Graph Drawing Beyond Planarity: Some Results and Open Problems

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(final – talk)
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\[ \sqrt{(final - talk)^2} \]
Outline

• Graph drawing beyond planarity
• Area-crossings trade-offs
• Inclusion relationships
• Open problems
GD beyond planarity
edge crossings significantly affect the readability (see, e.g., Sugiyama et al., Warshall, North et al., Batini et al., mid 80s) - confirmed by cognitive experimental studies (Purchase et al., 2000-2002)
Readability and Crossings

edge crossings significantly affect the readability (see, e.g., Sugiyama et al., Warshall, North et al., Batini et al., mid 80s) - confirmed by cognitive experimental studies (Purchase et al., 2000-2002)

rich body of graph drawing techniques assume the input is a planar (planarized) graph
The planarization handicap

for dense enough or constrained enough drawings, many edge crossing are unavoidable
The planarization handicap

for dense enough or constrained enough drawings, many edge crossing are unavoidable

FlyCircuit Database, NTHU
Mutzel’s intuition about crossings

34 crossings:
minimum “skewness”
(number of edges whose deletion makes it planar)

24 crossings:
minimum number of crossings
Experiments of Eades, Hong, Huang

Observations from eye tracking

- **No crossings**: eye movements were smooth and fast.
- **Large crossing angle**: eye movements were smooth, but a little slower.
- **Small crossing angle**: eye movements were very slow and no longer smooth (back-and-forth movements at crossing points).
Example

[Didimo, L., Romeo, “A Graph Drawing Application to Web Site Traffic Analysis”, JGAA 2011]
Beyond planarity

the visual complexity not only depends on the number of crossings but also on the type of crossings
Beyond planarity

the *visual complexity* not only depends on the number of crossings but also on the *type of crossings*

**challenge:** compute drawings where some “bad” crossing configurations are forbidden (minimized)
Drawings with forbidden crossing configurations
Drawings with forbidden crossing configurations
Drawings with forbidden crossing configurations

RAC

SKEWNESS-\(h\) (\(h=1\))
Drawings with forbidden crossing configurations

RAC

SKEWNESS-h (h=1)

h-PLANAR (h=3)
Drawings with forbidden crossing configurations

RAC

SKEWNESS-h (h=1)

h-PLANAR (h=3)

h-QUASI-PLANAR (h=3)
Drawings with forbidden crossing configurations

Strong 1-visibility drawing

Weak 1-visibility drawing
Most explored research directions
Most explored research directions

Turán-type: find upper bounds on the edge density
Most explored research directions

**Turán-type:** find upper bounds on the edge density

**Recognition:** how hard is it to test whether a graph admits a drawing with a forbidden configuration?
Most explored research directions

**Turán-type:** find upper bounds on the edge density

**Recognition:** how hard is it to test whether a graph admits a drawing with a forbidden configuration?

**Fáry-type:** given a drawing (with jordan arcs), is there a straight-line drawing that preserves the given topology?
New research directions
New research directions

study trade-offs between crossing complexity and other aesthetic criteria
New research directions

study trade-offs between crossing complexity and other aesthetic criteria

study the combinatorial relationships between different families of nearly planar graphs
Area Requirement Beyond Planarity
A result by Angelini et al.

RAC straight-line drawings of planar graphs may require quadratic area

(Angelini et al., JGAA 2011)
Area req. of h-planar drawings

$h$-planar (constant $h$) straight-line drawings (and RAC straight-line drawings) of planar graphs may require \textit{quadratic area} [Di Giacomo et al., 2012]
Area req. of h-planar drawings

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Area req. of h-planar drawings

*h-planar (constant h)* straight-line drawings (and RAC straight-line drawings) of planar graphs may require quadratic area [Di Giacomo et al., 2012]
Area req. of skewness-$h$ drawings

skewness-$h$ (constant $h$) straight-line drawings of planar graphs may require quadratic area
Area req. of skewness-$h$ drawings

skewness-$h$ (constant $h$) straight-line drawings of planar graphs may require quadratic area
linear area upper bound
linear area upper bound

h-quasi-planar drawings
linear area upper bound

4-quasi-planar

h-quasi-planar drawings
Bounded treewidth

G has treewidth $\leq k \iff G$ is a partial $k$-tree
Bounded treewidth

G has treewidth $\leq k \iff G$ is a partial $k$-tree
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G has treewidth $\leq k \iff G$ is a partial $k$-tree
Bounded treewidth

G has treewidth $\leq k \Leftrightarrow$ G is a partial $k$-tree
Bounded treewidth

