Hamilton cycles in truncated triangulations and the maximum genus of a graph

Martin Škoviera

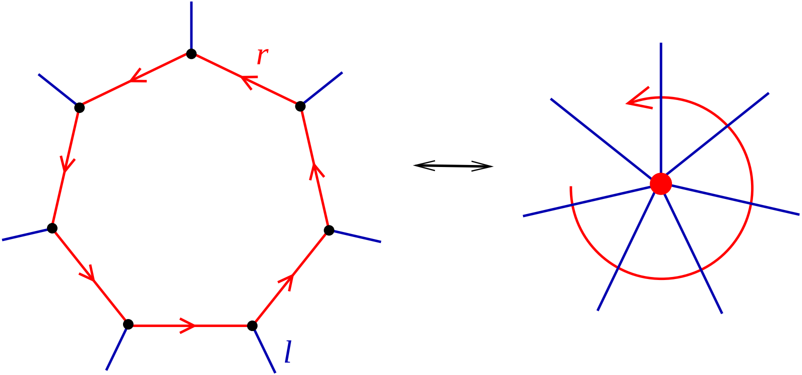
Comenius University, Bratislava, Slovakia

joint work with
Michal Kotrbčík & Roman Nedela

EuroGIGA Final Conference

Freie Universität Berlin, 17–21 February 2014

Truncation of an embedded graph



Graph symmetry and hamiltonicity

Question (Lovász, 1969)

Does every connected vertex-transitive graph have a Hamilton path, i. e., a simple path going through all vertices?

Graph symmetry and hamiltonicity

Question (Lovász, 1969)

Does every connected vertex-transitive graph have a Hamilton path, i. e., a simple path going through all vertices?

Only four connected vertex-transitive graphs with no Hamilton cycle are known (but they do have a Hamilton path).

Hamilton cycles in Cayley graphs

All known hon-hamiltonian vertex-transitive graphs are cubic, and none of them is a Cayley graph.

Conjecture (Folklore)

Every Cayley graph (of order ≥ 3) has a Hamilton cycle.

Hamilton cycles in Cayley graphs

All known hon-hamiltonian vertex-transitive graphs are cubic, and none of them is a Cayley graph.

Conjecture (Folklore)

Every Cayley graph (of order ≥ 3) has a Hamilton cycle.

Counter-conjecture (Babai, 1995)

For some $c > 0$, there are infinitely many vertex-transitive graphs G, even Cayley graphs, without cycles of length $> (1 - c)|G|$.

Hamilton cycles in Cayley graphs: current status

Hamilton cycles or paths are known in the following types of vertex-transitive graphs:

- Graphs of specific order:
  p, $2p$, $3p$, $4p$, $5p$, $6p$, $2p^2$, ... (where p a prime)
- Cayley graphs of p-groups
- Cayley graphs of groups with a cyclic commutator subgroup of order p^k.

Hamilton cycles in Cayley graphs: current status

Hamilton cycles or paths are known in the following types of vertex-transitive graphs:

- Graphs of specific order:
  p, $2p$, $3p$, $4p$, $5p$, $6p$, $2p^2$, ... (where p a prime)
- Cayley graphs of p-groups
- Cayley graphs of groups with a cyclic commutator subgroup of order p^k.

Until 2009, very little was known about Hamilton cycles or paths in cubic vertex-transitive graphs.

Let $H = \langle r, l \rangle$ be a $(2,3,s)$-presented finite group; i.e., $r^s = l^2 = (rl)^3 = 1$.

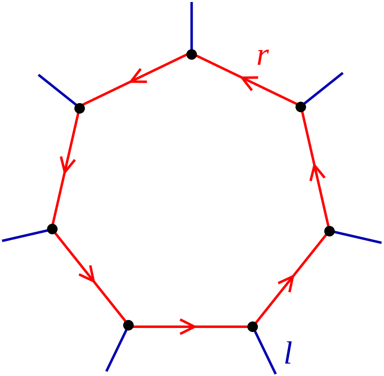
Then H is a finite quotient of the modular group $PSL(2,\mathbb{Z})$.

