The least-squares partial-matching Voronoi diagram

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*Freie Universität Berlin
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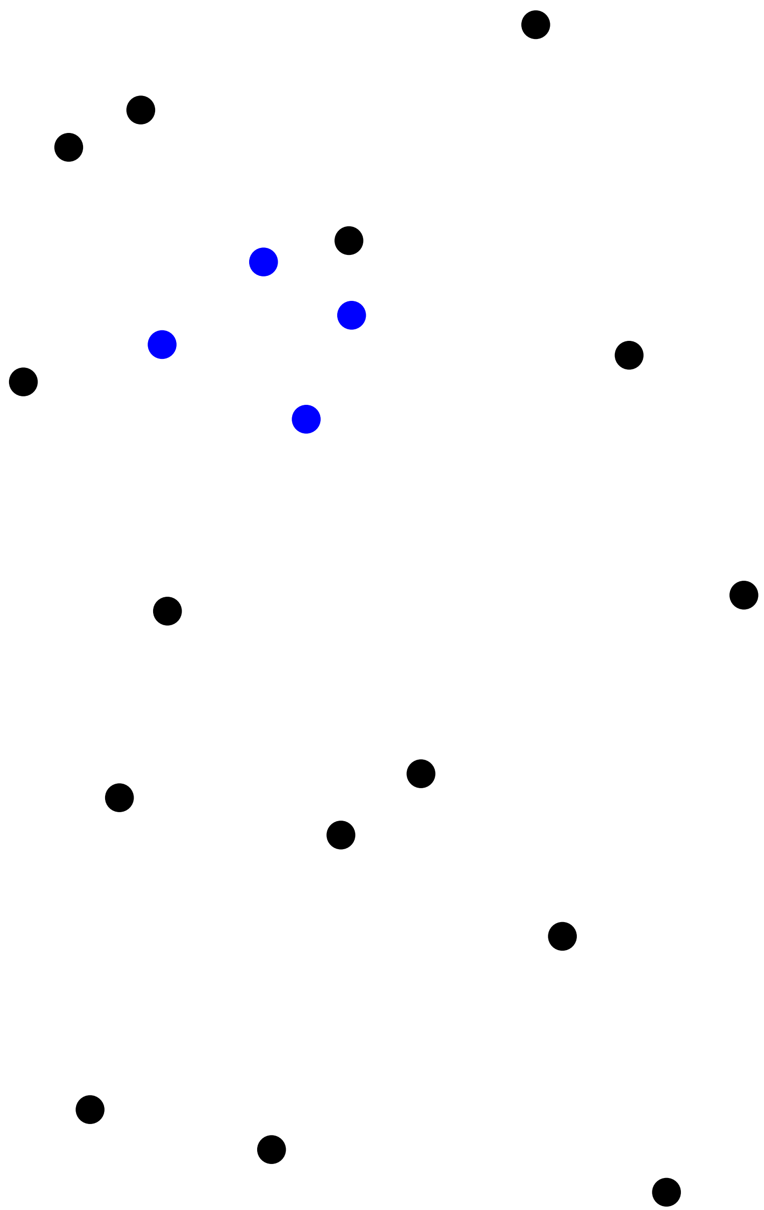
least-squares partial-matching Voronoi diagram



$A \subset \mathbb{R}^d$

$B \subset \mathbb{R}^d$

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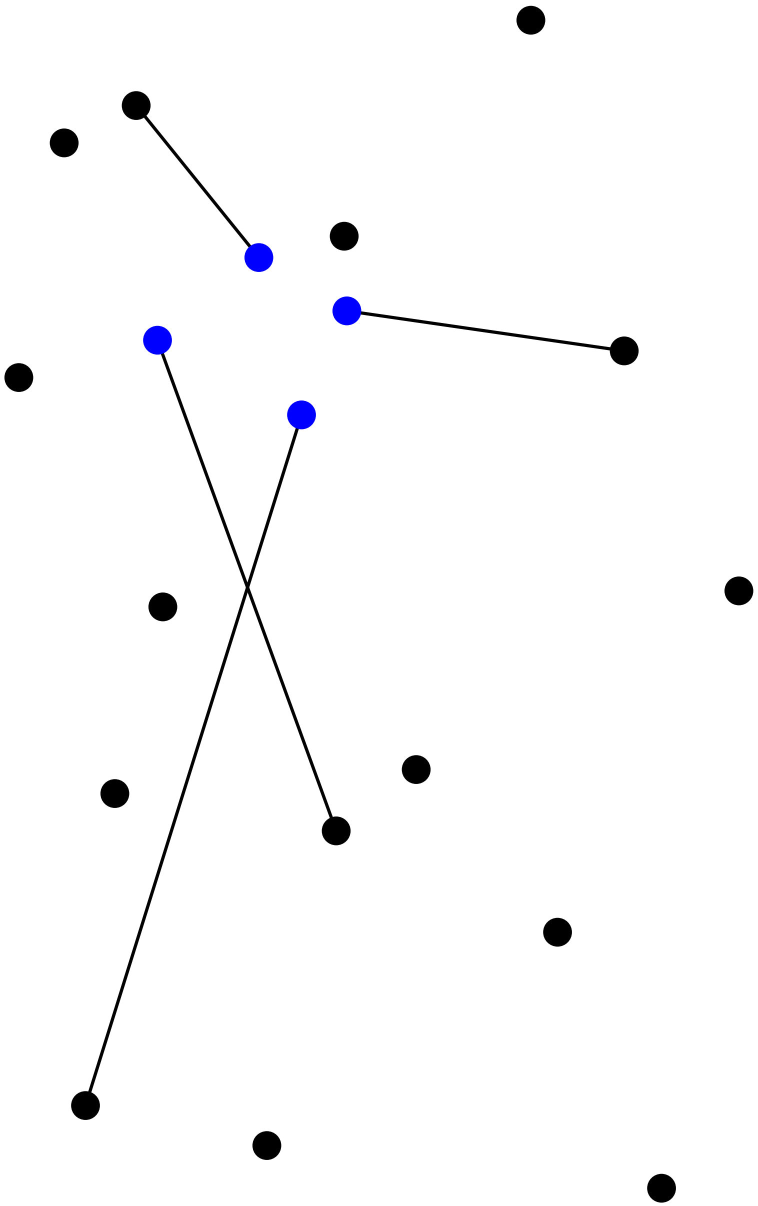

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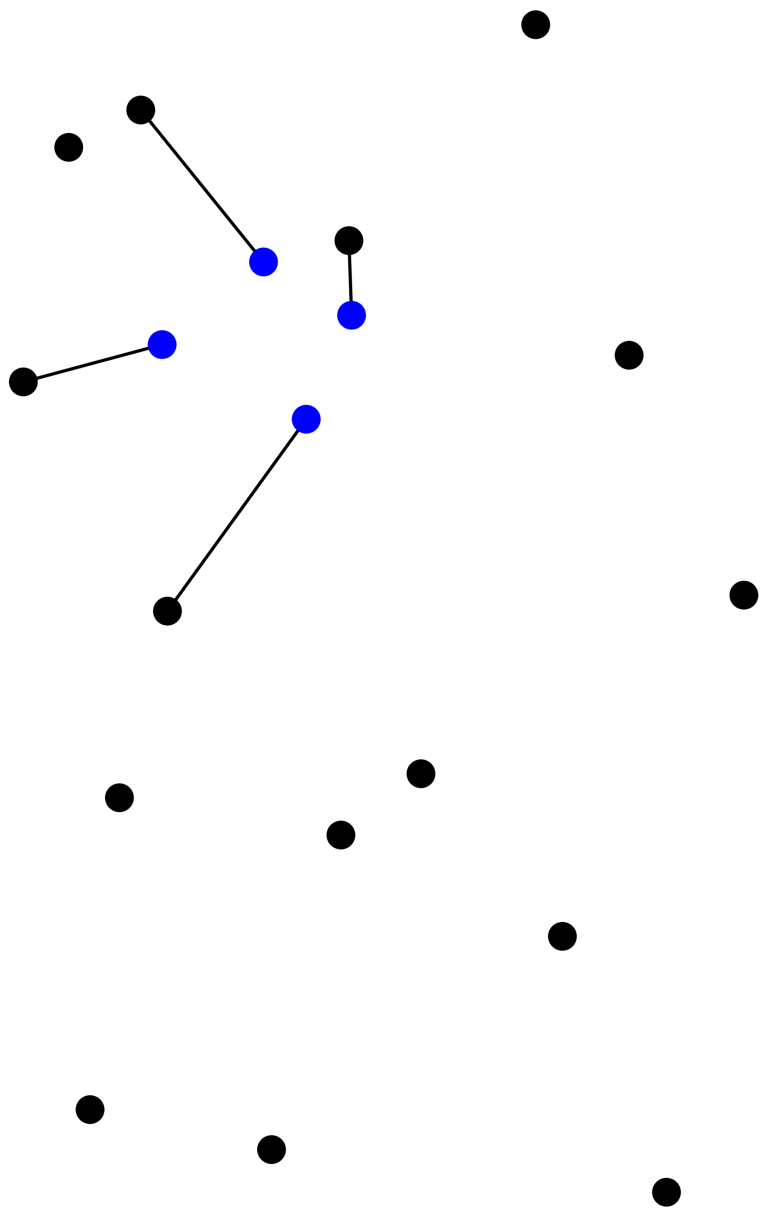

