EuroGIGA Final Conference
"Geometry in Graphs and Algorithms"

Empty triangles in good drawings of the complete graph

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- drawing of a simple graph $G = (V, E)$ on S^2 or E^2

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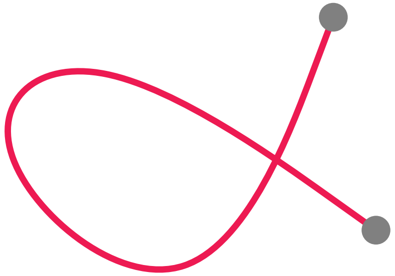

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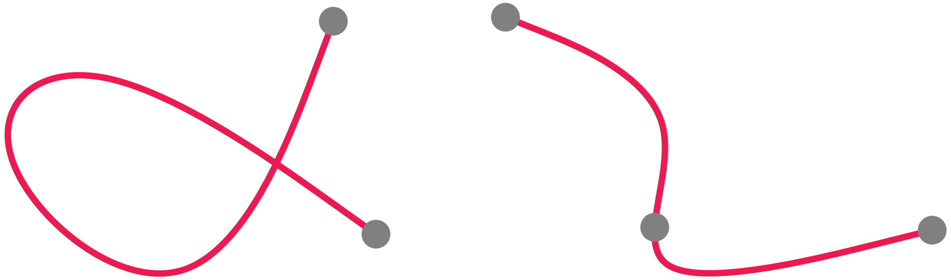

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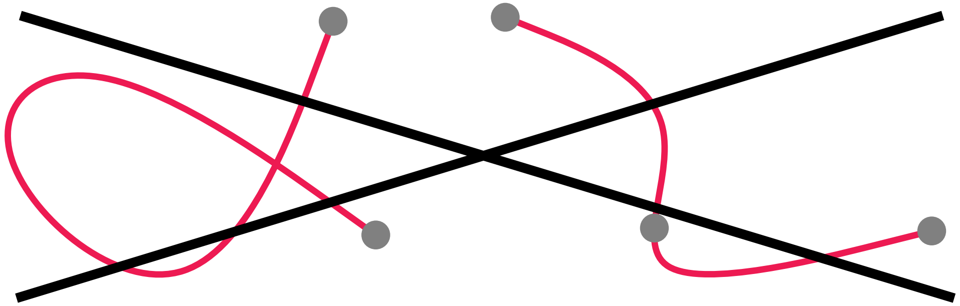

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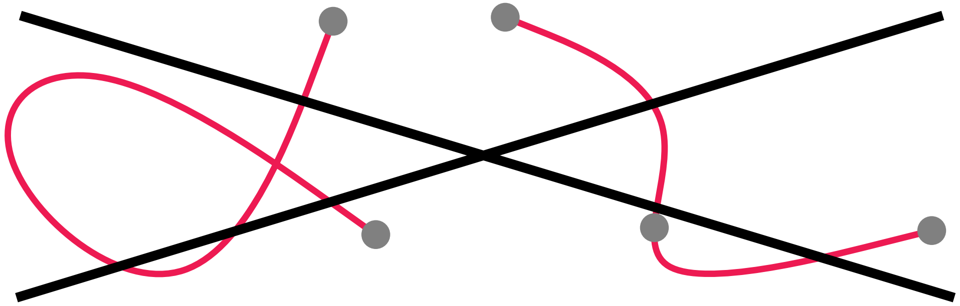

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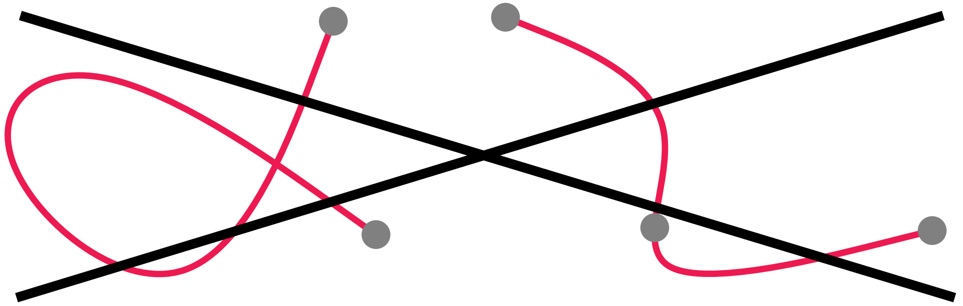