Theorem (Glover & Marušič, 2009)

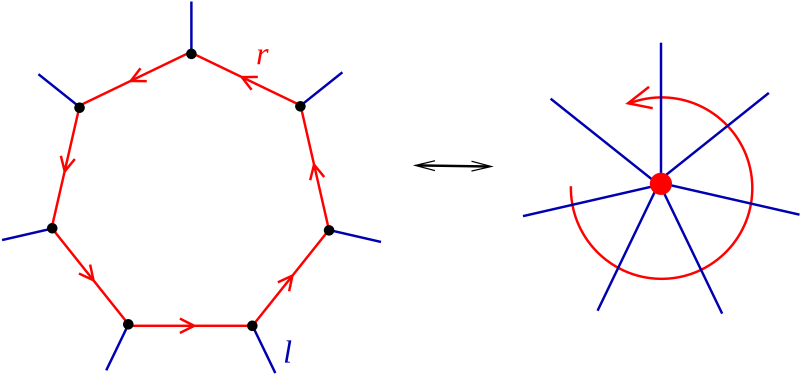
Let $K = \mathrm{Cay}(H;\, r, r^{-1}, l)$ be a cubic Cayley graph, where $H = \langle r, l \mid r^s = l^2 = (rl)^3 = 1, \dots \rangle$ is a finite quotient of the modular group $PSL(2,\mathbb{Z})$. Then K has a Hamilton path. Moreover,

- *if $|H| \equiv 2 \pmod 4$, then K has a Hamilton cycle*
- *if $|H| \equiv 0 \pmod 4$, then K has a cycle through all but two adjacent vertices.*

Proof I: Cayley graph and the corresponding triangulation



Proof I: Cayley graph and the corresponding triangulation



Proof II: How to find a Hamilton cycle

Idea: Construct a Hamilton cycle in $K = \mathrm{Cay}(H;\, r, r^{-1}, l)$ as the topological boundary $\partial(\bigcup \mathcal{F})$ of a set of faces $\mathcal{F}$.

Proof II: How to find a Hamilton cycle

Idea: Construct a Hamilton cycle in $K = \mathrm{Cay}(H;\, r, r^{-1}, l)$
as the topological boundary $\partial(\bigcup \mathcal{F})$ of a set of faces $\mathcal{F}$.

To get a Hamilton cycle

$\partial(\bigcup \mathcal{F})$ must be connected and must cover all vertices of K.

Proof II: How to find a Hamilton cycle

Idea: Construct a Hamilton cycle in $K = \mathrm{Cay}(H;\, r, r^{-1}, l)$
as the topological boundary $\partial(\bigcup \mathcal{F})$ of a set of faces $\mathcal{F}$.

To get a Hamilton cycle

$\partial(\bigcup \mathcal{F})$ must be connected and must cover all vertices of K.

Two types of faces:

1. red faces of size s corresponding to the generator r
2. alternating red-blue hexagons corresponding to rl.

Proof II: How to find a Hamilton cycle

Idea: Construct a Hamilton cycle in $K = \mathrm{Cay}(H;\, r, r^{-1}, l)$ as the topological boundary $\partial(\bigcup \mathcal{F})$ of a set of faces $\mathcal{F}$.

To get a Hamilton cycle

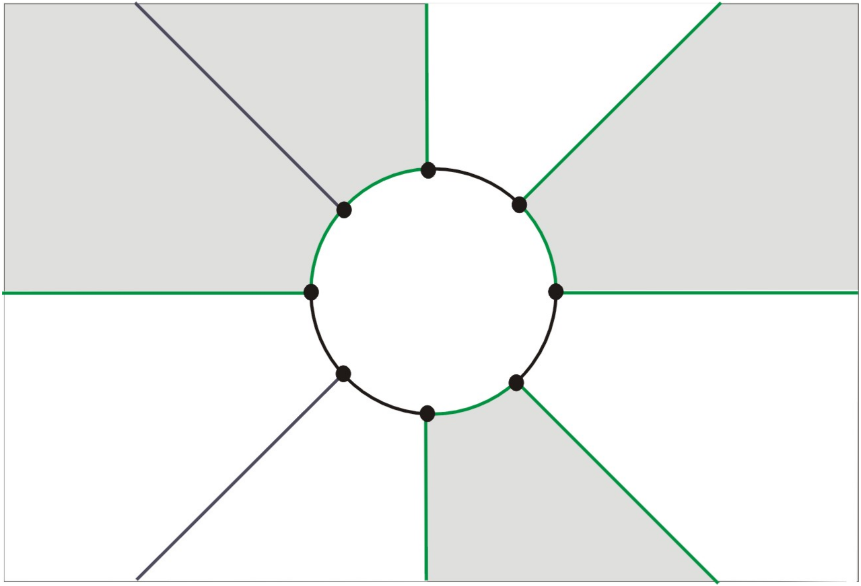
$\partial(\bigcup \mathcal{F})$ must be connected and must cover all vertices of K.

