Generalized β-skeletons

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β-skeletons

The **β-skeletons $\{G(V)\}_\beta$** for a point set V is a hierarchy of graphs on V based upon a natural notion of „neighborlines" parameterized by real number $\beta \geq 0$. They are both important and popular because of many practical applications which span a spectrum of areas from geographic information systems and wireless ad hoc networks to shape recognition and machine learning.

Two types of β–skeletons are especially well-known, Gabriel Graph (GG) for $\beta = 1$ and the Relative Neighborhood Graph (RNG) for $\beta = 2$.

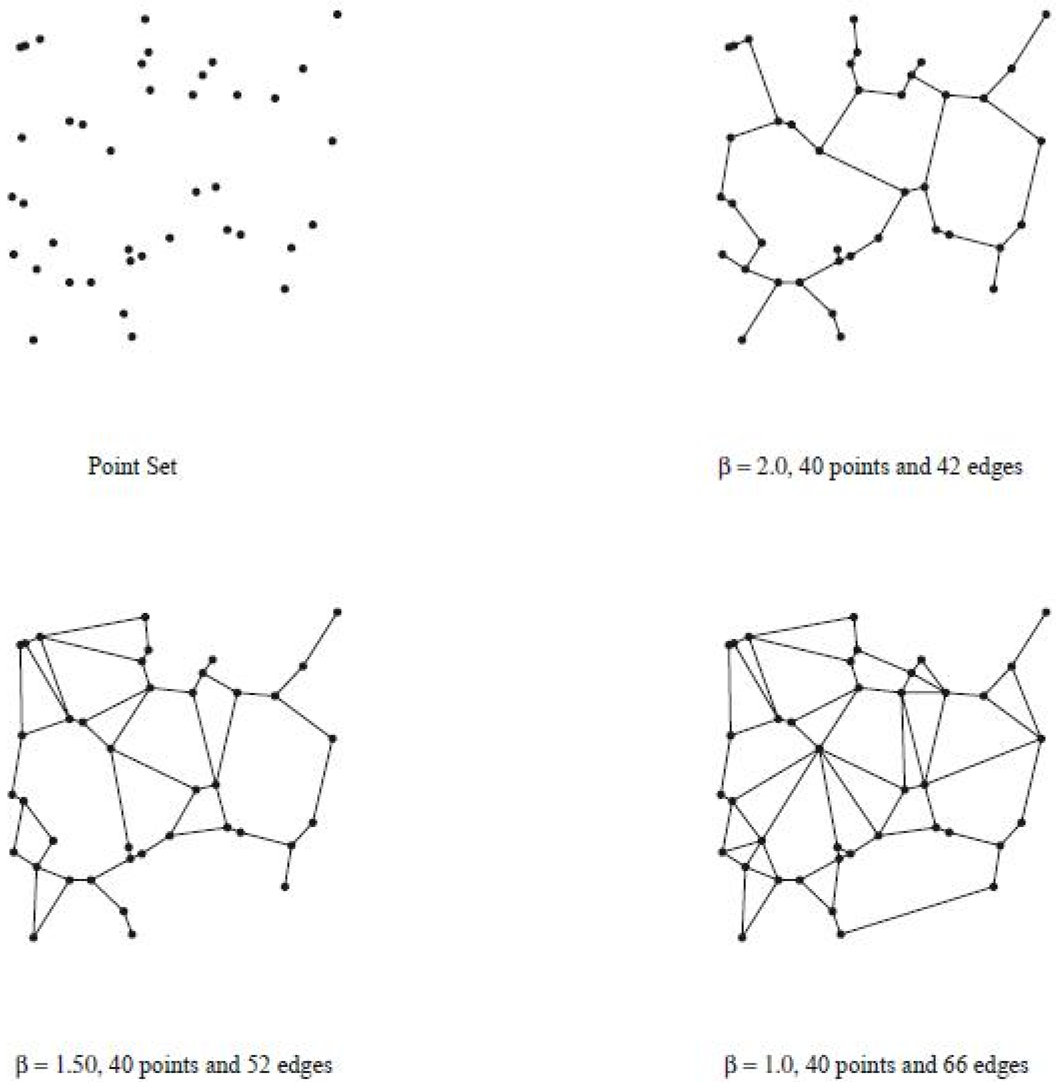


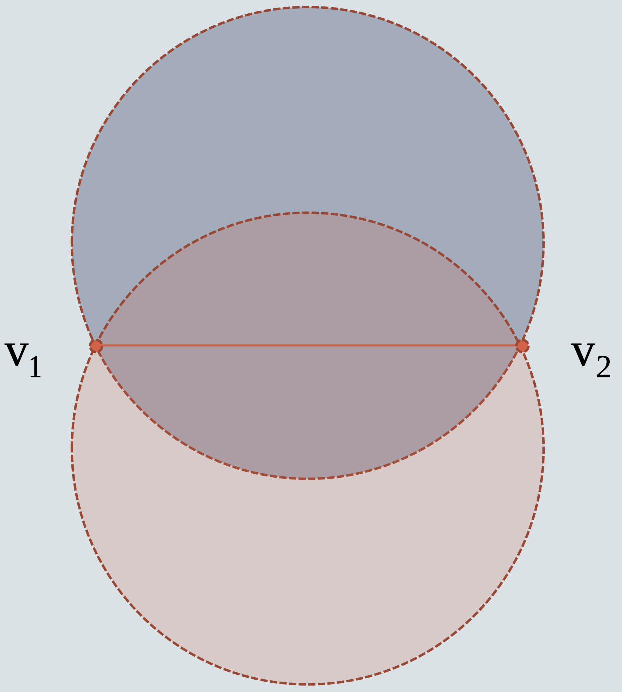
Figure 2: β-skeletons for various values of β