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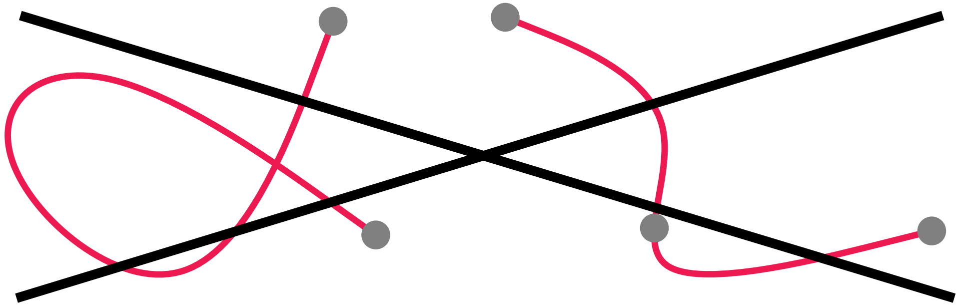

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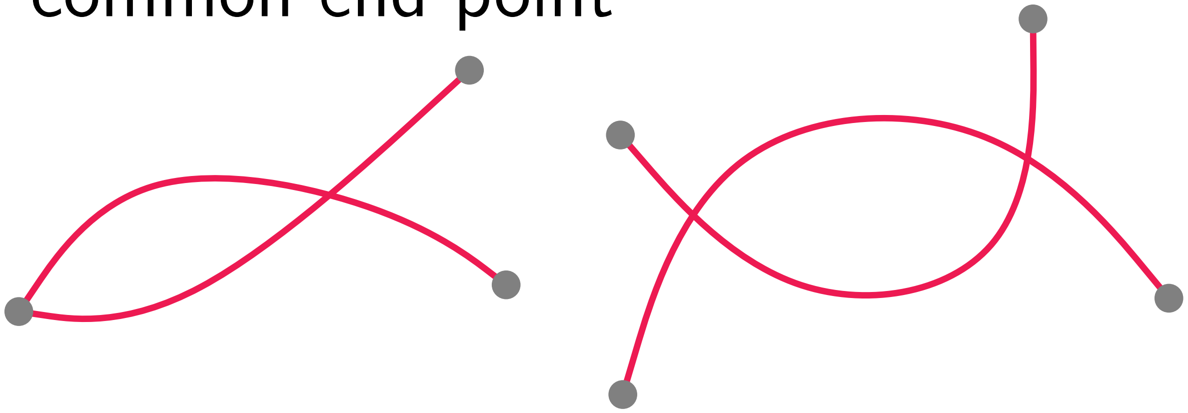

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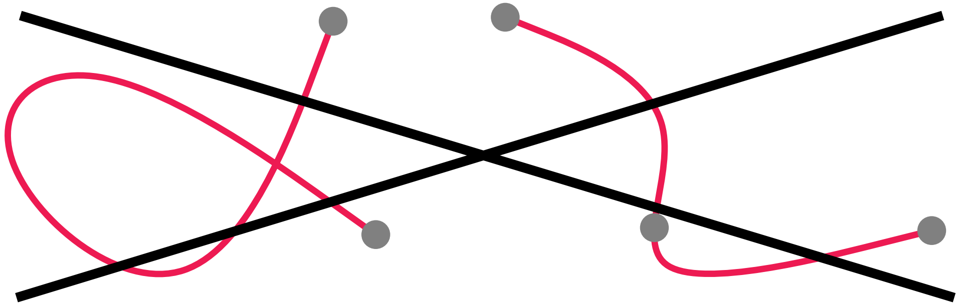