Two types of faces:

1. red faces of size s corresponding to the generator r
2. alternating red-blue hexagons corresponding to rl.

We choose $\mathcal{F}$ to consist of hexagonal faces.

Proof III: Partition of hexagonal faces



Proof III: Partition of hexagonal faces

- In order to get every vertex covered, the complementary set of hexagons must be 'independent'.

Proof III: Partition of hexagonal faces

- In order to get every vertex covered, the complementary set of hexagons must be 'independent'.
- In order to get a connected boundary, the faces in $\mathcal{F}$ must form a 'tree' (i.e., $\partial(\bigcup \mathcal{F})$ must be homologically trivial.)

Proof III: Partition of hexagonal faces

- In order to get every vertex covered, the complementary set of hexagons must be 'independent'.
- In order to get a connected boundary, the faces in $\mathcal{F}$ must form a 'tree' (i.e., $\partial(\bigcup \mathcal{F})$ must be homologically trivial.)

Note:

hexagonal faces of the embedded $K = \mathrm{Cay}(H;\, r, r^{-1}, l)$
$\longleftrightarrow$ faces of the triangulation $\mathcal{T}$
$\longleftrightarrow$ vertices of the underlying cubic graph G^* of the dual map $\mathcal{T}^*$

Proof III: Partition of hexagonal faces

- In order to get every vertex covered, the complementary set of hexagons must be 'independent'.
- In order to get a connected boundary, the faces in $\mathcal{F}$ must form a 'tree' (i.e., $\partial(\bigcup \mathcal{F})$ must be homologically trivial.)

Note:

hexagonal faces of the embedded $K = \mathrm{Cay}(H;\, r, r^{-1}, l)$
$\longleftrightarrow$ faces of the triangulation $\mathcal{T}$
$\longleftrightarrow$ vertices of the underlying cubic graph G^* of the dual map $\mathcal{T}^*$

Summing up:

We need to find a partition of $V(G^*)$ into two sets A and J, where A induces a tree and J is independent.

Proof IV: Dual of the triangulation

Theorem (Payan & Sakarovitch, 1975)

Let G be a cyclically 4-edge-connected cubic graph with n vertices. Then the following hold:

(i) *If $n \equiv 2 \pmod 4$, then $V(G)$ has a partition $\{A, J\}$ where A induces a tree and J is independent.*

(ii) *If $n \equiv 0 \pmod 4$, then $V(G)$ has a partition $\{A, J\}$ where either A induces a tree and J induces a graph with a single edge, or A induces a forest with two components and J is independent.*

Proof IV: Dual of the triangulation

Theorem (Payan & Sakarovitch, 1975)

Let G be a cyclically 4-edge-connected cubic graph with n vertices. Then the following hold:

(i) *If $n \equiv 2 \pmod 4$, then $V(G)$ has a partition $\{A, J\}$ where A induces a tree and J is independent.*

(ii) *If $n \equiv 0 \pmod 4$, then $V(G)$ has a partition $\{A, J\}$ where either A induces a tree and J is near-independent, or A induces a forest with two components and J is independent.*

Proof IV: Dual of the triangulation

Theorem (Payan & Sakarovitch, 1975)

Let G be a cyclically 4-edge-connected cubic graph with n vertices. Then the following hold:

(i) *If $n \equiv 2 \pmod 4$, then $V(G)$ has a partition $\{A, J\}$ where A induces a tree and J is independent.*

(ii) *If $n \equiv 0 \pmod 4$, then $V(G)$ has a partition $\{A, J\}$ where either A induces a tree and J is near-independent, or A induces a forest with two components and J is independent.*

Case (i) implies a Hamilton cycle in $\mathrm{Cay}(H;\, r, r^{-1}, l)$

Case (ii) implies a Hamilton path in $\mathrm{Cay}(H;\, r, r^{-1}, l)$

Proof IV: Dual of the triangulation

Theorem (Payan & Sakarovitch, 1975)