Lune-based β-skeletons

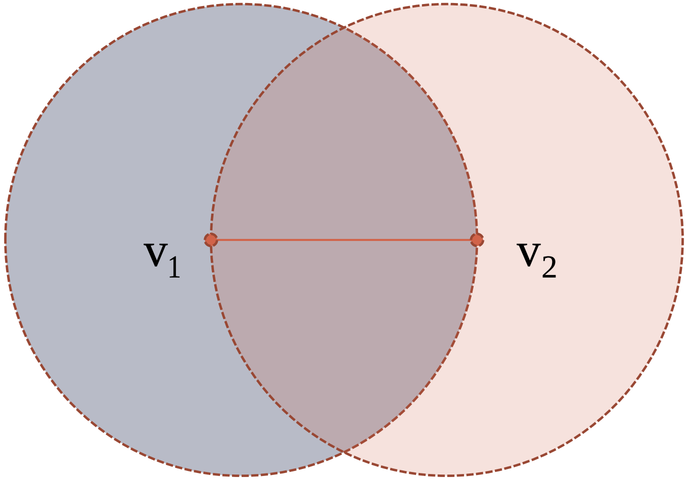
For a given set of points $V=\{v_1,v_2,\ldots,v_n\}$ in $\mathbb{R}^2$ and parameters $\beta\geq 0$ and p we define graph $G_\beta(V)$- called a *lune-based β-skeleton* –as follows:
two points v_1,v_2 are connected with an edge if and only if no point from $V\setminus\{v_1,v_2\}$ belongs to the set $N_p(v_1,v_2,\beta)$ where:

1. for $\beta=0$ the set $N_p(v_1, v_2, \beta)$ is segment v_1v_2;

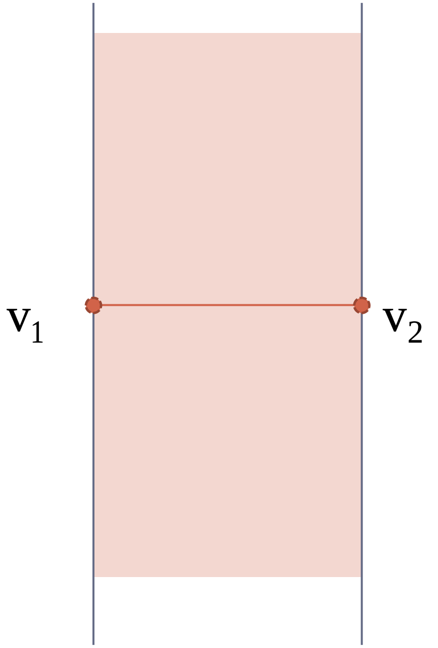
2. if $0<\beta<1$ then $N_p(v_1, v_2, \beta)$ is an intersection of two discs in l_p, each with radius $|v_1v_2|/2\beta$, whose boundaries contain both v_1 and v_2;



3. for $1 \leq \beta < \infty$ set $N_p(v_1, v_2, \beta)$ is an intersection of two discs in l_p metric, each with radius $\beta|v_1v_2|/2$, whose centers are in points $(\beta/2)v_1+(1-\beta/2)v_2$ and $(1-\beta/2)v_1+(\beta/2)v_2$;



4. for $\beta=\infty$, $N_p(v_1, v_2, \beta)$ is the unbound strip between lines perpendicular to the segment v_1v_2 that contain v_1 and v_2 respectively.



Motivation

Fact 1

Let us assume that that points in V are in general position.
For $1 \le \beta \le \beta' \le 2$ following inclusions are true:

$$MST(V) \subseteq RNG(V) \subseteq G_{\beta'}(V) \subseteq G_{\beta}(V) \subseteq GG(V) \subseteq DT(V).$$

Notice, that in some metric spaces (like for example in weighted graphs) it is hard to interpret the linear combination of analyzed two points from the definition of the lune-based β-skeleton, so we want to instead use the fact that we know the distances of centers of discs defining the lune from the ends of the edge we are checking.

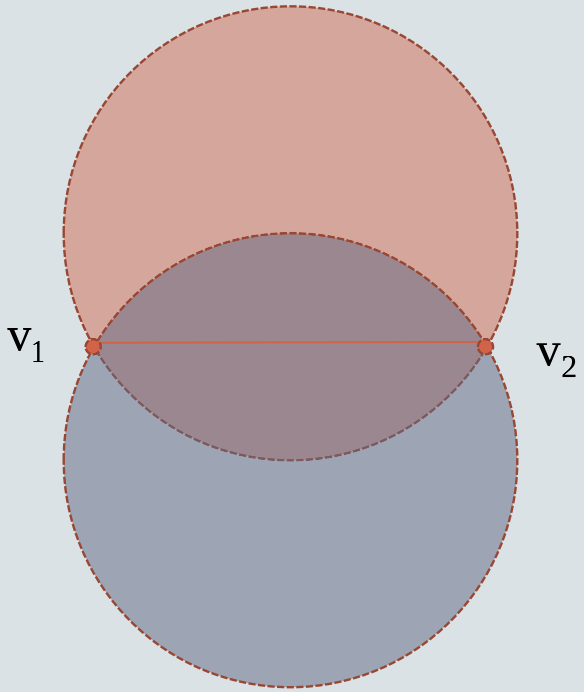
We are interested in creating a global definition, based only on a distance criterion, that will allow us to define β-skeletons for a bigger class of events and that will satisfy the above inclusions.

Circle-based β-skeletons

By changing the definition of lune-based β-skeletons for $\beta \geq 1$ we get the family of circle-based β-skeletons.

1. for $1 \leq \beta < \infty$ set $N_p(v_1, v_2, \beta)$ is an union of two discs in l_p each with radius $\beta\,|v_1 v_2|/2$, whose boundaries contain both v_1 and v_2;

2. for $\beta = \infty$, $N_p(v_1, v_2, \beta)$ is an union of segment $v_1 v_2$ and two open hyperplanes defined by the line passing through $v_1 v_2$.