G has treewidth $\leq k \iff G$ is a partial $k$-tree
The good news

every n-vertex graph with bounded treewidth admits an $h$-quasi planar straight-line drawing in linear area such that the value of $h$ does not depend on $n$

[Di Giacomo, Didimo, L., Montecchiani, 2012]
Applying the result

Every $h$-colorable graph has a linear area s.l. drawing [Wood, CGTA, 2005]
Applying the result

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[Wood, CGTA, 2005]
Applying the result

every $h$-colorable graph has a linear area s.l.drawing [Wood, CGTA, 2005]

Di Giacomo et al., 2012
Ingredients

study the relationship between \((c,t)\)-track layouts and h-quasi planar straight-line drawings

new technique to compute a \((2,t)\)-track layout of a partial k-tree
(c, t)-track layout

vertex layering and edge coloring such that no X-crossing between two edges of the same colour occurs
vertex layering and edge coloring such that no X-crossing between two edges of the same colour occurs

\((c,t)\)-track layout
(c, t)-track layout

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vertex layering and edge coloring such that no \textbf{X-crossing} between two edges of the same colour occurs

\textbf{\((c,t)\)-track layout}\n
\textbf{(2,3)-track layout}
vertex layering and edge coloring such that no X-crossing between two edges of the same colour occurs

(c, t)-track layouts have been studied in the context of 3D graph drawing
assume to have a \((c,t)\)-track layout. It is shown how to compute a \([c(t-1) + 1]\)-quasi planar straight-line drawing in \(O(t^3n)\) area
The proof idea

assume to have a \((c,t)\)-track layout. It is shown how to compute a \([c(t-1) + 1]\)-quasi planar straight-line drawing in \(O(t^3n)\) area.

it is proved that every partial \(k\)-tree has a \((2,t)\)-track layout where \(t\) does not depend on \(n\).
The proof idea

assume to have a \((c,t)\)-track layout. It is shown how to compute a \([c(t-1) + 1]\)-quasi planar straight-line drawing in \(O(t^3n)\) area

it is proved that every partial \(k\)-tree has a \((2,t)\)-track layout where \(t\) does not depend on \(n\).

\(\Rightarrow\) every partial \(k\)-tree has a \(O(1)\)-quasi planar straight-line drawing with area \(O(n)\).
Proof technique at a glance
From \((c, t)\)-track layouts to \([c(t-1)+1]\)-quasi planar drawings
From \((c, t)\)-track layouts to \([c(t-1)+1]\)-quasi planar drawings

redraw the vertices along the tracks
From \((c, t)\)-track layouts to \([c(t-1)+1]\)-quasi planar drawings

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From \((c, t)\)-track layouts to \([c(t-1)+1]\)-quasi planar drawings

redraw the vertices along the tracks

\[O(t^2n)\]
From $(c, t)$-track layouts to $[c(t-1)+1]$-quasi planar drawings

redraw the vertices along the tracks

$O(t^2n)$

$A=O(t^3n)$
From \((c,t)\)-track layouts to 
\([c(t-1)+1]\)-quasi planar drawings
From \((c, t)\)-track layouts to \([c(t-1)+1]\)-quasi planar drawings
From \((c, t)\)-track layouts to 
\([c(t-1)+1]\)-quasi planar drawings

- The drawing in a trapezoid is 
  \([c+1]\)-quasi planar

- Trapezoid
From \((c, t)\)-track layouts to \([c(t-1) + 1]\)-quasi planar drawings

1. A trapezoid.
2. The drawing in a trapezoid is \([c+1]\)-quasi planar.
3. How large is a “bundle” of mutually intersecting trapezoids?
From \((c,t)\)-track layouts to \([c(t-1)+1]\)-quasi planar drawings

counting mutually crossing trapezoids
From \((c, t)-\)track layouts to \([c(t-1)+1]-\)quasi planar drawings.
From \((c,t)\)-track layouts to \([c(t-1)+1]\)-quasi planar drawings

at most \((t-1)\)
mutually crossing trapezoids

counting mutually
crossing trapezoids
From \((c,t)\)-track layouts to \([c(t-1)+1]\)-quasi planar drawings

at most \((t-1)\) mutually crossing trapezoids

the drawing is \([c(t-1)+1]\)-quasi planar and has \(O(t^3n)\) area

counting mutually crossing trapezoids

\[ O(t^2n) \]
Computing a $(2,t)$-track layout

partial 2-tree
Computing a $(2,t)$-track layout
Computing a (2,t)-track layout
Computing a $(2,t)$-track layout
Computing a \((2,t)\)-track layout