Let G be a cyclically 4-edge-connected cubic graph with n vertices. Then the following hold:

(i) *If $n \equiv 2 \pmod 4$, then $V(G)$ has a partition $\{A, J\}$ where A induces a tree and J is independent.*

(ii) *If $n \equiv 0 \pmod 4$, then $V(G)$ has a partition $\{A, J\}$ where either A induces a tree and J is near-independent, or A induces a forest with two components and J is independent.*

Case (i) implies a Hamilton cycle in $\mathrm{Cay}(H;\, r, r^{-1}, l)$

Case (ii) implies a Hamilton path in $\mathrm{Cay}(H;\, r, r^{-1}, l)$

Symmetry is only used to guarantee sufficient cyclic connectivity (!!)

Proof IV: Dual of the triangulation

Theorem (Payan & Sakarovitch, 1975)

Let G be a cyclically 4-edge-connected cubic graph with n vertices. Then the following hold:

(i) *If $n \equiv 2 \pmod 4$, then $V(G)$ has a partition $\{A, J\}$ where A induces a tree and J is independent.*

(ii) *If $n \equiv 0 \pmod 4$, then $V(G)$ has a partition $\{A, J\}$ where either A induces a tree and J is near-independent, or A induces a forest with two components and J is independent.*

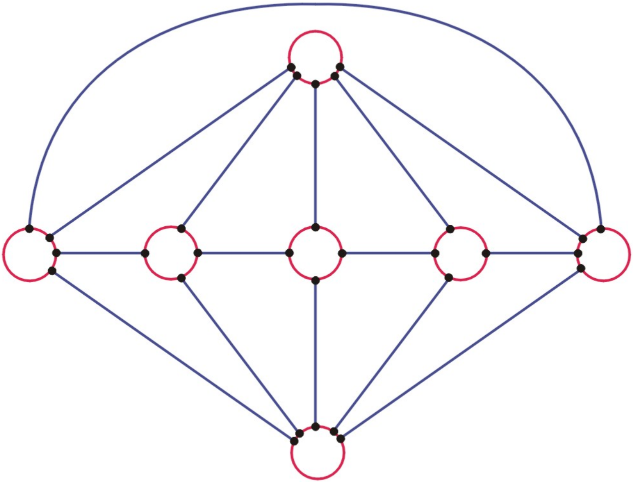
Case (i) implies a Hamilton cycle in $\mathrm{Cay}(H; r, r^{-1}, l)$

Case (ii) implies a Hamilton path in $\mathrm{Cay}(H; r, r^{-1}, l)$

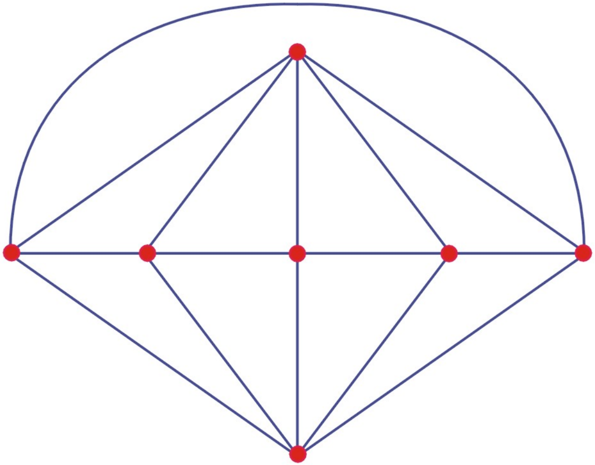
Symmetry is only used to guarantee sufficient cyclic connectivity (!!)

This method can be substatially generalised.

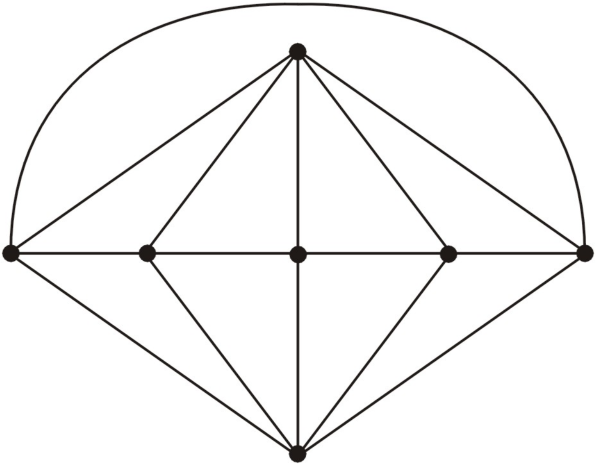
Example: Construction of a Hamilton cycle in $t(\mathcal{T})$