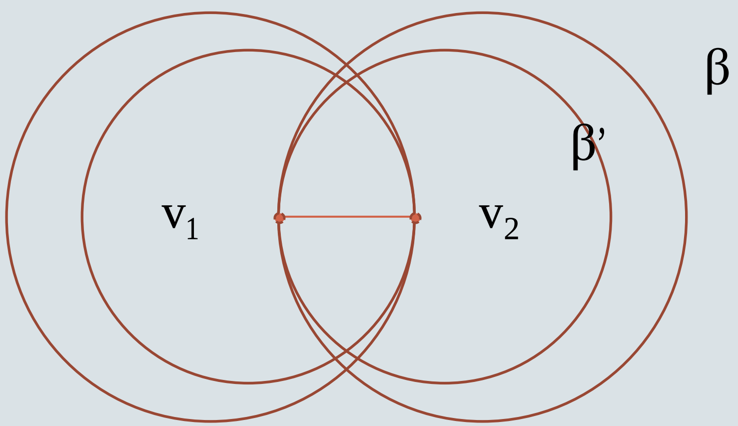
Defining β-skeletons using a distance criterion

For a given set V and given parameters $\beta \geq 0$ and $1 < p < \infty$ we define a graph $G_\beta^*(V)$, where an edge exists between two points v_1 and v_2 iff no point from $V \setminus \{v_1, v_2\}$ belongs to set $N_p^*(v_1, v_2, \beta)$ where:

1. for $0 \leq \beta < 1$ we have $N_p^*(v_1, v_2, \beta) = N_p(v_1, v_2, \beta)$ defined as before;
2. for $1 \leq \beta < \infty$ the set $N_p^*(v_1, v_2, \beta)$ is an intersection of two l_p discs , each with radius $\beta\,|v_1v_2|/2$, whose centers are in such points c_1 and c_2 that the distance between v_1 and c_1 (v_2 and c_2 respectively) is $\beta\,|v_1v_2|/2$, the distance between v_1 and c_2 (v_2 and c_1 respectively) is $|1-\beta/2||v_1v_2|$ and the distance between c_1 and c_2 is $(\beta-1)\,|v_1v_2|$;

3. If $\beta=\infty$ we define C_∞ as a set of all discs c such that there exists a sequence of discs $\{c(\beta)|c(\beta)$ is a disc that we used to define set $N_d(v_1,v_2,\beta)$ $\}$ convergent to c with $\beta \rightarrow \infty$;
then $N_d(v_1, v_2, \infty)$ is intersection of any two different discs from C_∞.

We rely here on the fact that for $\beta \geq \beta' \geq 1$ lune defined by discs $D_p(c_1(\beta'), \beta'|v_1v_2|/2)$) and $D_p(c_2(\beta'), \beta'|v_1v_2|/2)$) is contained in a similiar lune defined for β.



Lemma 1

Let V be a set of points in R^2 with l_p metric , where $1<p<\infty$. Then, centers of the discs determining lunes are uniquely defined and $G_\beta(V)=G_\beta^*(V)$.

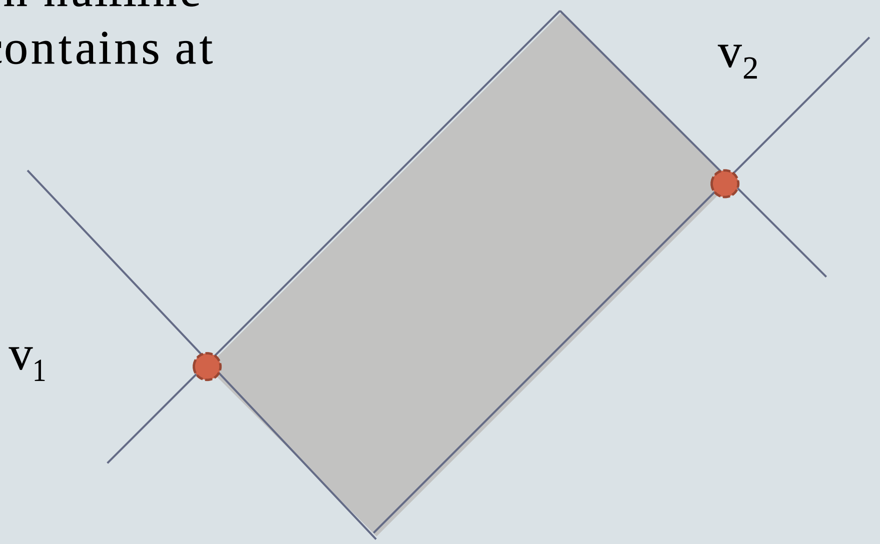
Note that in l_1 and l_∞ Lemma 1 is not true, since it is possible to find in those metrics internally tangent circles with different centers that instersect at many point and so the centers of the discs determining lunes are not uniquely defined.

We are interested in changing the definition so that we also include the cases where the centers of the discs defining the lunes for a given edge are not uniquely defined.

Problem 1: Not uniquely defined centers of the discs determining lunes

In l_p metric, where $p \in \{1,\infty\}$, for a given set V and given parameter $\beta \geq 0$ we define a graph G_β^(V), where an edge exists between two points v_1 and v_2 iff no point from $V \setminus \{v_1, v_2\}$ belongs to of of the sets N_p^(v_1, v_2, β) where:

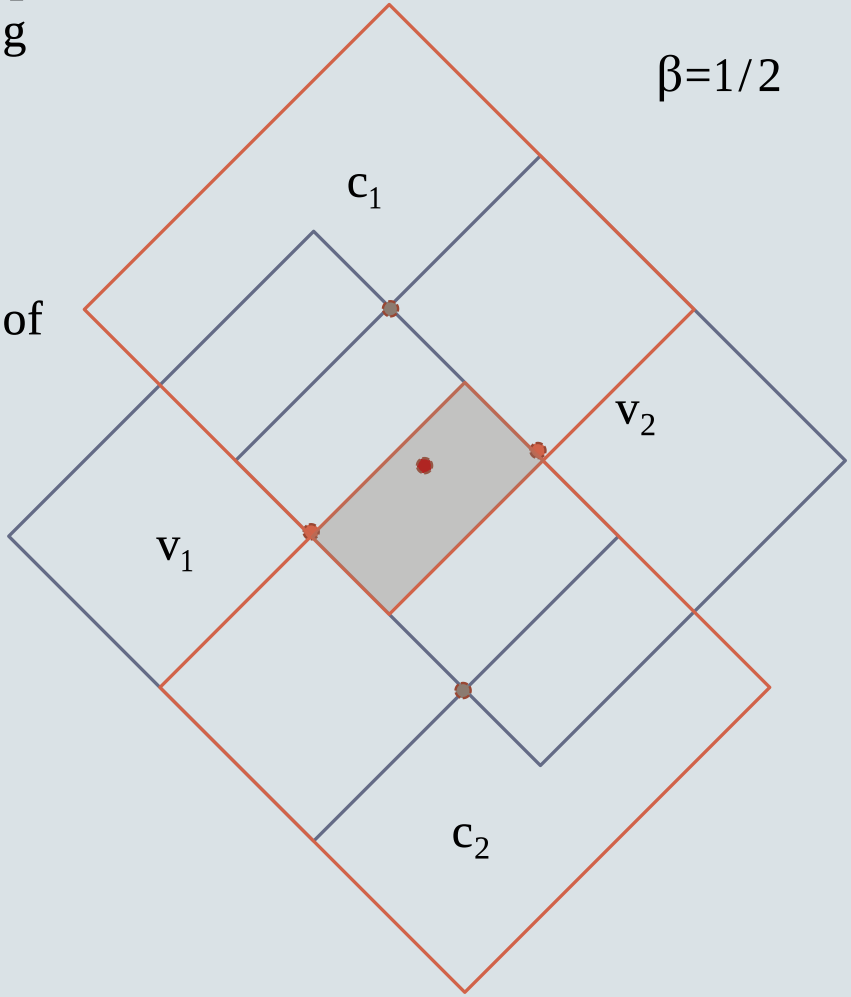
1. for β=0 the set N_p^(v_1, v_2, β) is an intersection of two quarter-planes (that borders are parallel to parts of circles in l_p), such that each halfline on the border of the quarter-planes contains at least one of points v_1 and v_2;



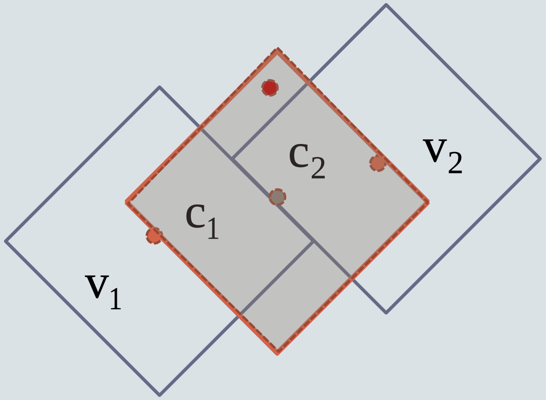
2. if for $0<\beta<1$ let points c_1 and c_2 be such points (if they exist) that: $d_p(c_1,v_1) = d_p(c_2,v_1) = d_p(c_1,v_2) = d_p(c_2,v_2) = d_p(v_1,v_2)/2\beta$ and all paths connecting those points with lenght shorter then $d_p(v_1,v_2)/\beta$ intersect shortest paths between v_1 and v_2:

then N_p^(v_1, v_2, β) is an intersection of discs $D_p(c_1, d_G(v_1,v_2)/2\beta)$ and $D_p(c_2, d_p(v_1,v_2)/2\beta)$;

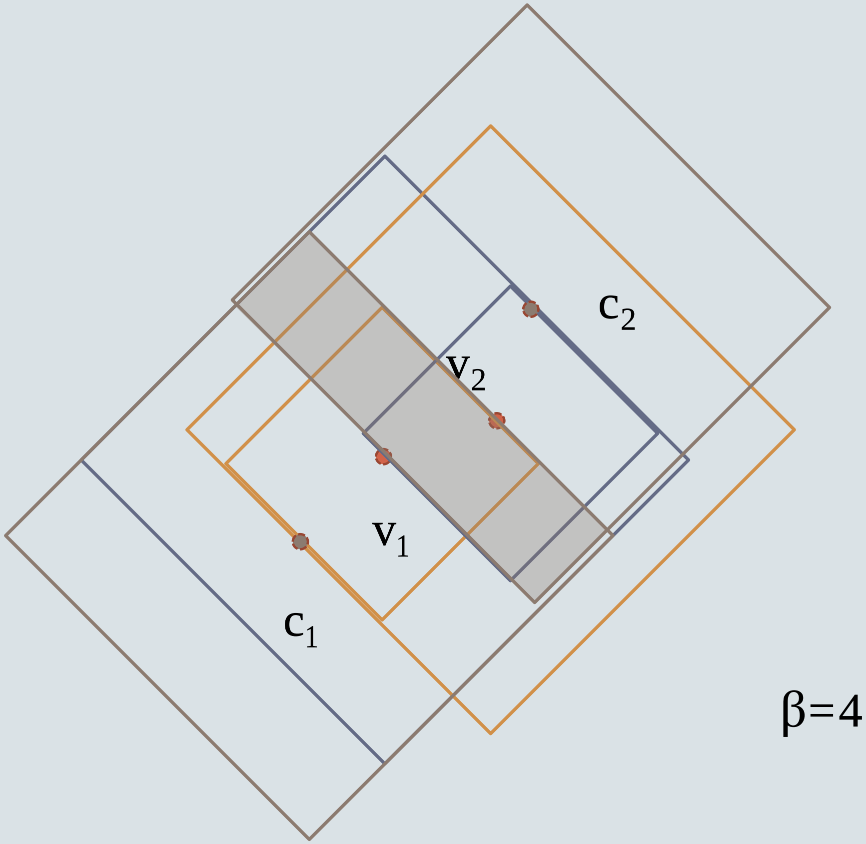
if points c_1 and c_2 don't exist then N_p^(v_1, v_2, β) is an empty set;