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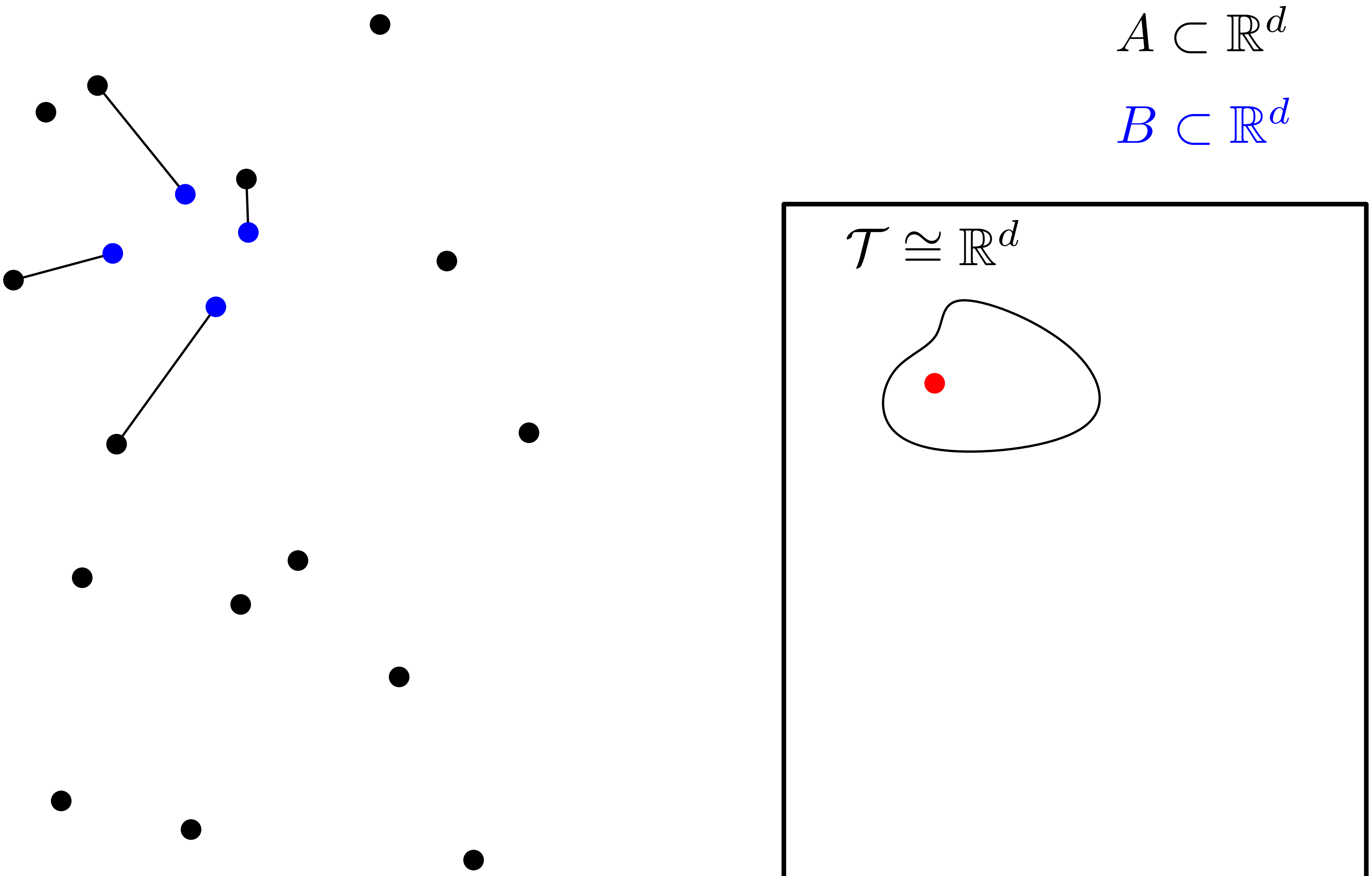

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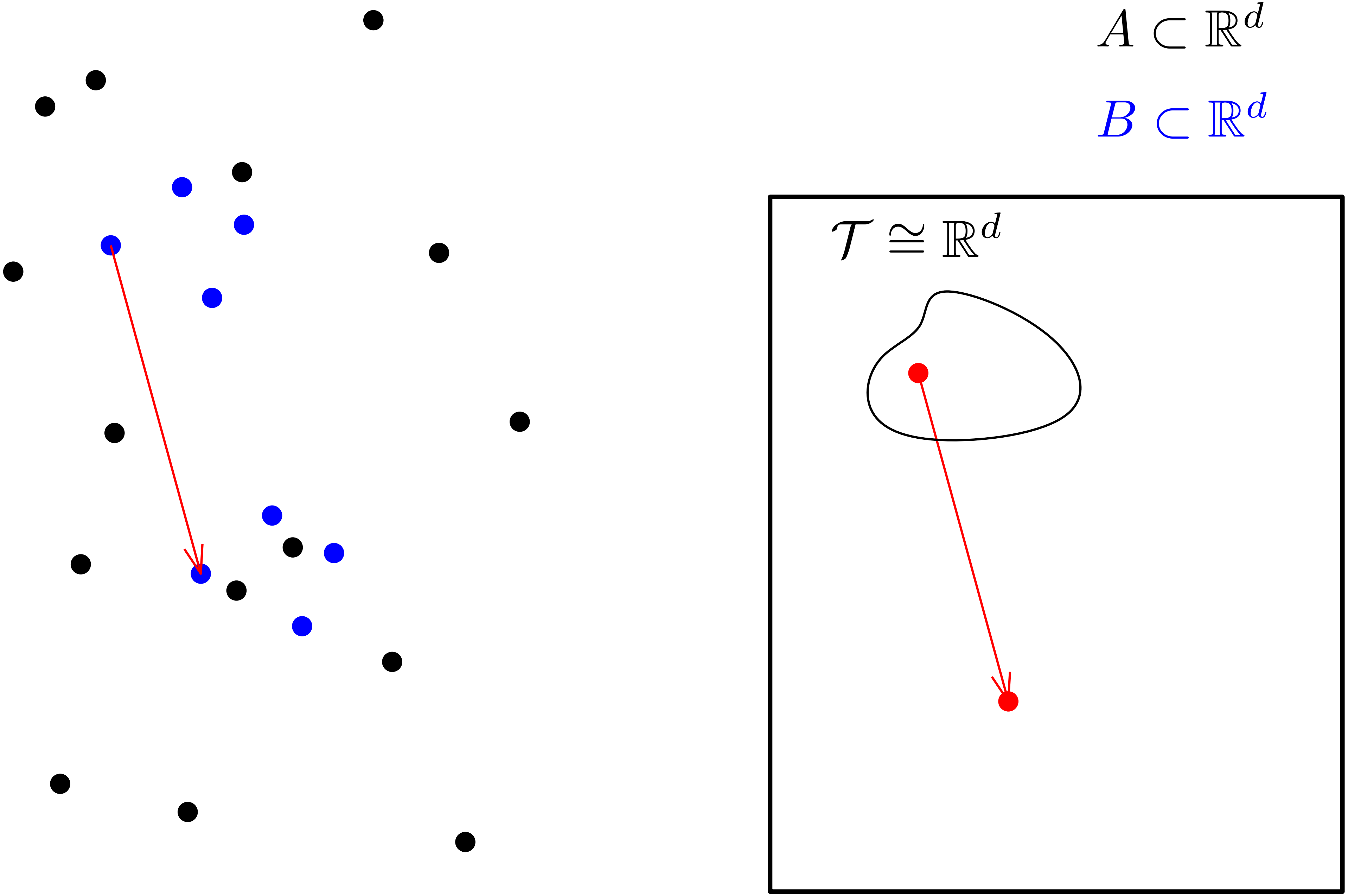

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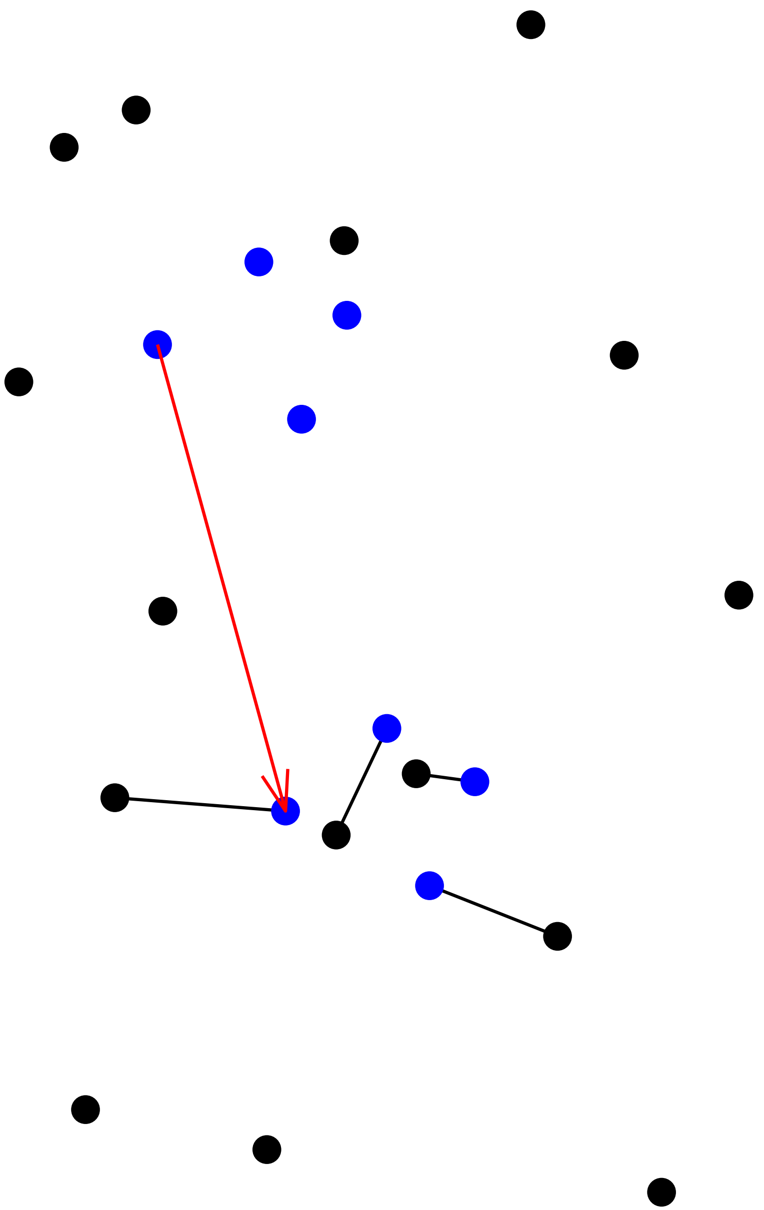