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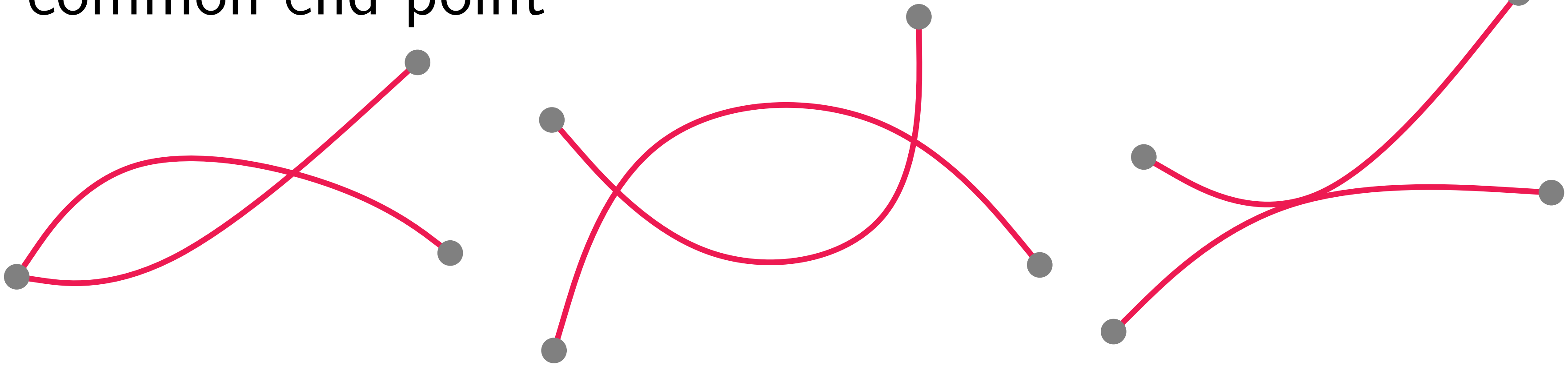

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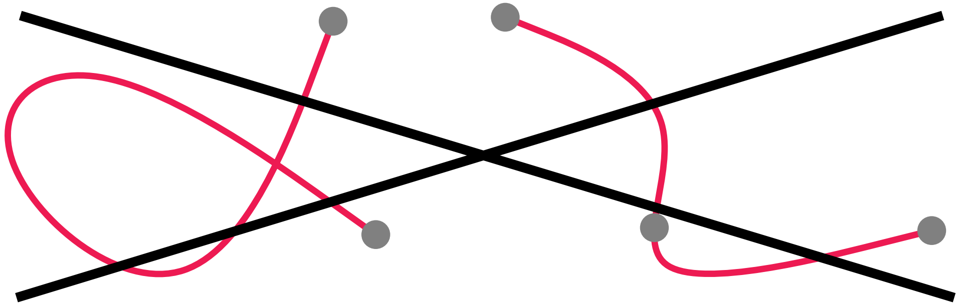