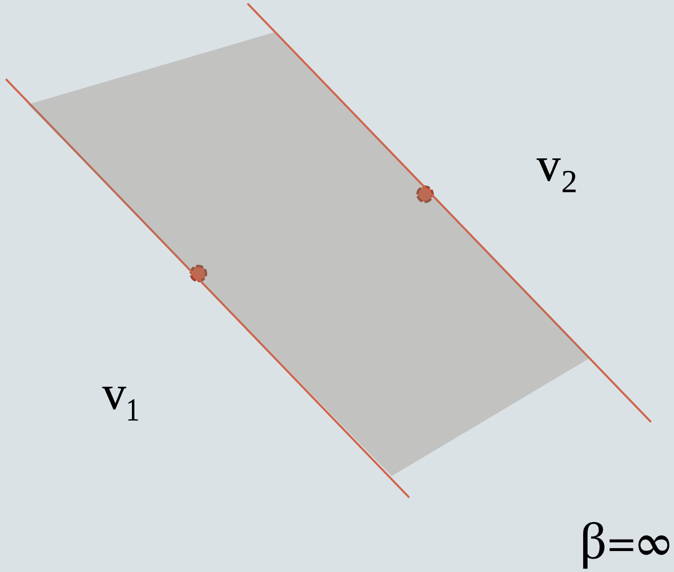


3. for $1 \leq \beta < \infty$ we define set C_1 (respectively C_2) of disc centers c_1 such that the distance between v_1 and c_1 (v_2 and c_2 respectively) is $\beta|v_1v_2|/2$ and the distance between v_1 and c_2 (v_2 and c_1 respectively) is $|1-\beta/2||v_1v_2|$; set N_p^(v_1, v_2, β) is an intersection of any two discs , each with radius $\beta\ |v_1v_2|/2$ centered in such points $c_1 \in C_1$ and $c_2 \in C_2$ (respectively) that the distance between c_1 and c_2 is $(\beta-1)\ |v_1v_2|$;



4. for $\beta=\infty$ we define C_∞ as a set of all discs c such that there exists a sequence of discs $\{c(\beta)|c(\beta)$ is a disc that we used to define set $N_d(v_1,v_2,\beta)\}$ convergent to c with $\beta\rightarrow\infty$;
then $N_d(v_1, v_2, \infty)$ is intersection of any two different discs from C_∞.





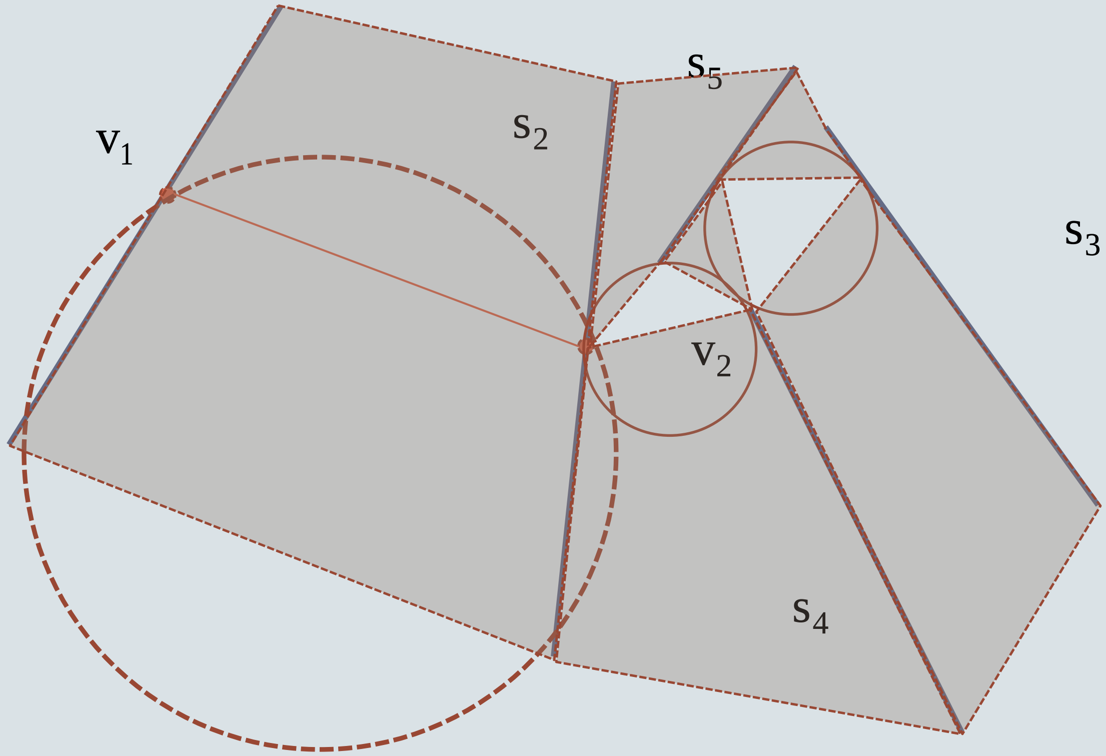
Problem 2: β-skeleton for a set of objects different then points

Let S be a set of segments in (R^2, l_e). From *Flip Algorithm for Segment Triangulations* by Brevilliers, Chevallier and Schmitt we have:

A **segment triangulation T** of S is a partition of the convex hull conv(S) of S in disjoint sites, edges, and faces such that:

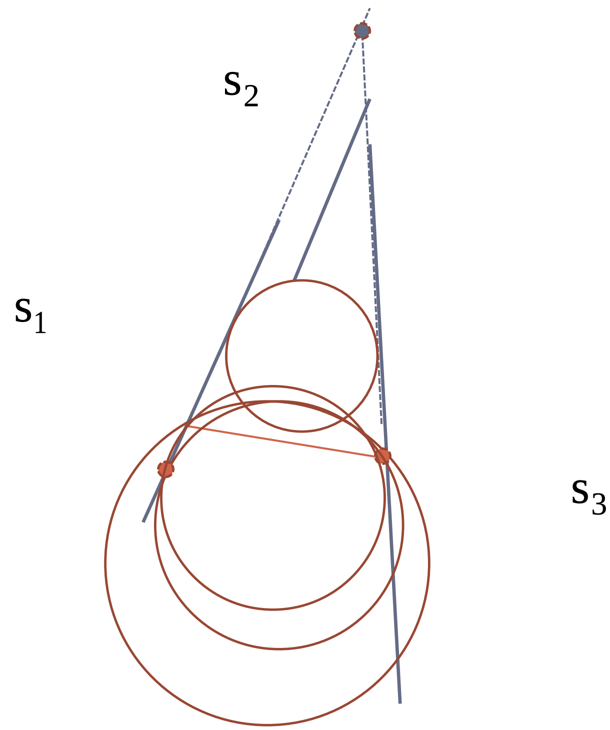
1. Every face of T is an open triangle whose vertices are in three distinct sites of S and whose open edges do not intersect S,
2. No face can be added without intersecting another one,
3. The edges of T are the (possibly two-dimensional) connected components of conv(S)\(F ∪S), where F is the union of faces of T.