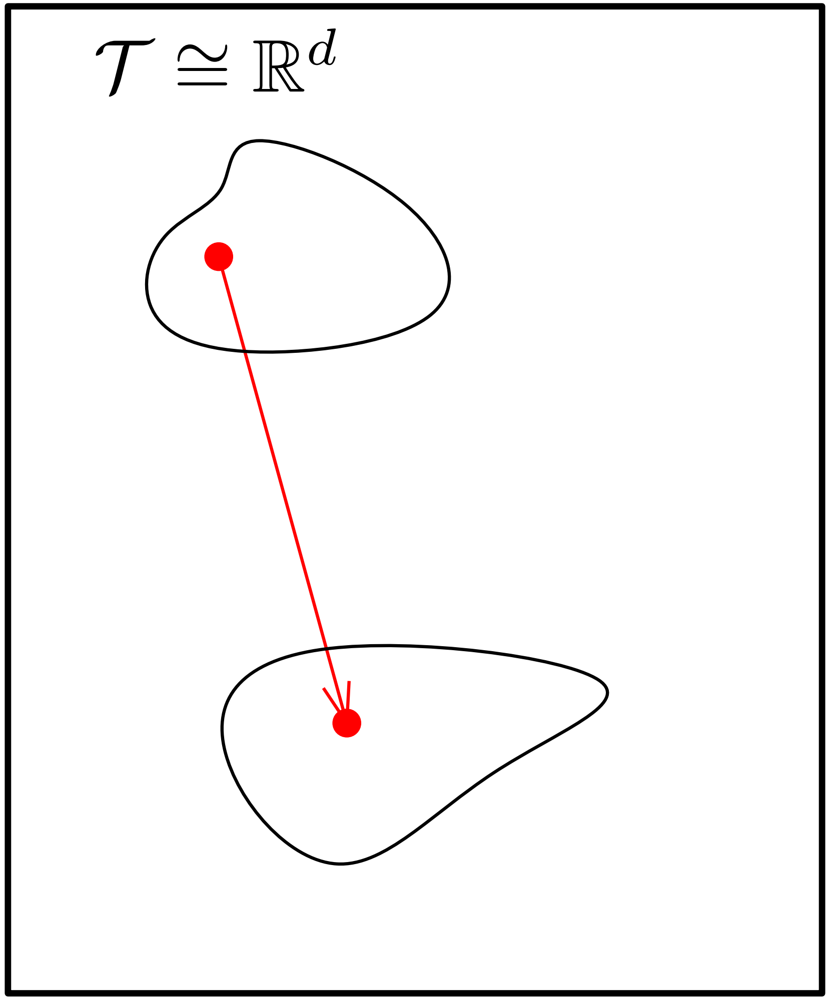

$$\min_{t,\pi} \quad \sum_{b \in B} \|b + t - \pi(b)\|^2$$

$$\text{s.t.} \quad \pi : B \hookrightarrow A$$

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$$V_\pi = \{t \in \mathbb{R}^d : c_\pi(t) \leq c_\sigma(t) \quad \forall \sigma : B \hookrightarrow A\}$$

$$\|b - \pi(b)\|^2 + 2t \cdot (b - \pi(b)) + \|t\|^2$$

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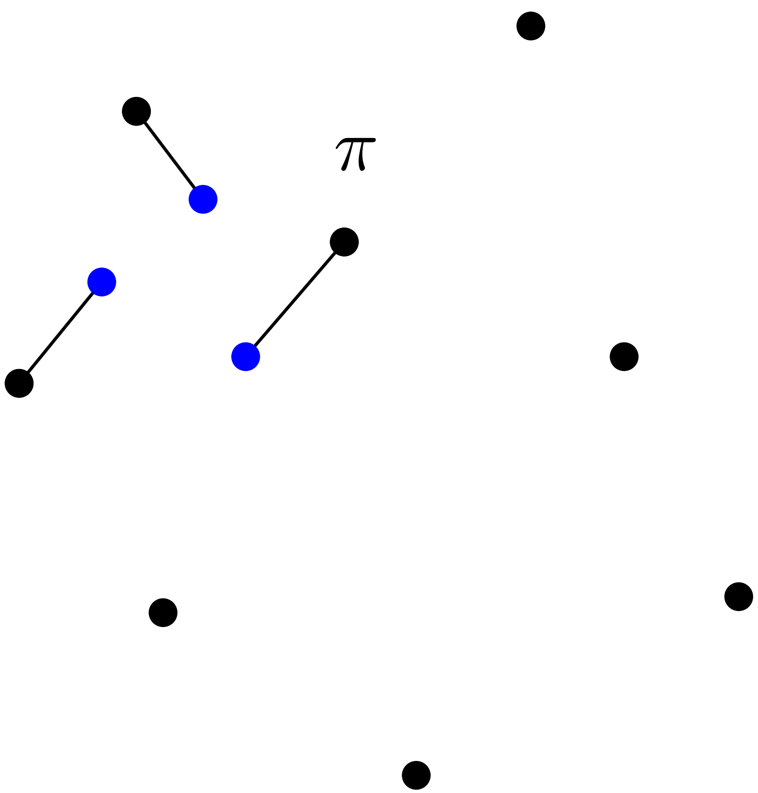

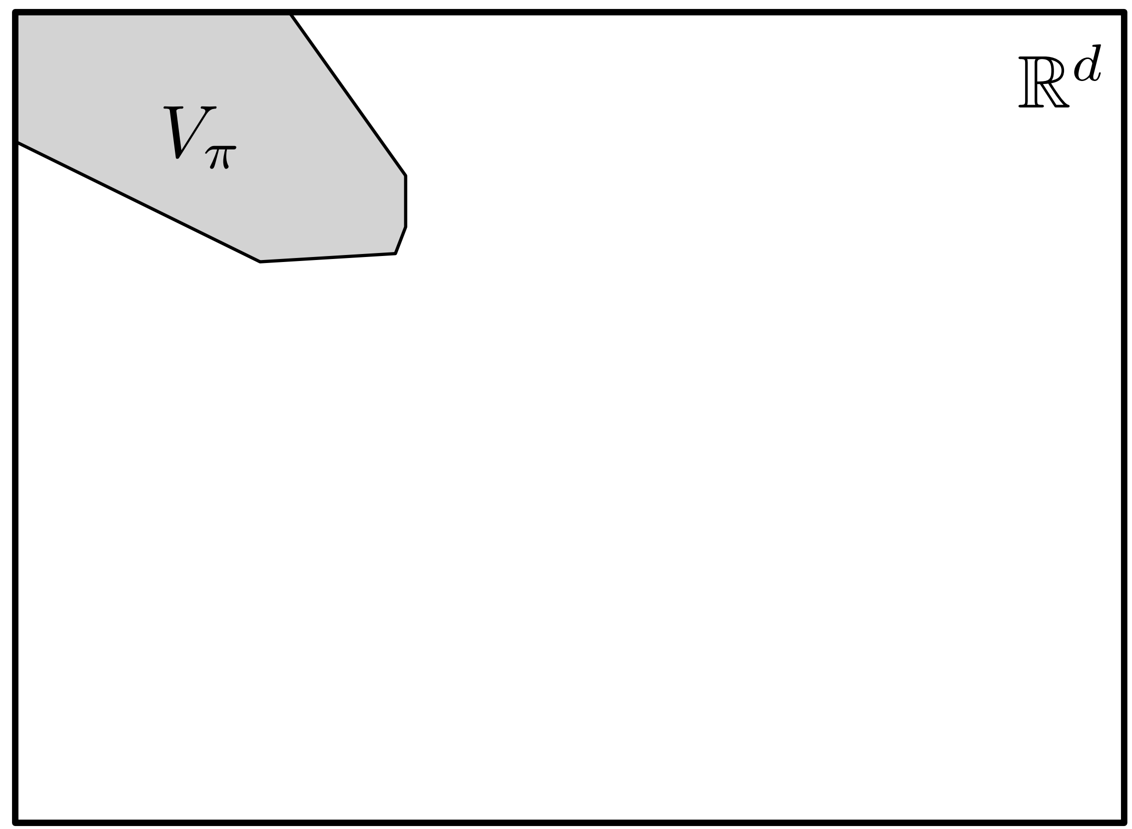

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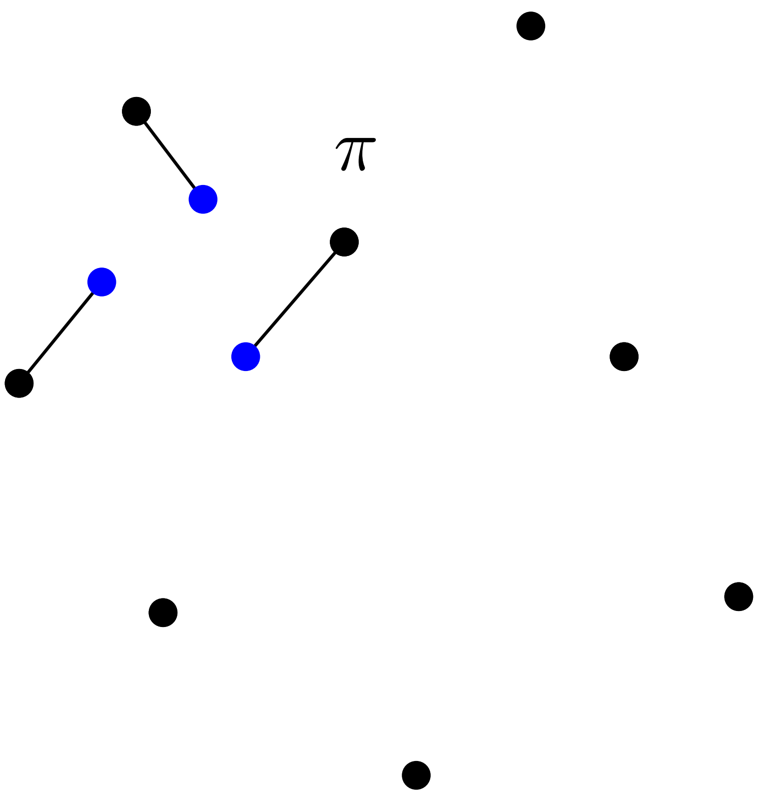