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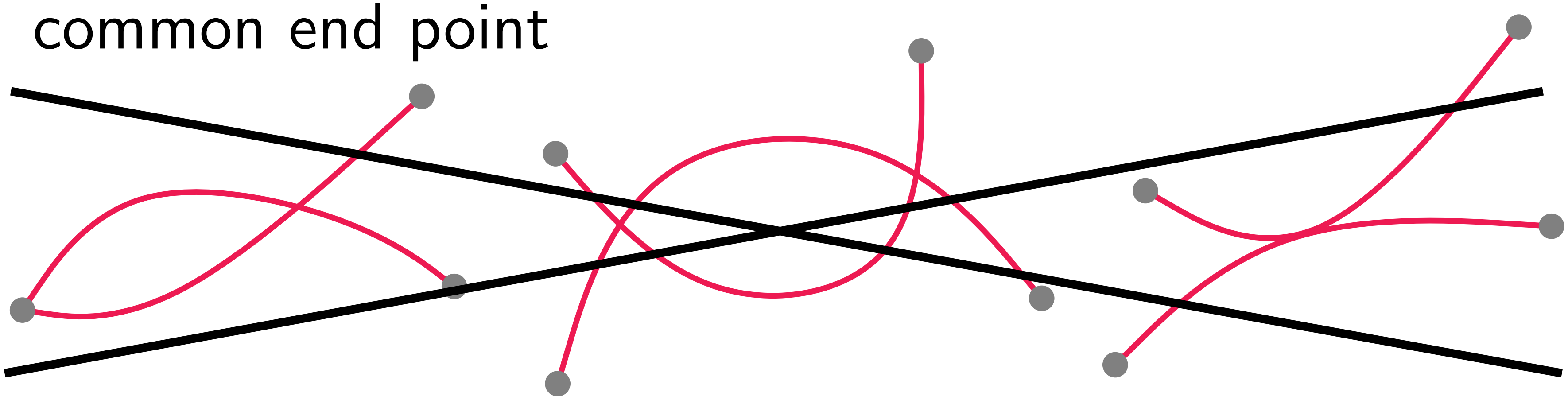

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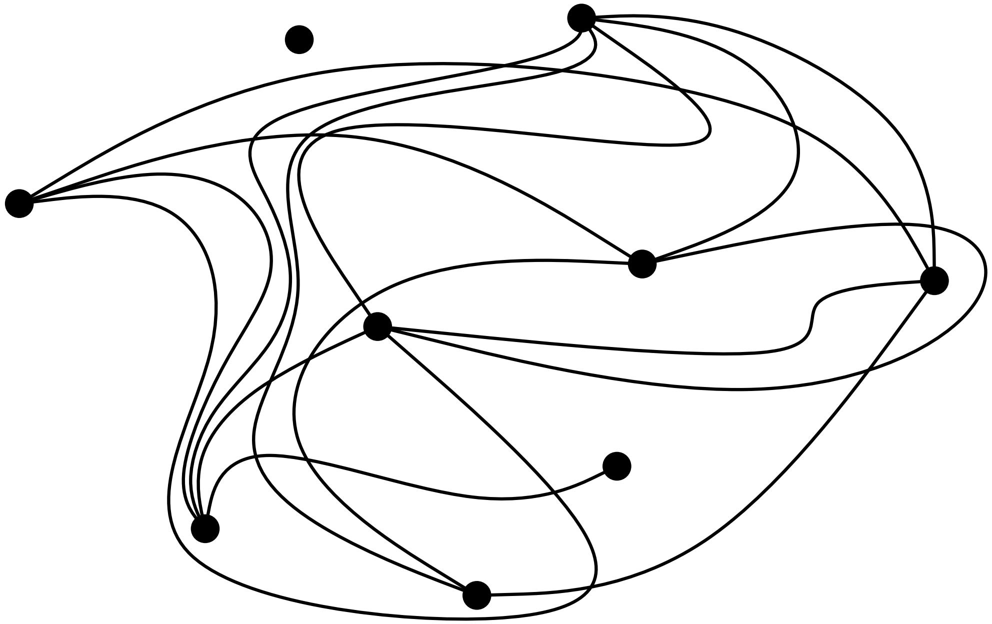