A segment triangulation of S is Delaunay DT(S) if the circumcircle of each face does not contain any point of S in its interior.



Now, for a given parameter $\beta \in [1,2]$ we define lune-based β-skeleton $M_\beta(S)$ as follows:

for segments s_1 and s_2 and edge exists between vertices $v_1 \in s_1$ and $v_2 \in s_2$ iff no point from segments $S \setminus \{s_1, s_2\}$ belongs to set $N_2(v_1, v_2, \beta)$ (defined like in the first or second defition).



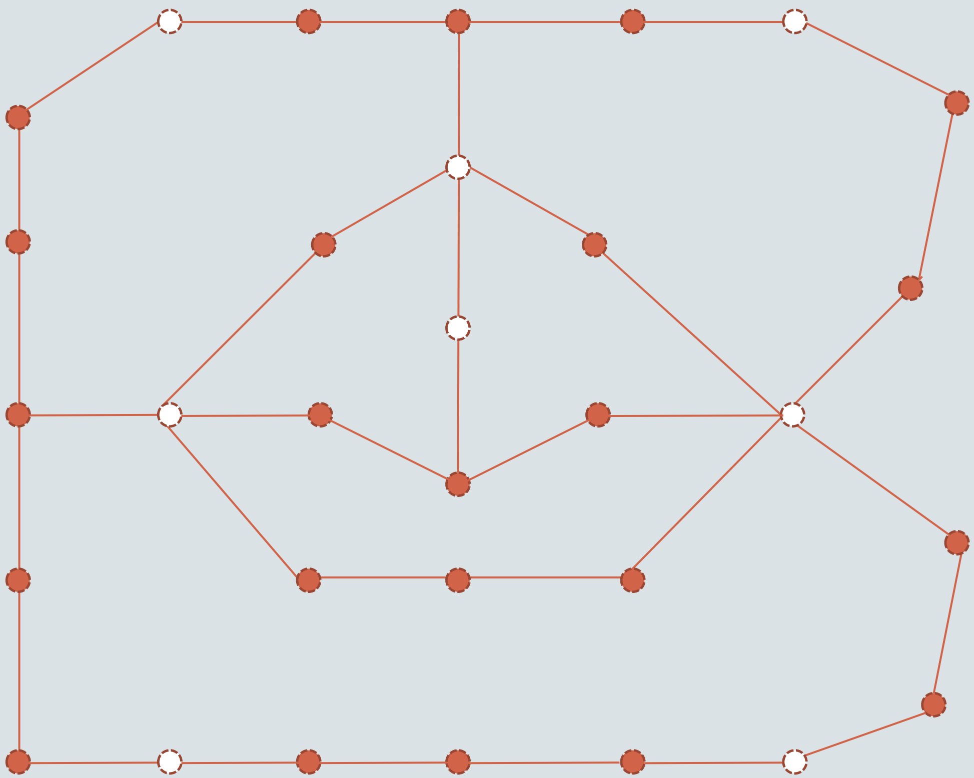
Observation 2

For 1≤β≤β'≤2 we have:

$$MST(S) \subseteq RNG(S) \subseteq M_{\beta'}(S) \subseteq M_{\beta}(S) \subseteq GG(S) \subseteq DT(S)$$

where $RNG(S)=M_2(S)$, $GG(S)=M_1(S)$ and MST(S) is a minimum spanning tree for the set of segments S.

β-skeletons in weighted graphs 1



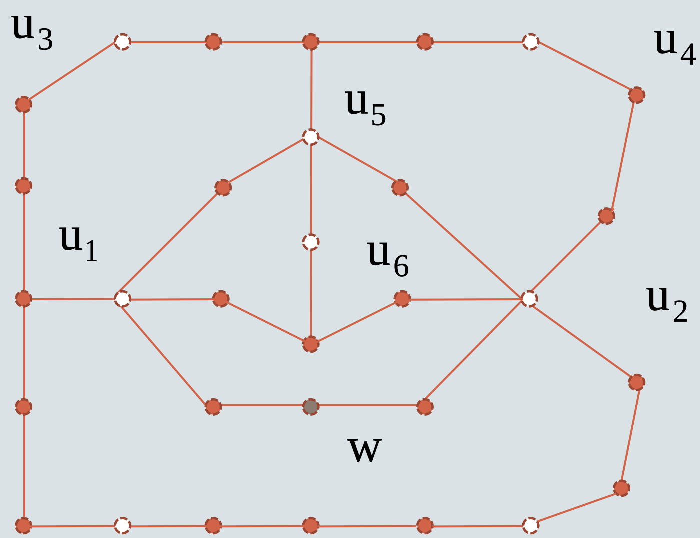
Graph G=(V,U,E) where all edges have a positive, finite weight

We can define a metric d_G here:
d_G(v,w)=|shortest path between v and w|

From *Proximity graphs in large weighted graphs* by Abrego et al. we get:

The **Voronoi region** for point $u_i \in U$ consists of those points $p \in G$ that satisfy $d_G(p,u_i) \leq d_G(p,u_k)$ for all $u_k \in U \setminus \{u_i\}$. **Voronoi Diagram of G**, denoted by VD(G) is the partition of G into Voronoi regions for points $u_i \in U$.