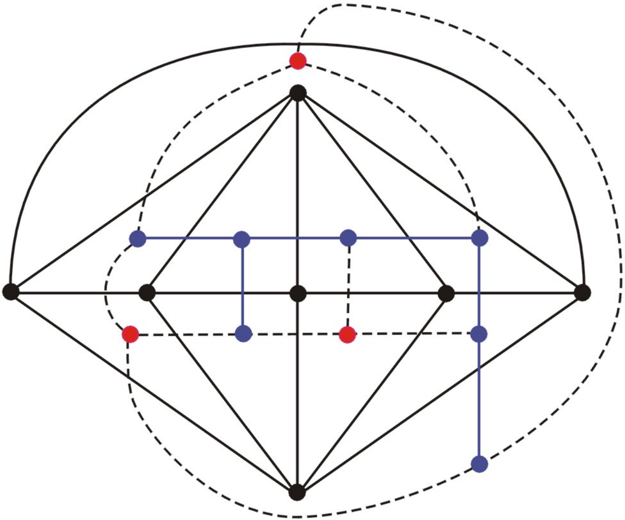
Example: Construction of a Hamilton cycle in $t(\mathcal{T})$



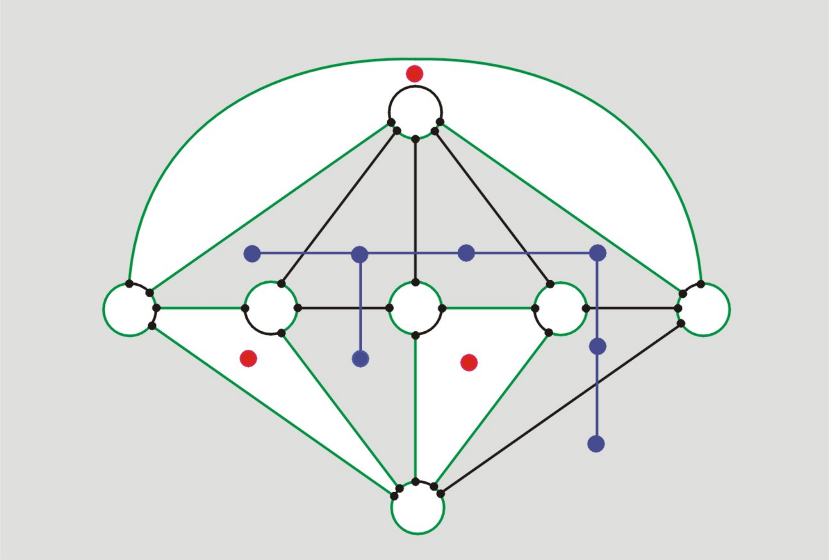
Example: Construction of a Hamilton cycle in $t(\mathcal{T})$



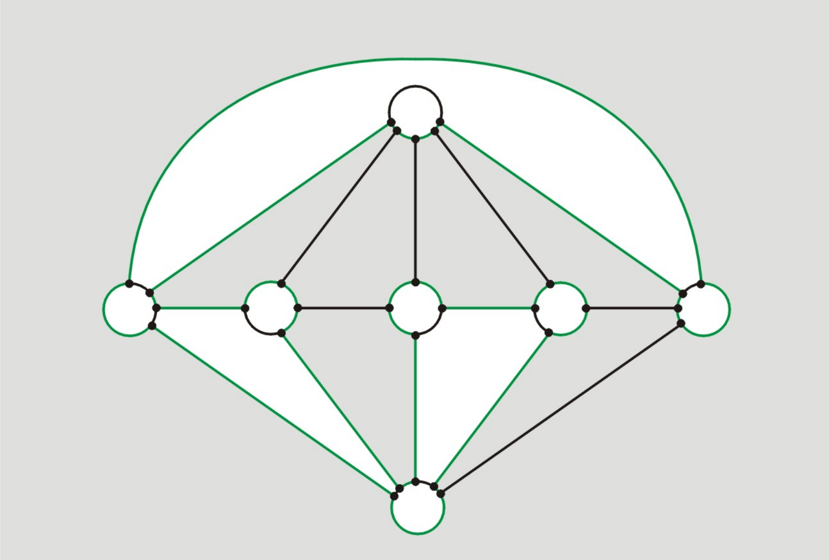
Example: Construction of a Hamilton cycle in $t(\mathcal{T})$