PERTINENT GRAPHS
Computing a (2,t)-track layout
Computing a \((2,t)\)-track layout
Computing a (2,t)-track layout
Computing a \((2,t)\)-track layout
Computing a (2,t)-track layout
Computing a (2,t)-track layout
Computing a (2,t)-track layout

PARENT 2-CLIQUES
Computing a \((2,t)\)-track layout
Computing a (2,t)-track layout
Computing a (2, t)-track layout
Computing a \((2,t)\)-track layout
Computing a $(2,t)$-track layout

$G_\alpha$
Computing a \((2, t)\)-track layout
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Computing a $(2,t)$-track layout
Computing a (2,t)-track layout
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Computing a (2,t)-track layout
Computing a (2,t)-track layout
Computing a $(2,t)$-track layout

$G_{\beta}$

$G_{\varepsilon}$

$G_{\gamma}$

$G_{\delta}$
Computing a \((2,t)\)-track layout
Computing a (2,t)-track layout
Computing a (2,t)-track layout
Computing a $(2,t)$-track layout
Computing a (2,t)-track layout
Computing a (2,t)-track layout

X-crossing
Computing a $(2,t)$-track layout

$X$-crossing

$(2,6)$-track layout of a partial 2-tree
Computing a (2,t)-track layout

X-crossing

(2,6)-track layout of a partial 2-tree

inductive argument for larger k
Computing a $(2,t)$-track layout: the scary math

A partial $k$-tree admits a $(2,t_k)$-track layout, where $t_k$ is given by the following recursive equation:

$$t_k = (c_{k-1,k} + 1)t_{k-1}$$

$$c_{k,i} = (c_{k-1,k} + 1)(c_{k-1,i} + \frac{c_{k-1,k}}{4} \sum_{j=1}^{i-1} c_{k-1,j} \cdot c_{k-1,i-j}) \quad (i = 1, \ldots, k + 1)$$

$$c_{k,k+2} = 0$$

with $t_1 = 2$ and $c_{1,1} = 4$ and $c_{1,2} = 2$. 
Computing a (2,t)-track layout: the scary math

A partial $k$-tree admits a $(2, t_k)$-track layout, where $t_k$ is given by the following recursive equation:

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$$c_{k,k+2} = 0$$

with $t_1 = 2$ and $c_{1,1} = 4$ and $c_{1,2} = 2$.

$\Rightarrow$ every partial $k$-tree has a $O(1)$-quasi planar straight-line drawing with area $O(n)$. 
Combinatorial relationships between nearly planar graphs
1-planarity, quasi-planarity and 1-visibility

K7

quasi-planar

1-planar

1-planarity

quasi-planarity

1-visibility
1-planarity, quasi-planarity and 1-visibility

[Evans et al., 2013]
1-planarity, quasi-planarity and 1-visibility

[Evans et al., 2013]
1-planarity, quasi-planarity and 1-visibility

[Evans et al., 2013]
RAC and 1-planarity

[Eades, L., 2013]
RAC graphs and 1-planarity

Theorem

A maximally dense RAC graph is 1-planar. Also, for every integer $i$ such that $i \geq 0$ there exists a 1-planar graph with $n = 8 + 4i$ vertices and $4n - 10$ edges that is not a RAC graph. Finally, for every integer $n > 85$, there exists a RAC graph with $n$ vertices that is not 1-planar. [Eades, L., 2013]
Some details about the proof of this last theorem
Preliminaries: edge coloring

- red edges do not cross
- each green edge crosses with a blue edge
Preliminaries: edge coloring

- Red edges do not cross.
- Each green edge crosses with a blue edge.
  - Red-blue (embedded planar) graph $= \text{red} + \text{blue}$ edges.

\[ G \quad G_{rb} \]
Preliminaries: edge coloring

- **red** edges do not cross

- **each green** edge crosses with a **blue** edge
  - red-blue (embedded planar) graph = red + blue edges
  - red-green (embedded planar) graph = red + green edges
Preliminaries: $G_{rb}$ and $G_{rg}$ in a maximally dense RAC graph

Each internal face of $G_{rb}$ ($G_{rg}$) has at least two red edges [Didimo, Eades, L., 2011]
Preliminaries: $G_{rb}$ and $G_{rg}$ in a maximal RAC graph

Notation:
- $m_r$, $m_b$, $m_g$ = number of red, blue, and green edges
- $f_{rb}$ = number of faces of the red-blue graph $G_{rb}$

Assumption:
- $m_g \leq m_b$
Maximally dense RAC graphs are 1-planar

Approach:

• suppose we can show that $G_{rb}$ and $G_{rg}$ are both maximal planar graphs; then:
Maximally dense RAC graphs are 1-planar