Let DT(G) be such a graph where points u_i and u_j are connected by an edge if and only of there exists a disc $D_G(v,r)$ enclosing u_i and u_j and containing no other points from U, where $v \in V$ and $r = \min\{r | D_G(v, r)$ contains u_i and $u_j\}$.
We call this graph **Delaunay Triangulation** of V



β-skeletons in weighted graphs 2

In graph $G=(V,U,E)$ with metric d_G for a given parameter $\beta \geq 0$ we define a graph $G_\beta(G)=(U,F)$, where an edge exists between two points u_1 and u_2 iff there exists a set $N_G(u_1, u_2, \beta)$ such that no point from $U \setminus \{u_1, u_2\}$ belongs to it where:

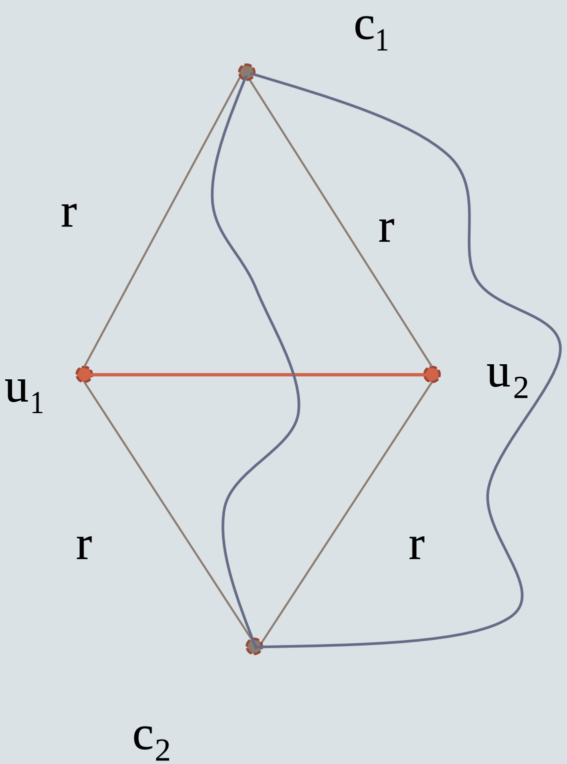
1. we don't define this $N_G(u_1, u_2, \beta)$ set for $\beta=0$ nor for $\beta=\infty$

We will explain in a moment why.

2. for $0<\beta<1$ let points c_1 and c_2 be such points (if they exist) that:
$d_G(c_1,u_1) = d_G(c_2,u_1) = d_G(c_1,u_2) = d_G(c_2,u_2) = d_G(u_1,u_2)/2\beta$ and all paths connecting those points with lenght shorter then $d_G(u_1,u_2)/\beta$ intersect shortest paths between u_1 and u_2:

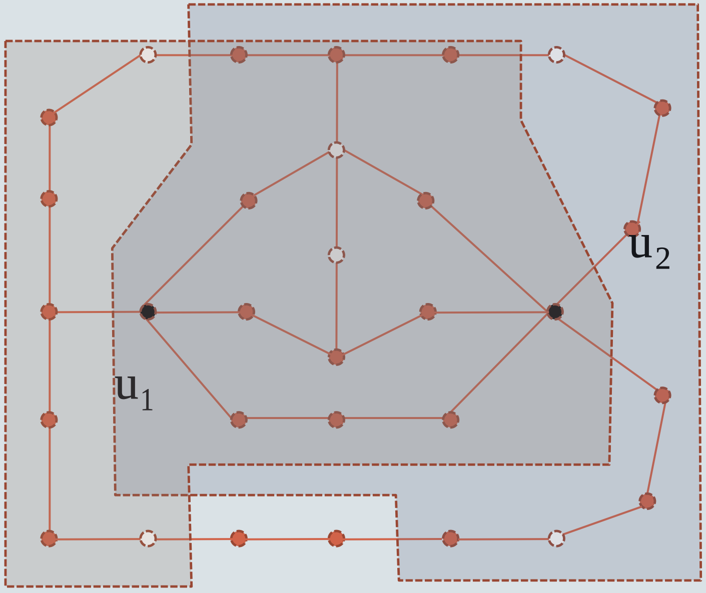
then $N_G(u_1, u_2, \beta)$ is an intersection of discs $D_G(c_1, d_G(u_1,u_2)/2\beta)$ and $D_G(c_2, d_G(u_1,u_2)/2\beta)$;

if points c_1 and c_2 don't exist then $N_G(u_1, u_2, \beta)$ is an empty set;



$d(v_1,v_2)/2\beta = r$

3. for $1 \le \beta < \infty$ we define set C_1 (respectively C_2) of disc centers c_1 such that the distance between u_1 and c_1 (u_2 and c_2 respectively) is $\beta d_G(u_1,u_2)/2$ and the distance between u_1 and c_2 (u_2 and c_1 respectively) is $|1-\beta/2| d_G(u_1,u_2)$; set $N_G(u_1, u_2, \beta)$ is an intersection of any two discs in G each with radius $\beta d_G(u_1,u_2)/2$, centered in such points $c_1 \in C_1$ and $c_2 \in C_2$ (respectively) tht the distance between c_1 and c_2 is $(\beta-1)|v_1 v_2|$;



$\beta=2$

We call graph $G_\beta(G)$ a **lune-based β-skeleton** for graph G.

Lemma

For each graph G=(V,U,E) and for $1 \le \beta \le \beta' \le 2$ following inclusions are true:

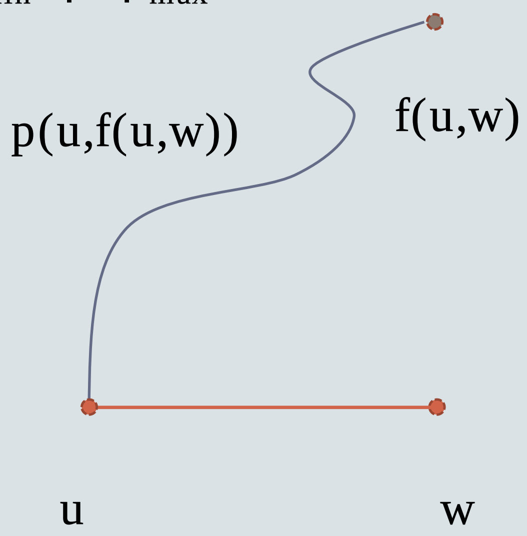
$$MST(G) \subseteq RNG(G) \subseteq G_{\beta'}(G) \subseteq G_{\beta}(G) \subseteq GG(G) \subseteq DT(G)$$

where $RNG(G)= G_2(G)$ and $GG(G)=G_1(G)$.