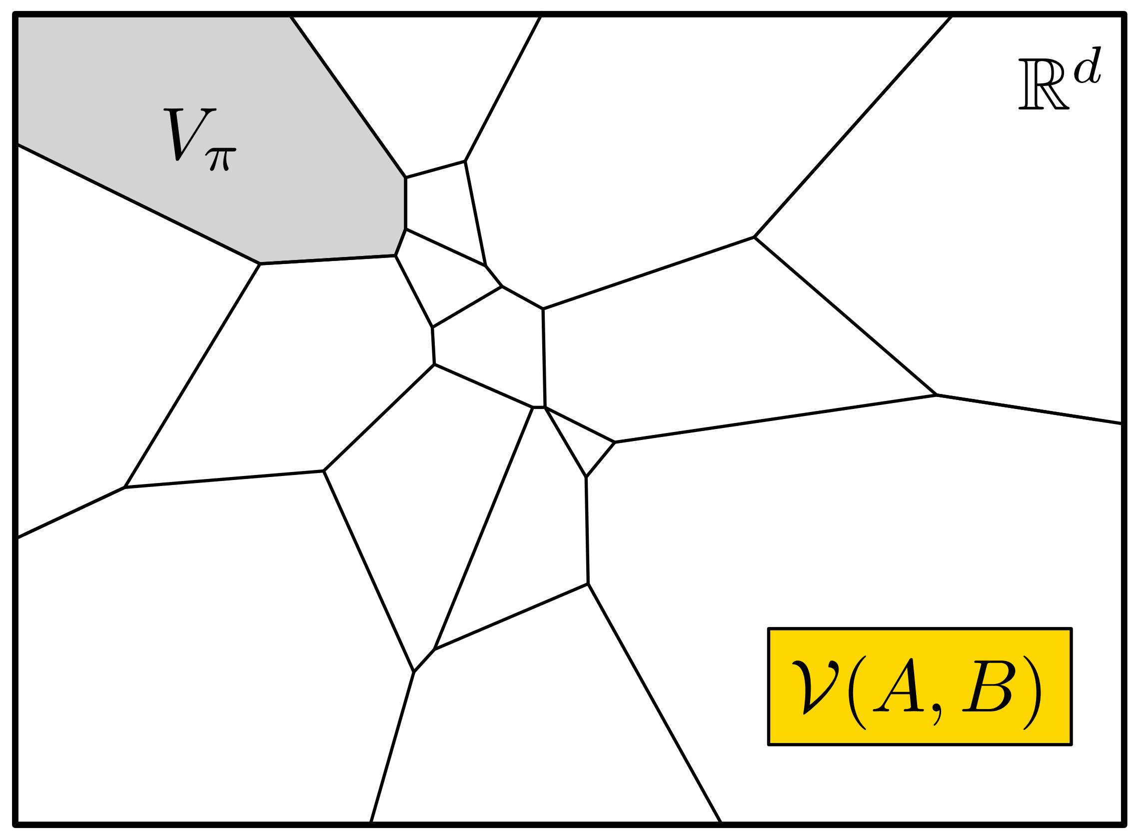

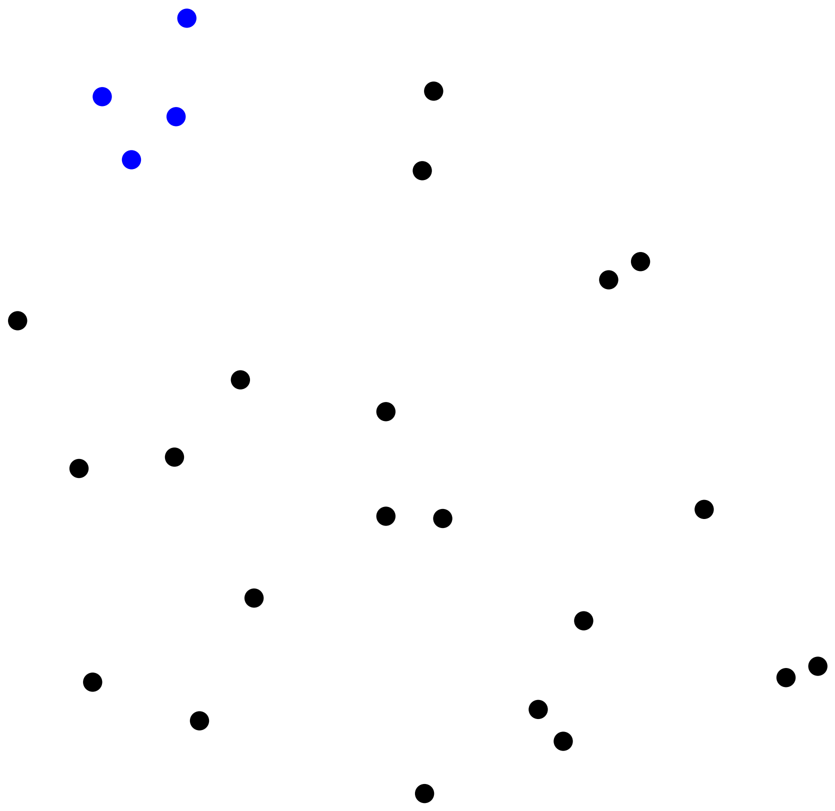
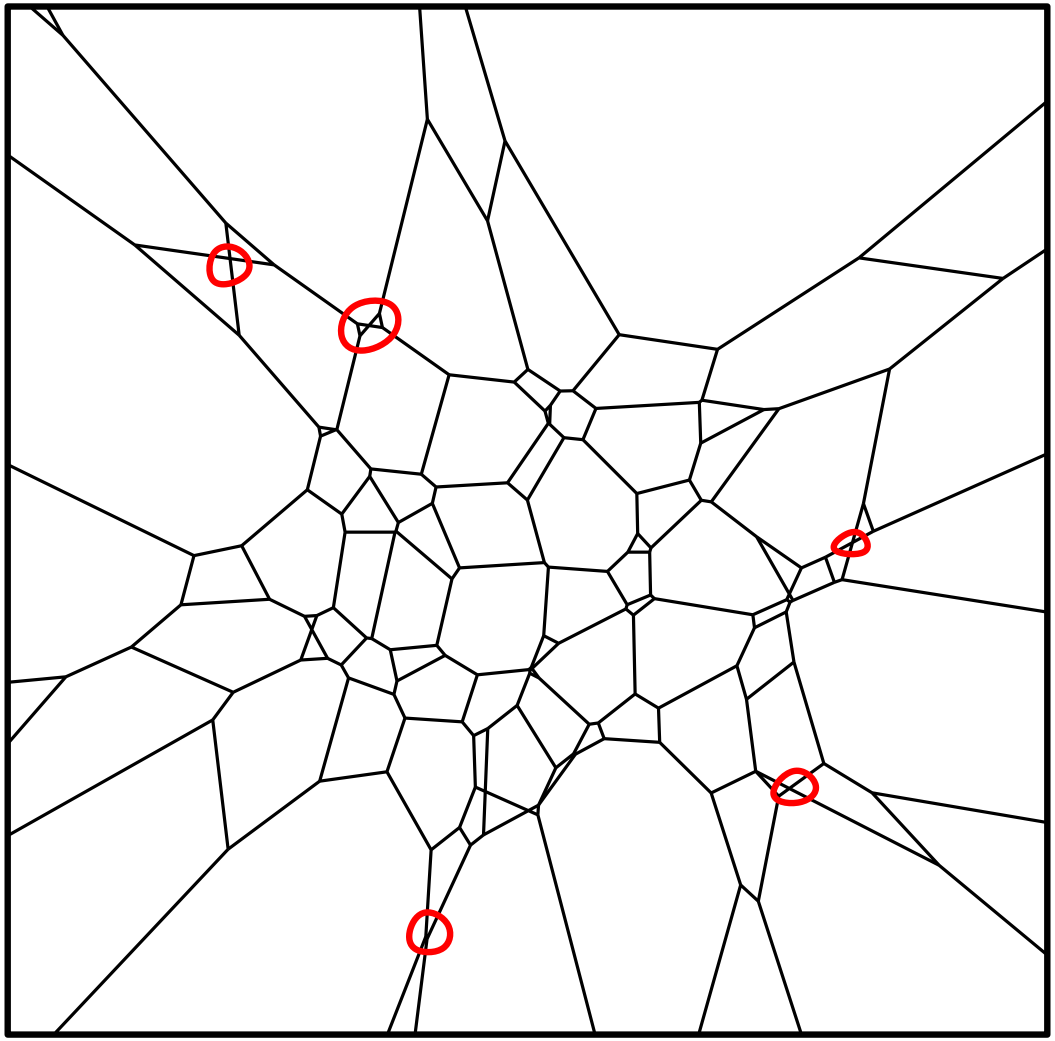

$$k = |B| \leq |A| = n$$

Theorem (Rote 2010): The tessellation $\mathcal{V}(A, B)$ is polyhedral and any line intersects at most $k(n-k)$ regions.