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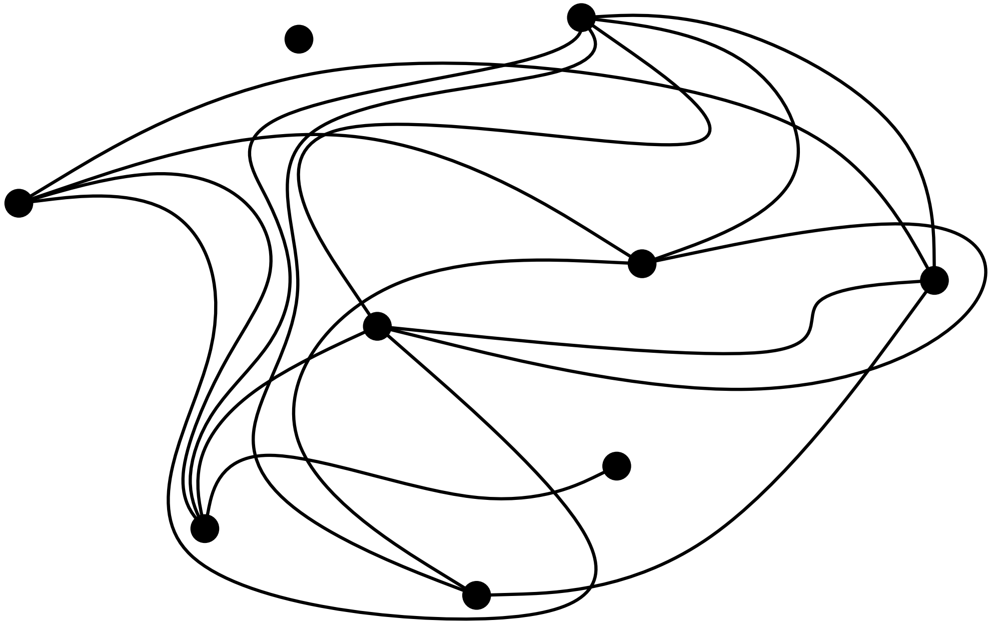
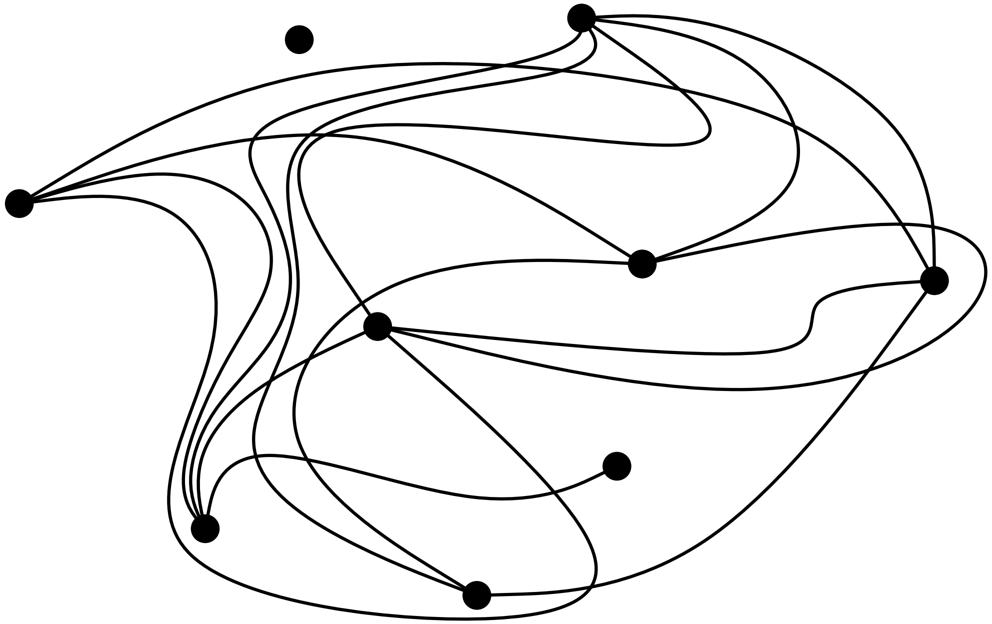