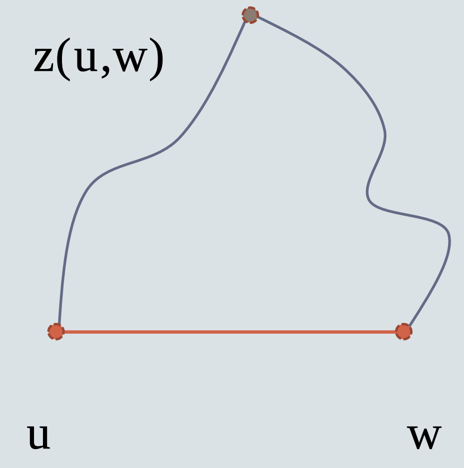
We cannot modify definiton of the lune-based beta skeleton for $\beta=0$ and for $\beta=\infty$ because if the weights of the edges of the graph G are finite then we can only define β-skeletons for some values of β.

Lemma

For graph G=(V,U,E) we can at most define $G_{\beta}(G)$ for $\beta_{min} \le \beta \le \beta_{max}$ where:



1. For a given pair of vertices u, w in graph G let f(u,w) be the point farthest from u in graph G\{all edges on shortest paths between u and w} and let p(u,f(u,w)) be the shortest path between u and f(u,w) in this graph.
   Then, $\beta_{max}=2[(\max_{u,w} |p(u, f(u,w))|/|uw|)]+2$.



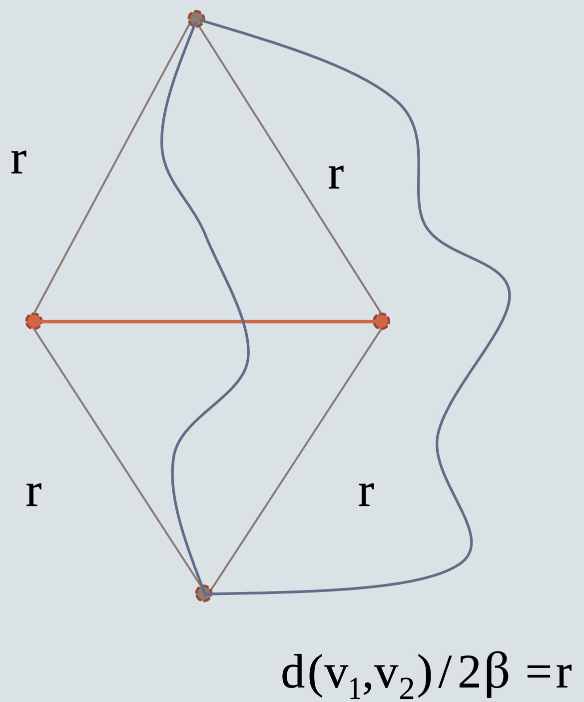
2. Now, let z(u, w) be the farthest point from u_1 and u_2 such that d(u,z(u,w))= d(w, z(u, w)) in graph G\{all edges on shortest paths between u and w}.

   Then, $\beta_{min} =\min_{u,w}[|uw|/2(d(u,z(u, w)))]$.

Generalized β-skeleton

For a given set of objects S in space L with metric d we define a graph $G_\beta(S)$, where an edge exists between two points v_1 from s_1 and v_2 from s_2 iff at least one set $N_d(v_1,v_2,\beta)$ is not intersected by any object from $S\backslash\{s_1, s_2\}$ where:

1. for $0<\beta<1$ let points c_1 and c_2 be such points (if they exist) that: $d(c_1,v_1) = d(c_2,v_1) = d(c_1,v_2) = d(c_2,v_2)$ and all paths connecting those points with lenght shorter then $d(v_1,v_2)/2\beta$ intersect shortest paths between v_1 and v_2: then $N_d(u_1, u_2, \beta)$ is an intersection of discs $D(c_1, d(v_1,v_2)/2\beta)$ and $D(c_2,d(v_1,v_2)/2\beta)$;
   if points c_1 and c_2 don't exist then $N_d(v_1,v_2, \beta)$ is an empty set;



2. for $1 \le \beta < \infty$ we define set C_1 (respectively C_2) of disc centers c_1 such that the distance between v_1 and c_1 (v_2 and c_2 respectively) is $\beta d(v_1,v_2)/2$ and the distance between v_1 and c_2 (v_2 and c_1 respectively) is $|1-\beta/2|d(v_1,v_2)$; set $N_d(v_1, v_2, \beta)$ is an intersection of any two discs in L, each with radius $\beta d(v_1,v_2)/2$, centered in such points $c_1 \in C_1$ and $c_2 \in C_2$ that the distance between c_1 and c_2 is $(\beta-1)\ |v_1 v_2|$;

3. for **$\beta = 0$** let C_0 be a set of all discs c such that there exists a sequence of discs $\{c(\beta)|c(\beta)$ is a disc that we used to define set $N_d(v_1, v_2, \beta)\}$ convergent to c with $\beta \to 0$;
   $N_d(v_1, v_2, 0)$ is an intersection of any two different discs from C_0;

4. for $\beta = \infty$ we define C_∞ as a set of all discs c such that there exists a sequence of discs $\{c(\beta)|c(\beta)$ is a disc that we used to define set $N_d(v_1, v_2, \beta)\}$ convergent to c with $\beta \to \infty$;
   then $N_d(v_1, v_2, \infty)$ is intersection of any two different discs from C_∞.

Conclusions and open problems

We showed a way to generalize β-skeletons basing on distance criterion. We focused only on few special cases but we think that they descibe well the idea if this general definition.

In a similiar way we could define β-skeletons for sets of polygons and it is also possible to generalize this definition to higher dimentions.

There is a couple of new problems regarding this definition. It would be interesting to check how those changes can influence the time of algorithms computing β-skeletons.

We can also analyse what interesting properties do β-skeletons for different objects have.

Thank You for Your Attention