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Polynomial-time algorithm?

$\mathcal{V}(A, B)$ might have $\binom{n}{k}k!$ cells...

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and $\Omega(n^d k^d)$.

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Apply to more general distances*

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*: $k!$ instead of $\log^k k$

$O(n^{2d}k^d \log^k k)$

$$\begin{pmatrix} 3 & 1 & 4 & 6 & 2 & 5 \\ 4 & 1 & 2 & 5 & 6 & 3 \\ 2 & 6 & 1 & 4 & 3 & 5 \\ 3 & 4 & 6 & 2 & 1 & 5 \end{pmatrix}$$

$|B| = k$ rows

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Preference matrix for fixed t

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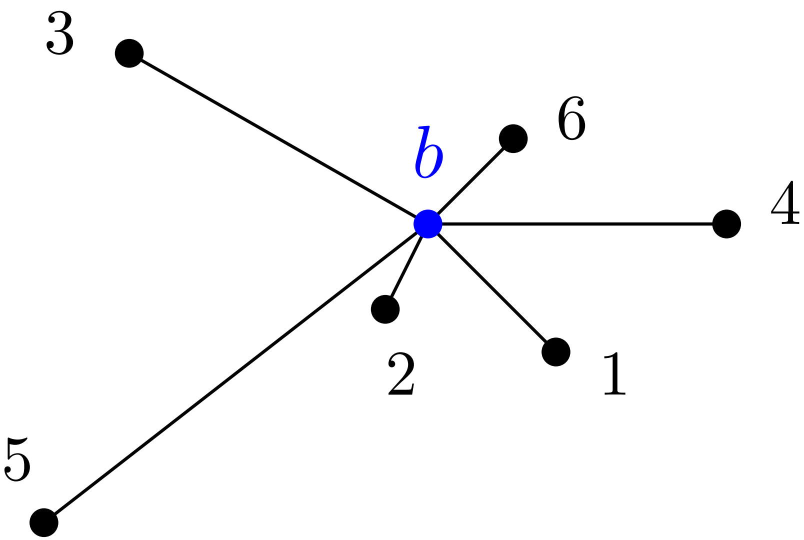

$$\|b+t-a_6\| < \|b+t-a_4\|$$

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A matching is **stable** if there is no other matching in which at least one point in B gets a closer point and no other gets a farther point.

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optimal $\Rightarrow$ stable

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stable matchings ?

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n houses, n agents...

Theorem (Abdulkadiroğlu, Sönmez):
Any matching in the core of a *House Allocation Problem* comes from a serial dictatorship.

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Theorem (Zikan 1991): For $|A| = |B|$, the least-squares optimal matching is independent of the translation.

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The number of different elements appearing in stable matchings is $O(k \log k)$.

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and $O(\binom{k \log k}{k}) = O(\log^k k)$.

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different preference matrices for $t \in \mathbb{R}^d$?

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different preference matrices for $t \in \mathbb{R}^d = O((kn^2)^d)$.

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In the plane: for every $b \in B$, the first k levels of the Voronoi arrangement have complexity $O(k^2 n)$.

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M. Sharir

$\#$ bisectors supporting an edge of a j-th order Voronoi diagram for some $j < k$. $= O(nk)$.

$O(n^{2d} k^d \log^k k)$

different preference matrices for $t \in \mathbb{R}^2 =$ $O(n^2 k^4)$

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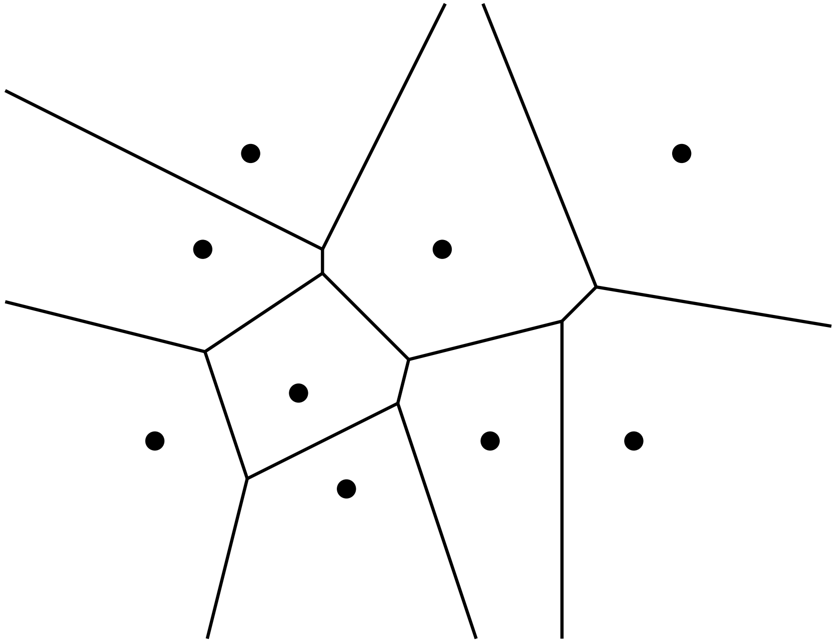

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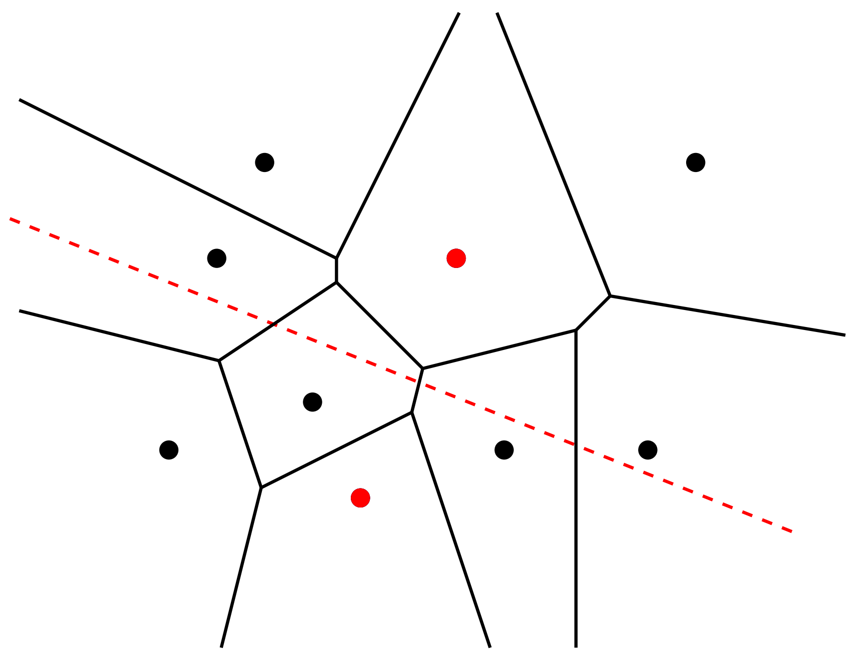

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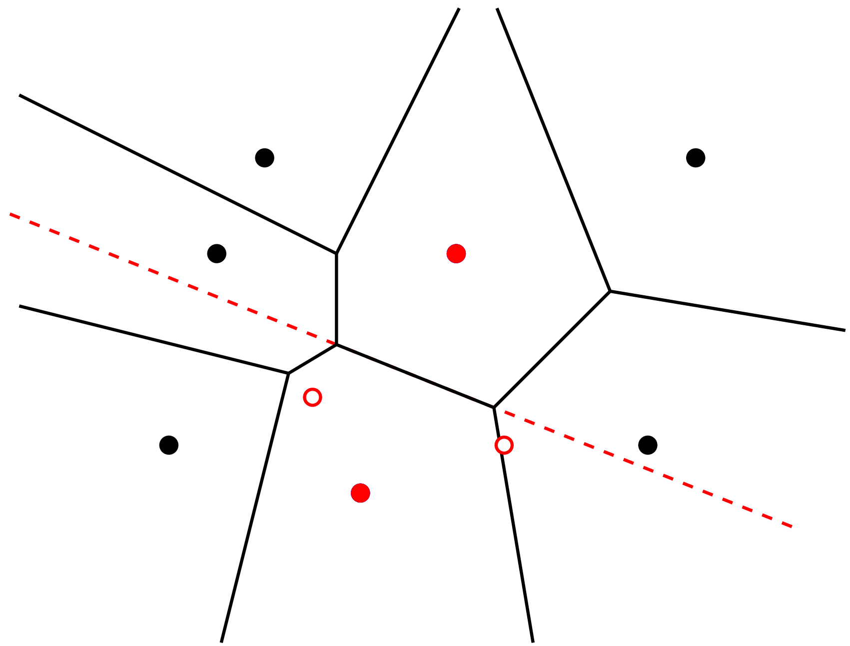

