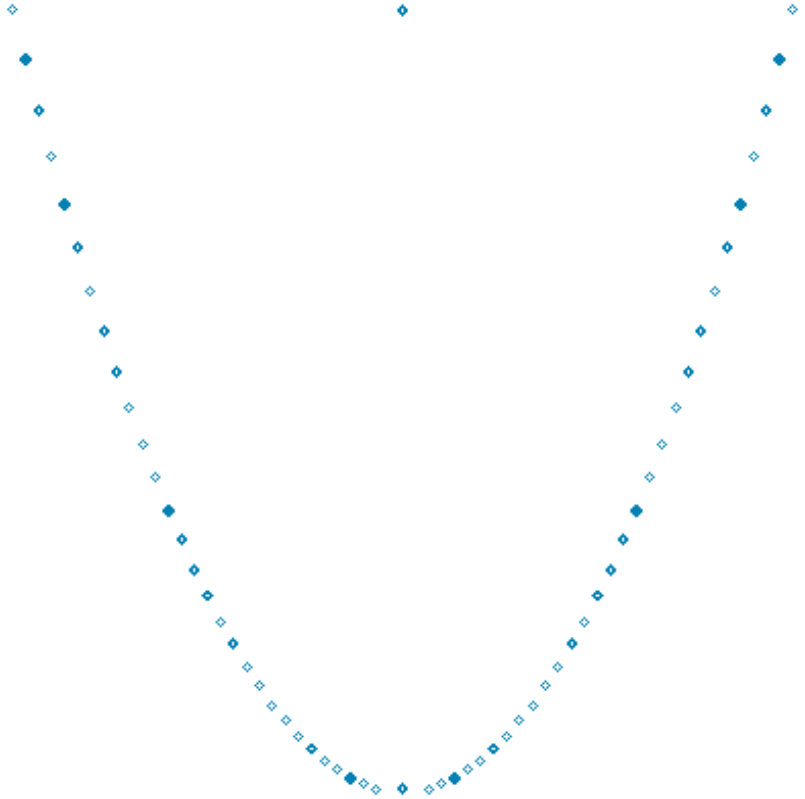


„Double circle“ of 60 points



„Double circle“ of 60 points

$$\sum_{0 \le i \le h} \binom{h}{i} (-1)^{h-i} C(h+i-2)$$

triangulations, where h=n/2,

$$C(k) = \frac{1}{k+1}\binom{2k}{k}$$

146048501608765489640471405 50 triangulations

1.46048… ee_28

„Double chain“ of 60 points

„Double chain“ of 60 points

$$C(h-2)^2\binom{2h-2}{h-1}$$

triangulations, where h=n/2 ,

$$C(k)=\frac{1}{k+1}\binom{2k}{k}$$

209156871987561535054674936116302001091878400 triang.s

2.09156… ee_45

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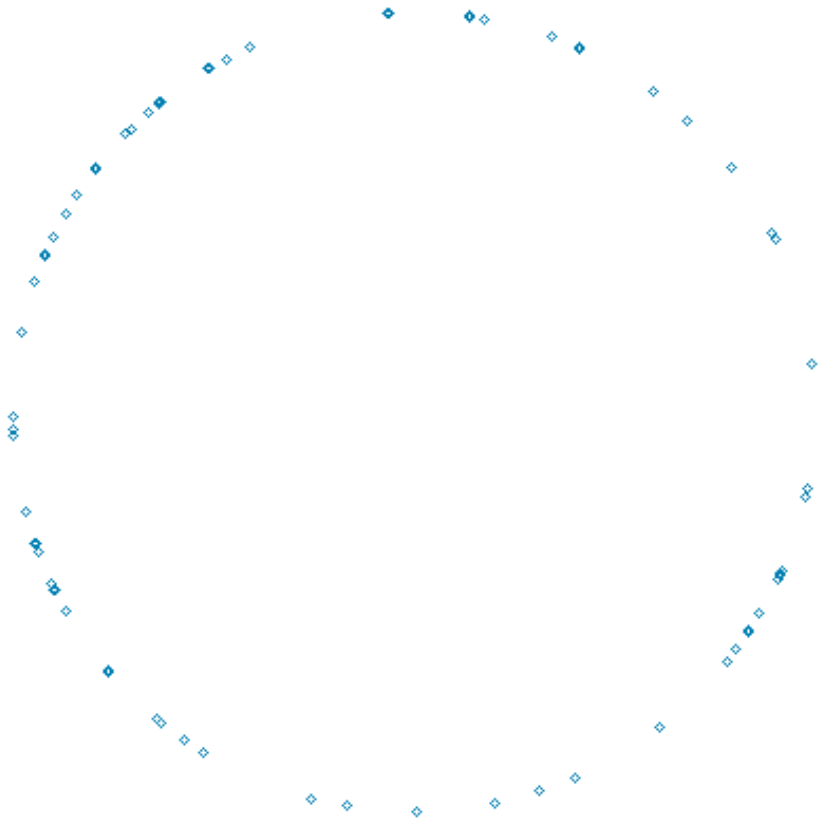
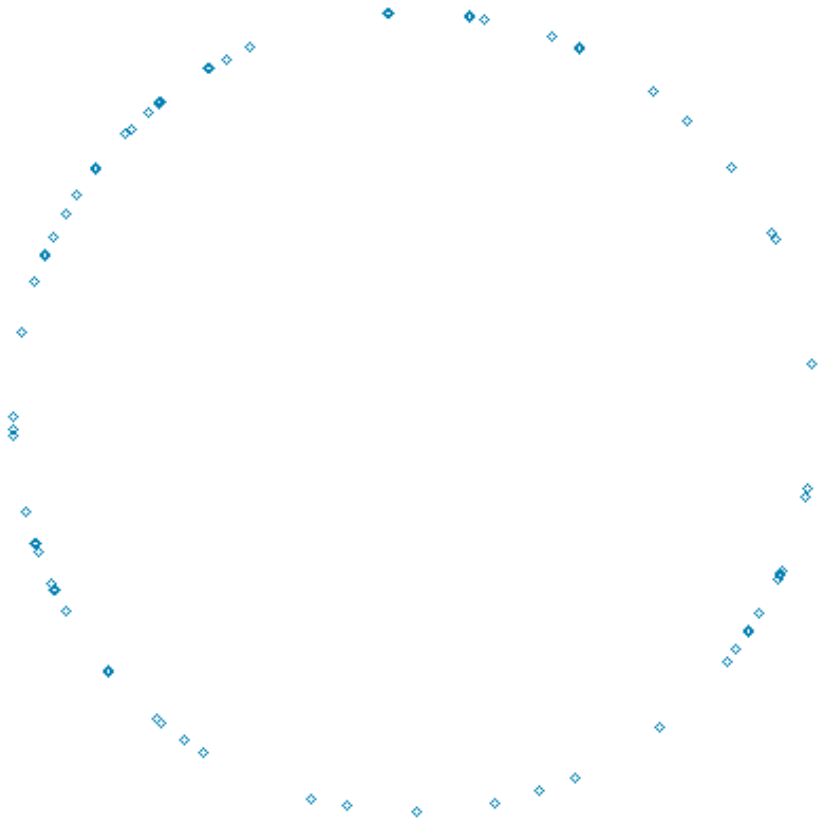


60 points in convex position



60 points in convex position

C(n-2) triangulations, where

$$C(k) = \frac{1}{k+1}\binom{2k}{k}$$

1040884602891223040334983188120 80 triangulations

1.040884... ee_32

Problems

- Find useful bounds for the path coupling coefficient for the DAGs arising in triangulation counting for

  (i) general point sets
  (ii) special classes of point sets

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  (i) general point sets
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- Design a non-averaging strategy for obtaining a guaranteed approximator from the path counting estimator for

  (i) general point sets
  (ii) special classes of point sets

Problems

- Design efficient methods or heuristics recognizing dead end nodes in the DAGs arising in triangulation counting.

  (recognize marked sweep chains that cannot be completed to a triangulation)

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- Design efficient methods or heuristics recognizing dead end nodes in the DAGs arising in triangulation counting.

  (recognize marked sweep chains that cannot be completed to a triangulation)

- Find other counting problems for which DAG path estimation can be profitably employed. (matching?)

- Where else has DAG path estimation been used, and under what name?

Easy Generalizations

Count triangulations of S that are constrained to contain certain edges.

Truly uniform random generation of triangulations of a given set S.

Fast unbiased estimation of tr(S).

Not so easy Generalizations

Counting non-crossing matchings, plane geometric graphs, spanning cycles, ….

Manuel Wettstein, MSc Thesis, ETH, April 2013
SOCG 2014

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Not so easy Generalizations

Counting non-crossing matchings, plane geometric graphs, spanning cycles, ….

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Talk this Friday at 10:30


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