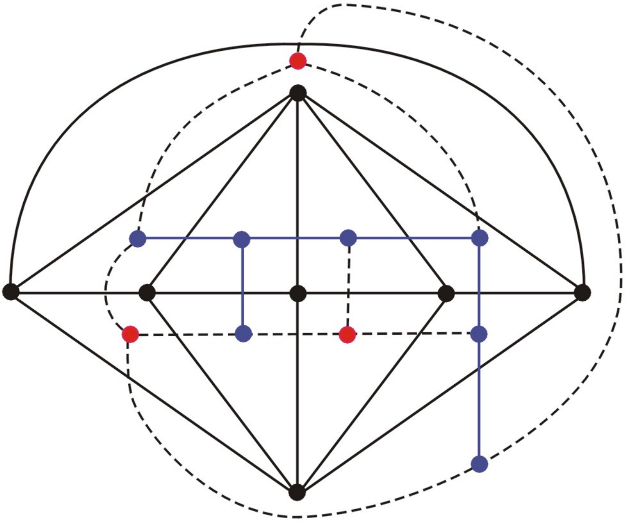
Example: Construction of a Hamilton cycle in $t(\mathcal{T})$



Example: The required Hamilton cycle



When does such a structure exist?



Maximum genus of a graph

Maximum genus of a graph

Definition. The maximum genus $\gamma_M(G)$ of a graph is the largest genus of an orientable surface in which G has a cellular embedding.

- By Euler-Poincaré Equation, $\gamma_M(G) \leq \lfloor \beta(G)/2 \rfloor$ where $\beta(G) = |E| - |V| + 1$ is the cycle rank of G.

Maximum genus of a graph

Definition. The maximum genus $\gamma_M(G)$ of a graph is the largest genus of an orientable surface in which G has a cellular embedding.

- By Euler-Poincaré Equation, $\gamma_M(G) \le \lfloor \beta(G)/2 \rfloor$ where $\beta(G) = |E| - |V| + 1$ is the cycle rank of G.

Definition. A graph G is upper-embeddable if $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$; equivalently, if it has an embedding with one or two faces.

Maximum genus of a graph

Definition. The maximum genus $\gamma_M(G)$ of a graph is the largest genus of an orientable surface in which G has a cellular embedding.

- By Euler-Poincaré Equation, $\gamma_M(G) \leq \lfloor \beta(G)/2 \rfloor$ where $\beta(G) = |E| - |V| + 1$ is the cycle rank of G.

Definition. A graph G is upper-embeddable if $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$; equivalently, if it has an embedding with one or two faces.

Upper-embeddability can be checked in a polynomial time!

Maximum genus of a graph

Definition. The maximum genus $\gamma_M(G)$ of a graph is the largest genus of an orientable surface in which G has a cellular embedding.

- By Euler-Poincaré Equation, $\gamma_M(G) \leq \lfloor \beta(G)/2 \rfloor$ where $\beta(G) = |E| - |V| + 1$ is the cycle rank of G.

Definition. A graph G is upper-embeddable if $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$; equivalently, if it has an embedding with one or two faces.

Upper-embeddability can be checked in a polynomial time!

In an upper-embeddable graph

- one-face embedding $\iff$ cycle rank is even
- two-face embedding $\iff$ cycle rank is odd

Maximum genus of a graph

Definition. The maximum genus $\gamma_M(G)$ of a graph is the largest genus of an orientable surface in which G has a cellular embedding.

- By Euler-Poincaré Equation, $\gamma_M(G) \leq \lfloor \beta(G)/2 \rfloor$ where $\beta(G) = |E| - |V| + 1$ is the cycle rank of G.

Definition. A graph G is upper-embeddable if $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$; equivalently, if it has an embedding with one or two faces.

Upper-embeddability can be checked in a polynomial time!