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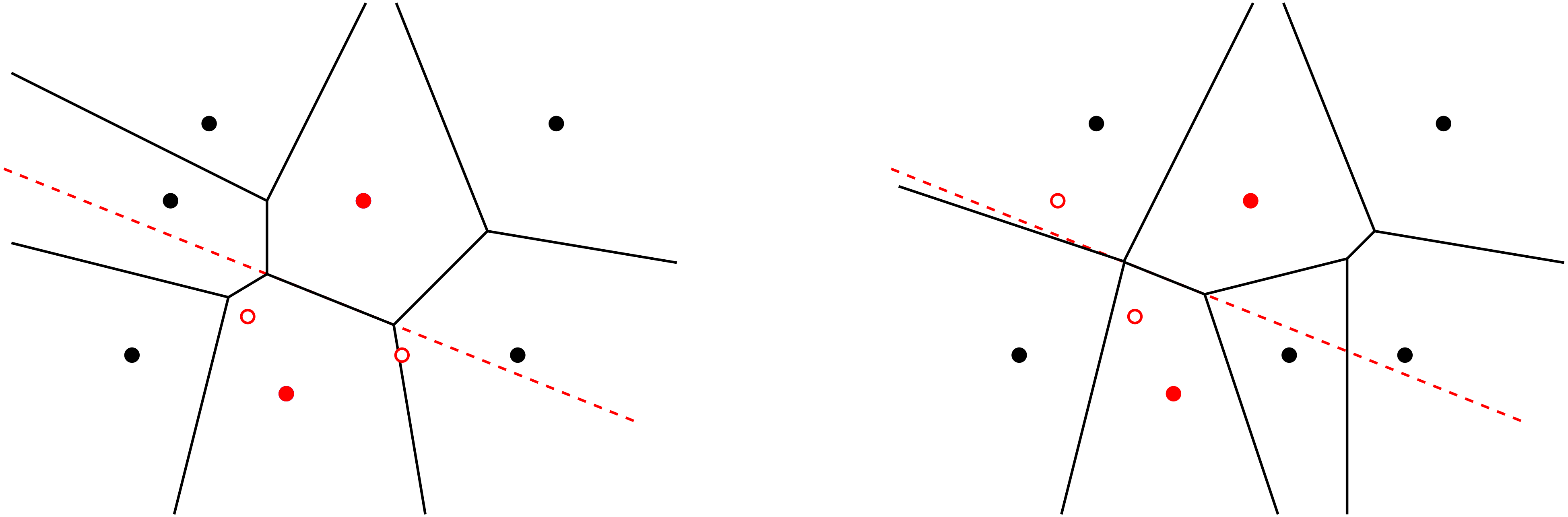

$\Omega(n^d k^d)$

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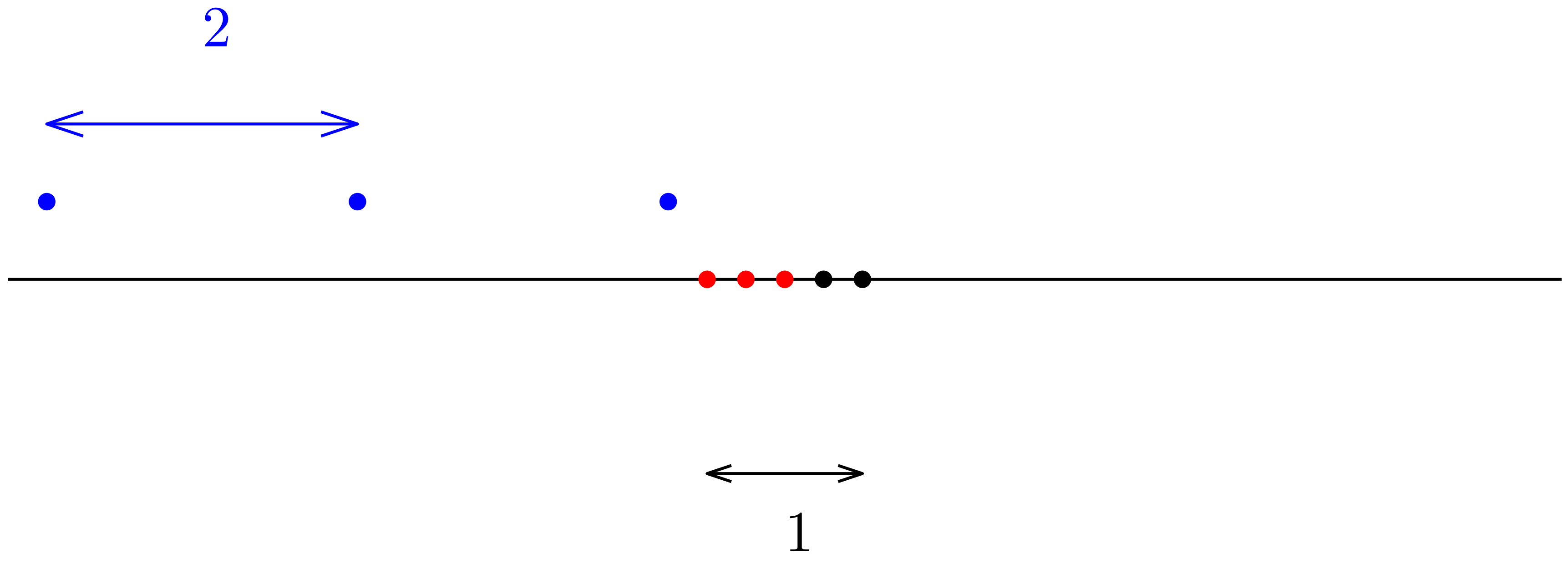

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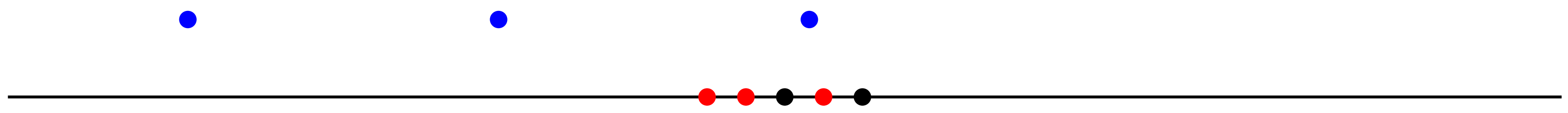

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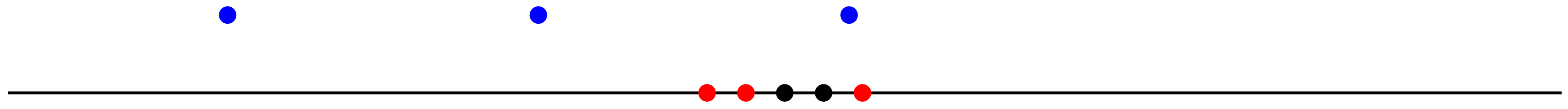