- also: **simple topological graphs**

Triangles

- triangle: three pairwise adjacent edges



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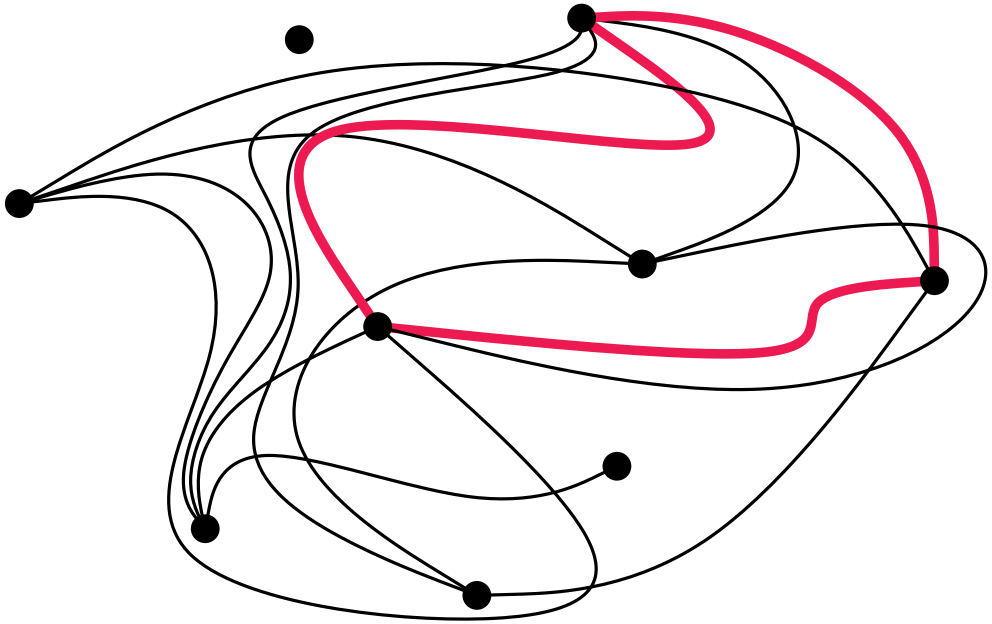

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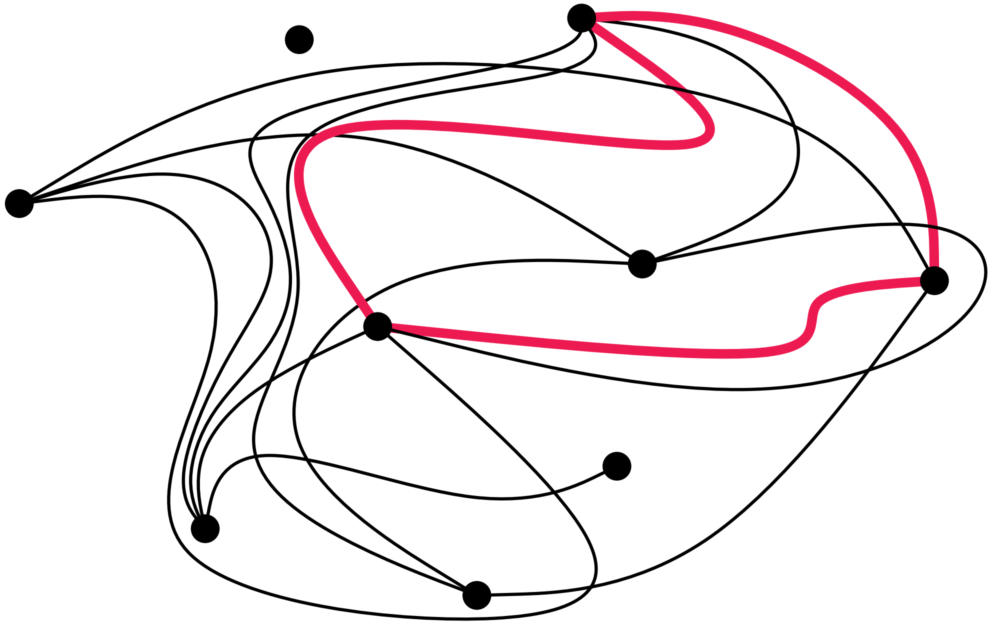

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  ⇒ partition of E^2 (or S^2) in two connected components