In an upper-embeddable cubic graph with n vertices

- one-face embedding $\iff$ $n \equiv 2 \pmod 4$
- two-face embedding $\iff$ $n \equiv 0 \pmod 4$

Upper-embeddable cubic graphs: even case

Theorem (K., N. & S., 2014+)

The following are equivalent for every connected cubic graph G.

(i) *G one-face-embeddable.*

(ii) *$V(G)$ has a partition $\{A, J\}$ where A induces a tree and J is independent.*

Upper-embeddable cubic graphs: even case

Theorem (K., N. & S., 2014+)

The following are equivalent for every connected cubic graph G.

(i) *G one-face-embeddable.*

(ii) *$V(G)$ has a partition $\{A, J\}$ where A induces a tree and J is independent.*

Corollary

Let $\mathcal{T}$ be a triangulation of a closed surface by f triangles. If the underlying graph of $\mathcal{T}^*$ is upper-embeddable and $f \equiv 2 \pmod 4$, then the truncation $t(\mathcal{T})$ has a Hamilton cycle.

Upper-embeddable cubic graphs: odd case

Theorem (K., N. & S., 2014+)

The following are equivalent for every connected cubic graph G.

(i) *G is two-face-embeddable.*

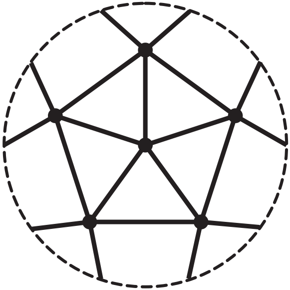
(ii) *$V(G)$ has a partition $\{A, J\}$ where either*

- *A induces a tree and J is near-independent, or*
- *A induces a forest with two components and J is independent.*

Corollary

Let $\mathcal{T}$ be a triangulation of a closed surface by f triangles. If the underlying graph of $\mathcal{T}^*$ is upper-embeddable and $f \equiv 0 \pmod 4$, then the truncation $t(\mathcal{T})$ has a Hamilton path.

Interesting example



Classes of upper-embeddable graphs

Many combinatorial properties properties are known to imply upper-embeddability, but only few apply to cubic graphs.

Classes of upper-embeddable graphs

Many combinatorial properties properties are known to imply upper-embeddability, but only few apply to cubic graphs.

- (odd) cyclic 4-edge-connectivity ($\Rightarrow$ Payan & Sakarovitch)
- edge-transitivity ("$\Rightarrow$" Glover & Marušič)
- existence of embeddings with "short" faces

Classes of upper-embeddable graphs

Many combinatorial properties properties are known to imply upper-embeddability, but only few apply to cubic graphs.

- (odd) cyclic 4-edge-connectivity ($\Rightarrow$ Payan & Sakarovitch)
- edge-transitivity ("$\Rightarrow$" Glover & Marušič)
- existence of embeddings with "short" faces

Upper-embeddability guarantees a Hamilton cycle in the even case and a Hamilton path in general . . .

. . . but it does not guarantee a long cycle in the odd case.

Ample upper-embeddability

Definition.

1. A cubic graph G is amply upper-embeddable if

(1) G is upper-embeddable

(2) $G - \{x, y\}$ remains upper-embeddable for a suitable pair of adjacent vertices.

2. An upper-embeddable cubic graph G is called tightly upper-embeddable if it is not amply upper-embeddable.

Ample upper-embeddability

Definition.

1. A cubic graph G is amply upper-embeddable if

(1) G is upper-embeddable

(2) $G - \{x, y\}$ remains upper-embeddable for a suitable pair of adjacent vertices.

2. An upper-embeddable cubic graph G is called tightly upper-embeddable if it is not amply upper-embeddable.

- Tightly upper-embeddable graphs seem to be very rare.
- Proving that a certain property implies ample upper embeddability is much more difficult than proving the usual upper embeddability.

Amply upper-embeddable cubic graphs: odd case

Theorem (K., N. & S., 2014+)

The following are equivalent for every connected cubic graph G.

(i) *G is amply two-face-embeddable.*

(ii) *$V(G)$ has a partition $\{A, J\}$ where A induces a tree and J is near-independent.*

Corollary

Let $\mathcal{T}$ be a triangulation of a closed surface by f triangles. If the underlying graph of $\mathcal{T}^*$ is amply upper-embeddable and $f \equiv 0 \pmod 4$, then the truncation $t(\mathcal{T})$ has a cycle through all but two adjacent vertices.