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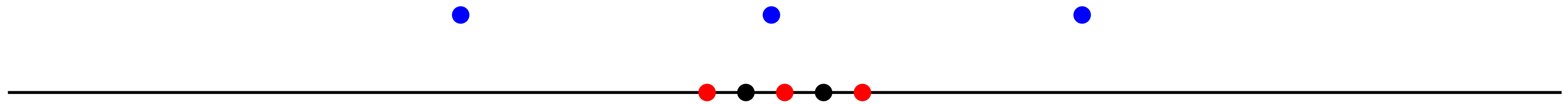
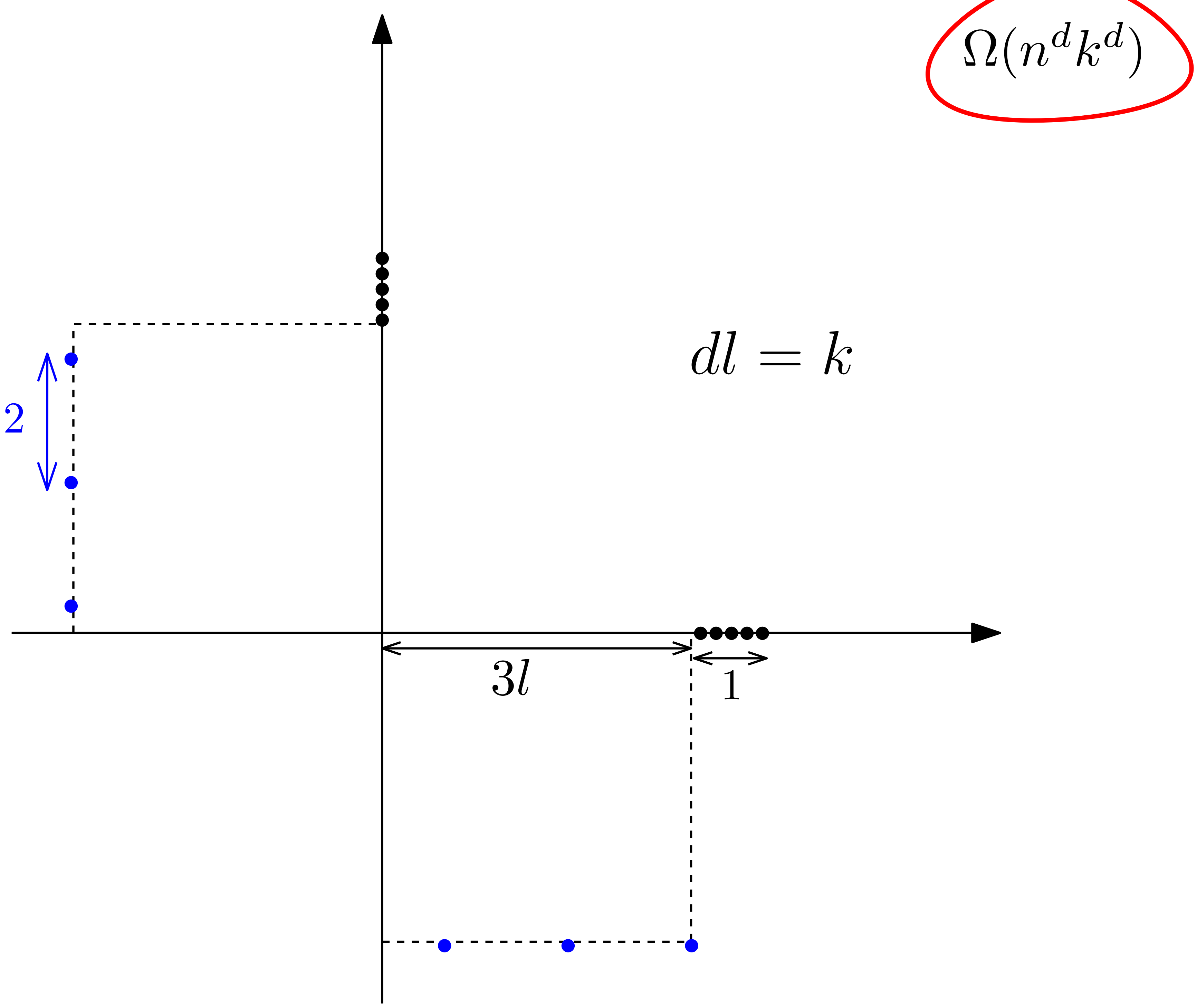
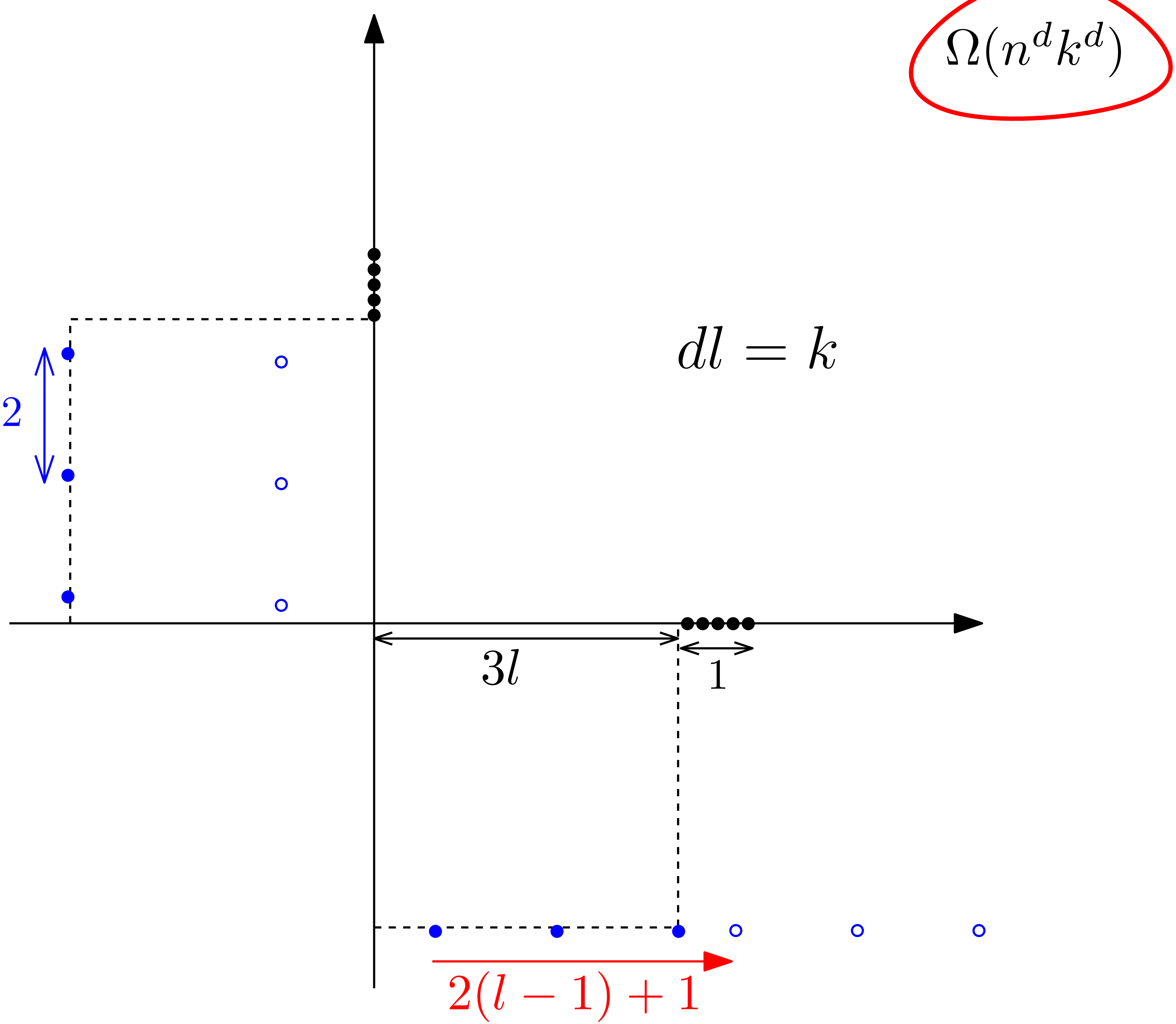



$\Omega(n^d k^d)$
$dl = k$
2
$3l$
1



$\Omega(n^d k^d)$
$dl = k$
2
$3l$
1
$2(l-1)+1$

Remarks

Generalizations

Remarks

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- algebraic surfaces of bounded degree

Remarks

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- lexicographic (polynomial)

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Structural properties of $\mathcal{V}(A, B)$

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Thank you for your attention!

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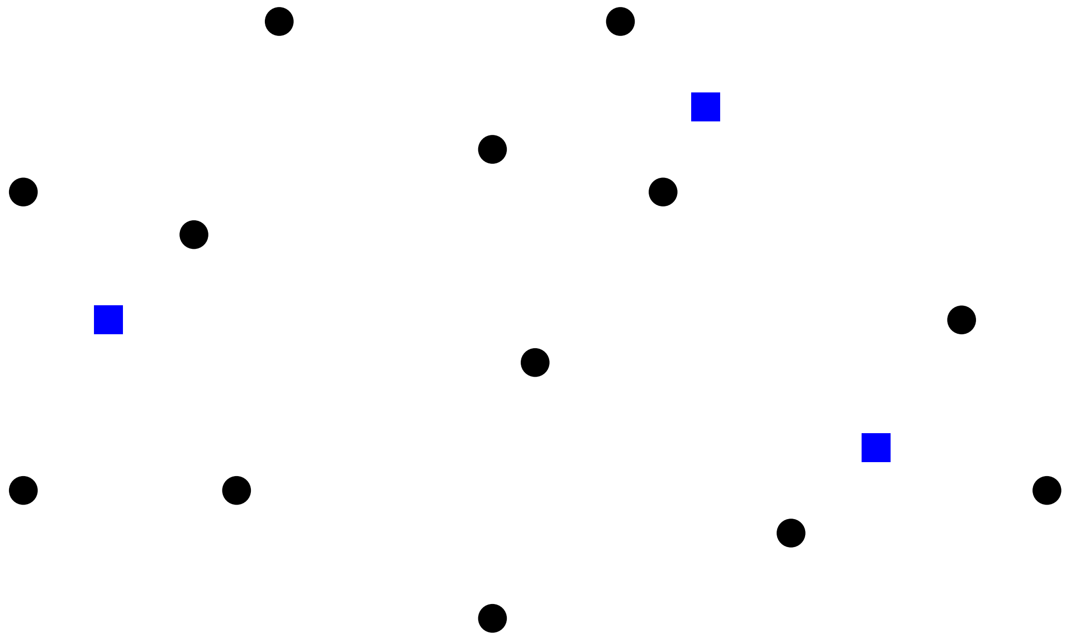
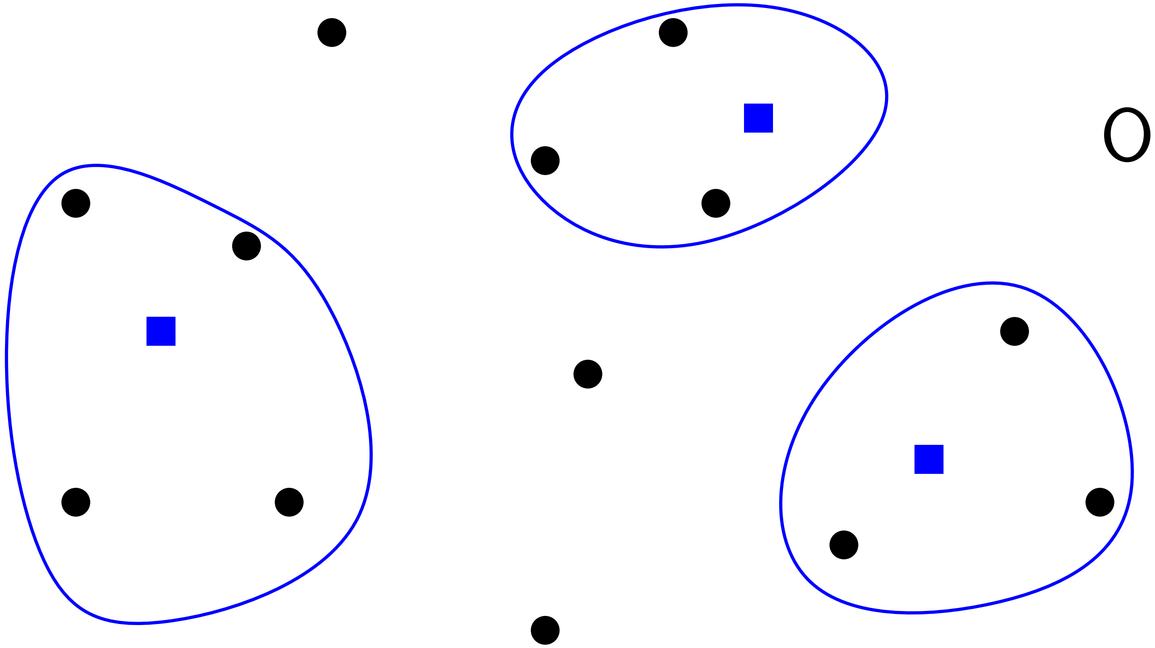
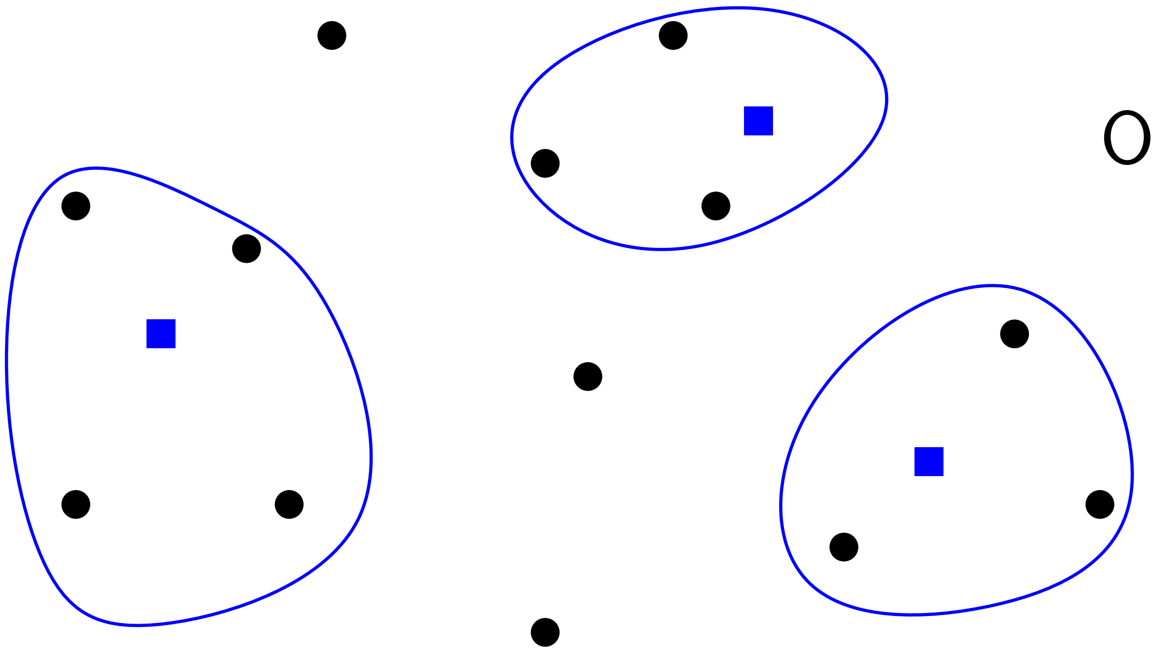
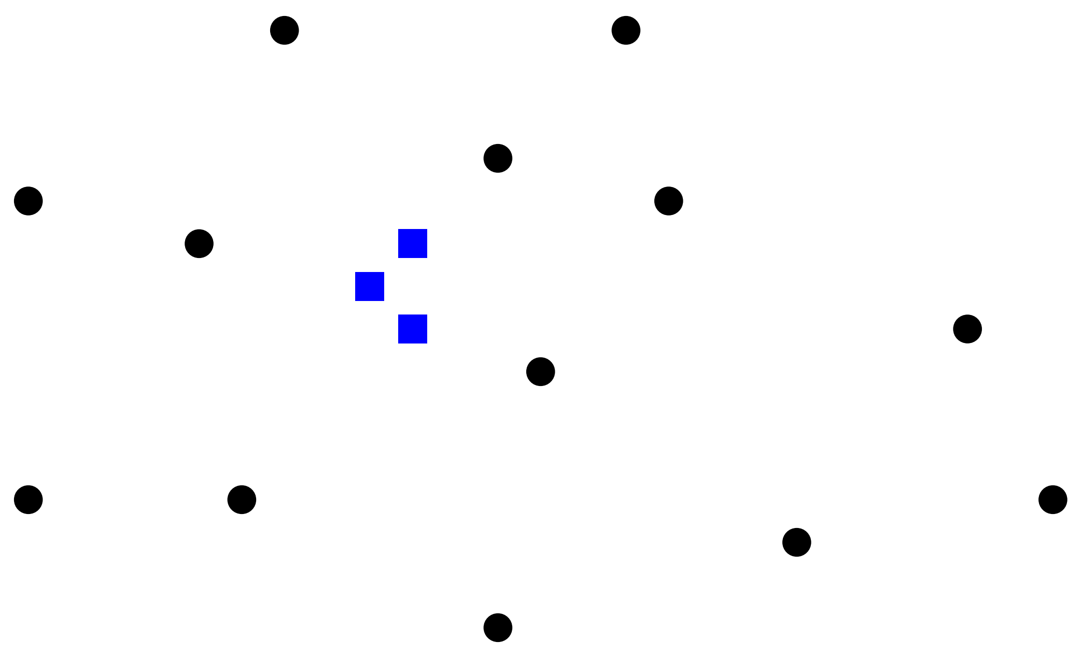