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![Diagram](image)
Maximally dense RAC graphs are 1-planar

**Approach:**

- suppose we can show that $G_{rb}$ and $G_{rg}$ are both maximal planar graphs; then:
$G_{rb}$ and $G_{rg}$ are maximal planar graphs (1)

- the following is proven first:

**Claim 1**: the external face of $G_{rb}$ and $G_{rg}$ is a 3-cycle

- then, we consider the internal faces of $G_{rb}$ that share at least one edge with the external face (fence faces)

![Diagram showing fence faces] there are at least 1 and at most 3 fence faces
$G_{rb}$ and $G_{rg}$ are maximal planar graphs (2)

...and prove the following

**Claim 2:** If $G$ is maximal, $G_{rb}$ has three fence faces and each fence face is a 3-cycle

• obs: at least two fence faces consist of red edges

\[ \alpha + \beta + \gamma \geq 360^\circ \]

\[ \alpha < 90^\circ \]

\[ \Rightarrow \beta \geq 90^\circ \text{ and } \gamma \geq 90^\circ \]
$G_{rb}$ and $G_{rb}$ are maximal planar graphs (3) since: (1) each internal face of $G_{rb}$ has at least 2 red edges; (2) the external face of $G_{rb}$ is a red 3-cycle; (3) at least two fence faces are red 3-cycles $\Rightarrow 2m_r \geq 2(f_{rb} - 3) + 3 + 3 + 3$

since $m_r$ and $f_{rb}$ are integers, we obtain

$2m_r \geq 2(f_{rb} - 3) + 3 + 3 + 3$

By Euler’s formula for planar graphs $\Rightarrow m_r + m_b \leq n + f_{rb} - 2$

$m_b \leq n - 4$

$m_r \geq f_{rb} + 2$
$G_{rb}$ and $G_{rb}$ are maximal planar graphs (4)

$m_b \leq n - 4$

G is a maximally dense RAC graph $\Rightarrow m_b + m_r + m_g = 4n - 10$

$m_r + m_g \geq 3n-6$

since by assumption $m_g \leq m_b$ and since both $G_{rg}$ and $G_{rb}$ are planar $\Rightarrow G_{rg}$ and $G_{rb}$ are both maximal planar graphs

Therefore a maximal RAC graph is 1-planar
Open problems
Characterize those 1-planar graphs that have a RAC drawing.
Inclusion properties and RAC graphs

Characterize those 1-planar graphs that have a RAC drawing

Recognizing those graphs that have a RAC drawing is NP-hard. Does this problem remain NP-hard for those graphs with $n$ vertices and $4n-10$ edges?
Do partial $k$-trees admit a $O(1)$-quasi planar straight line drawing in linear area and constant aspect ratio?
Area-crossing complexity trade-offs

Do partial $k$-trees admit a $O(1)$-quasi planar straight line drawing in linear area and constant aspect ratio?

For, example, do outerplanar graphs admit a 4-quasi planar straight line drawing in linear area and constant aspect ratio?
Area-crossing complexity trade-offs

Do partial $k$-trees admit a $O(1)$-quasi planar straight line drawing in linear area and constant aspect ratio?

For, example, do outerplanar graphs admit a $3$-quasi planar straight line drawing in linear area and constant aspect ratio?

Do all planar graphs have a sub-quadratic area $h$-quasi planar straight-line drawing with constant $h$?
## Other problem categories

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<thead>
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<th>Turan-type</th>
<th>Recognition</th>
<th>Fary-type</th>
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<tbody>
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<td><strong>RAC</strong></td>
<td>$O(n)$</td>
<td>NP-hard (linear-time for 2-layer)</td>
<td>-</td>
</tr>
<tr>
<td><strong>1-planar</strong></td>
<td>$O(n)$</td>
<td>NP-hard (linear time for given rot. syst.)</td>
<td>character. test, drawing</td>
</tr>
<tr>
<td><strong>3-quasi-planar</strong></td>
<td>$O(n)$</td>
<td>??</td>
<td>??</td>
</tr>
<tr>
<td><strong>skewness-1</strong></td>
<td>$O(n)$</td>
<td>polynomial</td>
<td>character. test, drawing</td>
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The crossing angle resolution problem

- Compute straight-line drawings of complete graphs where the smallest crossing angle is maximized
The crossing angle resolution problem

- Compute straight-line drawings of complete graphs where the smallest crossing angle is maximized

- Conjecture: the crossing angle resolution for $K_8$ and $K_9$ is at most $\pi/3$, while it is at most $\pi/4$ for $K_{10}$
Thank you!!

GD in the jurassic era