Classes of amply upper-embeddable cubic graphs

Theorem (K., N. & S., 2014+)

Every cyclically 4-edge-connected cubic graph is amply upper-embeddable.

Theorem (K., N. & S., 2014+)

Every connected edge-transitive cubic graphs is amply upper-embeddable.

Corollaries (samples)

Theorem (K., N. & S., 2014+)

Let $\mathcal{T}$ be a triangulation of a closed surface by f triangles which is either edge-transitive or has no separating cycle of length ≤ 3. Then $t(\mathcal{T})$ has a Hamilton path. Moreover,

- *if $f \equiv 2 \pmod 4$, then $t(\mathcal{T})$ has a Hamilton cycle, and*
- *if $f \equiv 0 \pmod 4$, then $t(\mathcal{T})$ has a cycle through all but two adjacent vertices.*

Corollary. Glover & Marušič

Corollaries (samples)

Theorem (K., N. & S., 2014+)

Let $\mathcal{T}$ be a triangulation of a closed surface by f triangles which is either edge-transitive or has no separating cycle of length ≤ 3. Then $t(\mathcal{T})$ has a Hamilton path. Moreover,

- *if $f \equiv 2 \pmod 4$, then $t(\mathcal{T})$ has a Hamilton cycle, and*
- *if $f \equiv 0 \pmod 4$, then $t(\mathcal{T})$ has a cycle through all but two adjacent vertices.*

Corollary. Glover & Marušič

Theorem (K., N. & S., 2014+)

Let $\mathcal{T}$ be a polyhedral triangulation of a closed surface by f triangles such that every vertex has valency ≤ 7. Then $t(\mathcal{T})$ has a Hamilton path, and if $f \equiv 2 \pmod 4$, then $t(\mathcal{T})$ has a Hamilton cycle.

Final remarks and problems

- Find more infinite classes of amply upper-embeddable graphs.

Final remarks and problems

- Find more infinite classes of amply upper-embeddable graphs.
- In particular:

Question

Is every simple cubic graph embeddable in some closed surface with all faces of length ≤ 7 amply upper-embeddable?

It is known that it is upper-embeddable (Huang & Liu, 2000).

Final remarks and problems

- Find more infinite classes of amply upper-embeddable graphs.
- In particular:

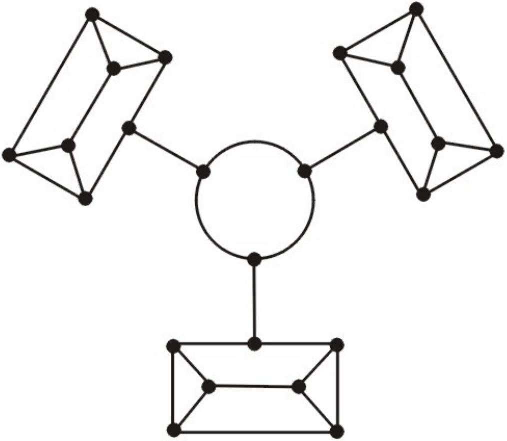
Question

Is every simple cubic graph embeddable in some closed surface with all faces of length ≤ 7 amply upper-embeddable?

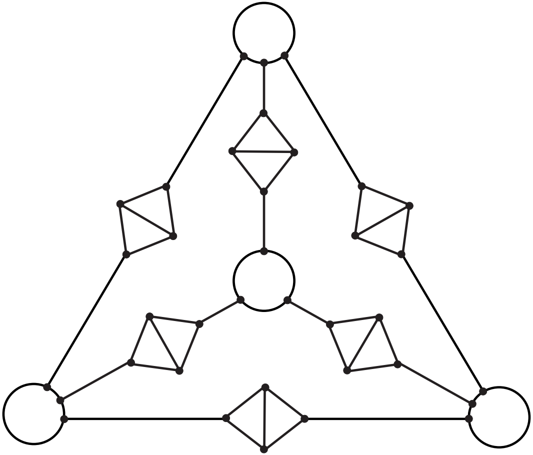
It is known that it is upper-embeddable (Huang & Liu, 2000).

- So far we know very little about tightly upper-embeddable graphs.

A tightly 2-face-embeddable graph



A tightly 2-face-embeddable graph



Final remarks and problems

Problem

Do there exist 3-connected tightly upper-embeddable cubic graphs?

Thank you!