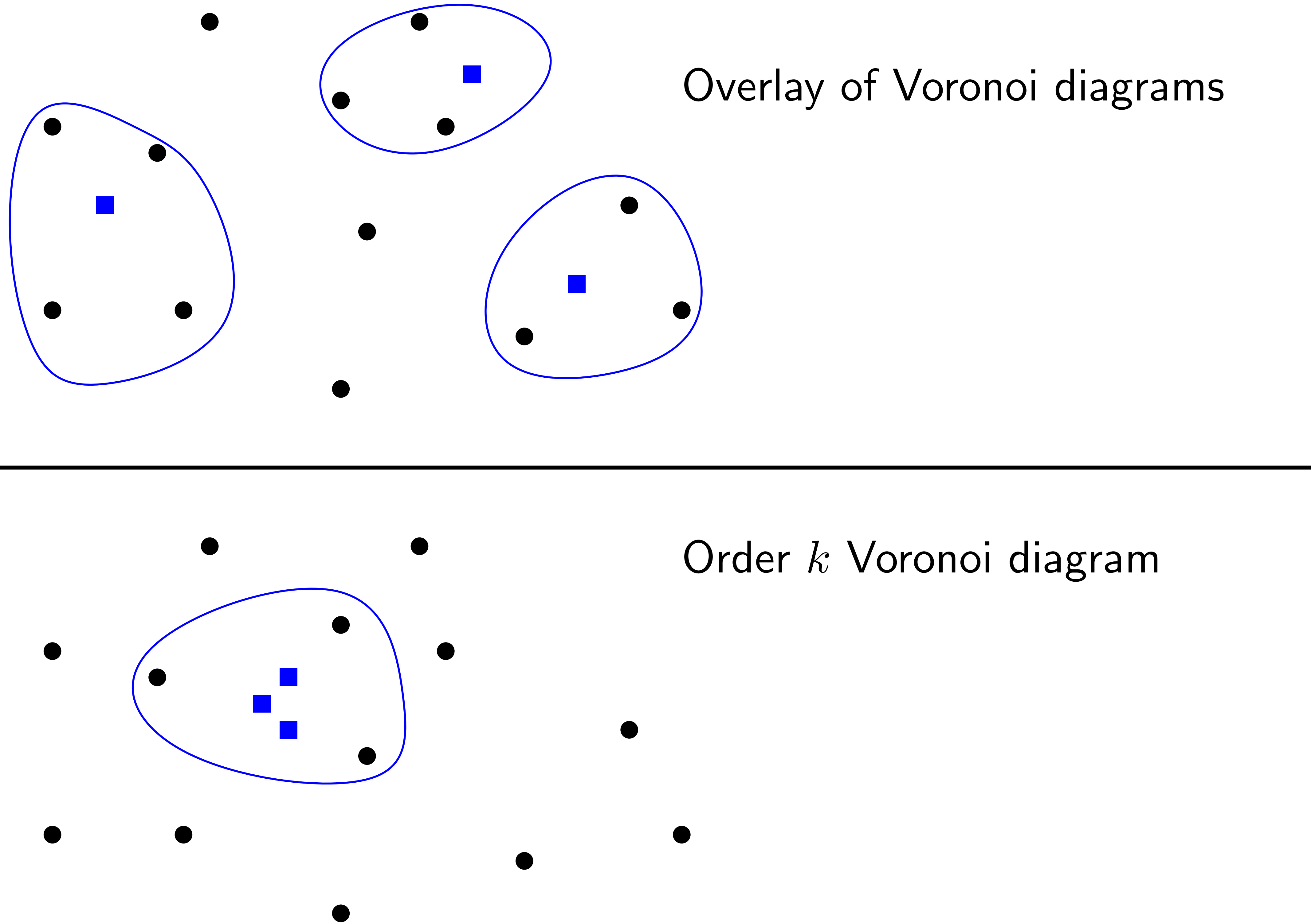
Overlay of Voronoi diagrams



Overlay of Voronoi diagrams






Overlay of Voronoi diagrams
Order k Voronoi diagram

$$
\mathcal{P}_k \qquad \Rightarrow \qquad \mathcal{P}_{2k}: \begin{array}{rcr l}
d_1 & : & 1 & \\
d_2 & : & 2 & \\
 & & \vdots & \mathcal{P}_k \\
d_{k-1} & : & k-1 & \\
d_k & : & k & \\
\hline
d_{k+1} & : & 1 & \\
d_{k+2} & : & 2 & \\
 & & \vdots & \mathcal{P}'_k \\
d_{2k-1} & : & k-1 & \\
d_{2k} & : & k &
\end{array}
$$

$$\begin{array}{llllll} d_1: & 1 & 2 & \dots & \lfloor \frac{k}{2} \rfloor & \lfloor \frac{k}{2} \rfloor + 1 \\ d_2: & 1 & 2 & \dots & \lfloor \frac{k}{2} \rfloor & \lfloor \frac{k}{2} \rfloor + 2 \\ & & \vdots & & \vdots & \\ d_k: & 1 & 2 & \dots & \lfloor \frac{k}{2} \rfloor & \lfloor \frac{k}{2} \rfloor + k \end{array}$$