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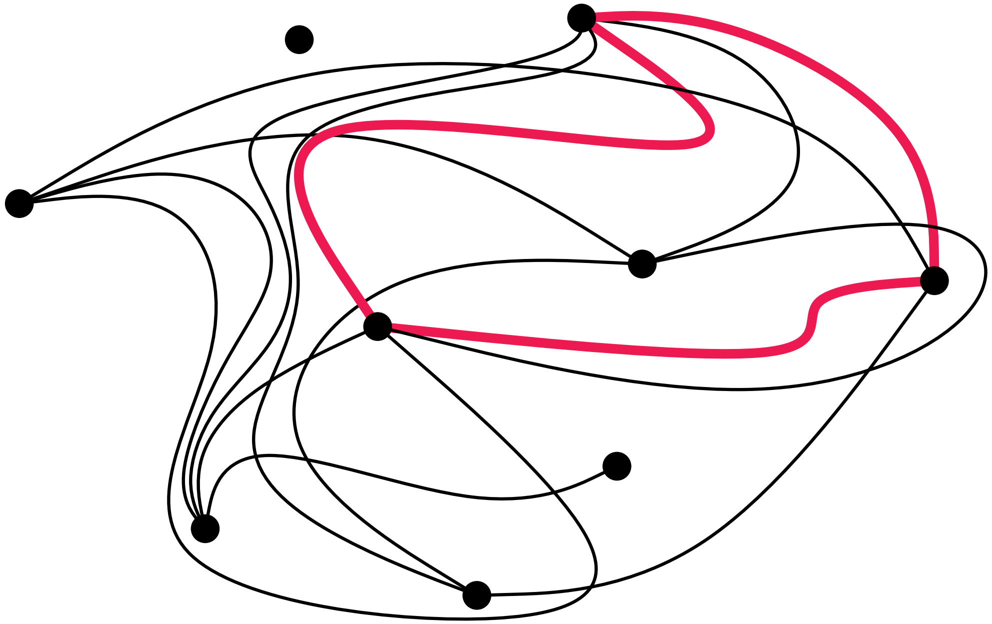

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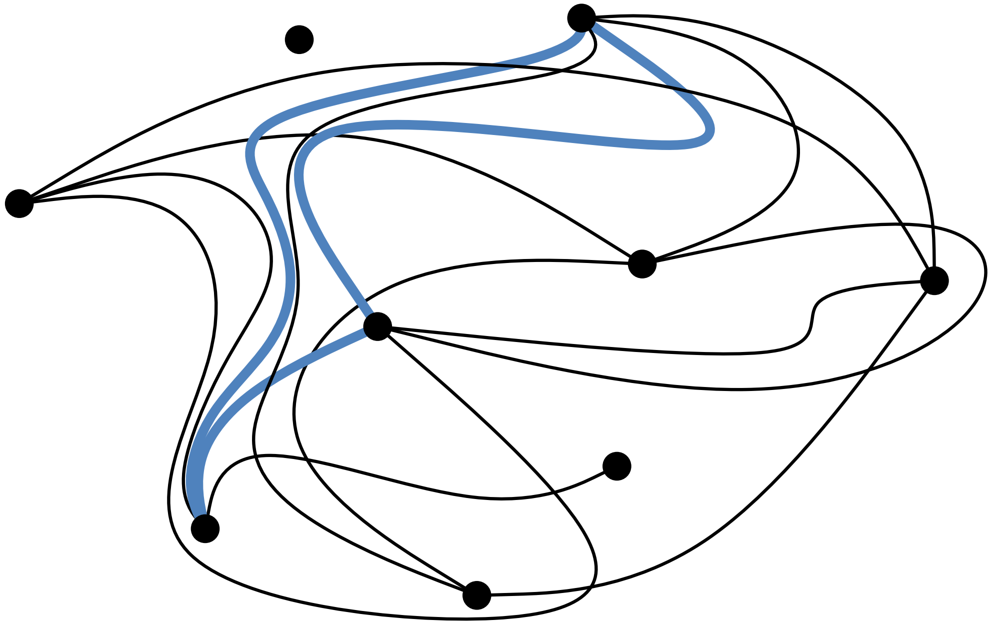

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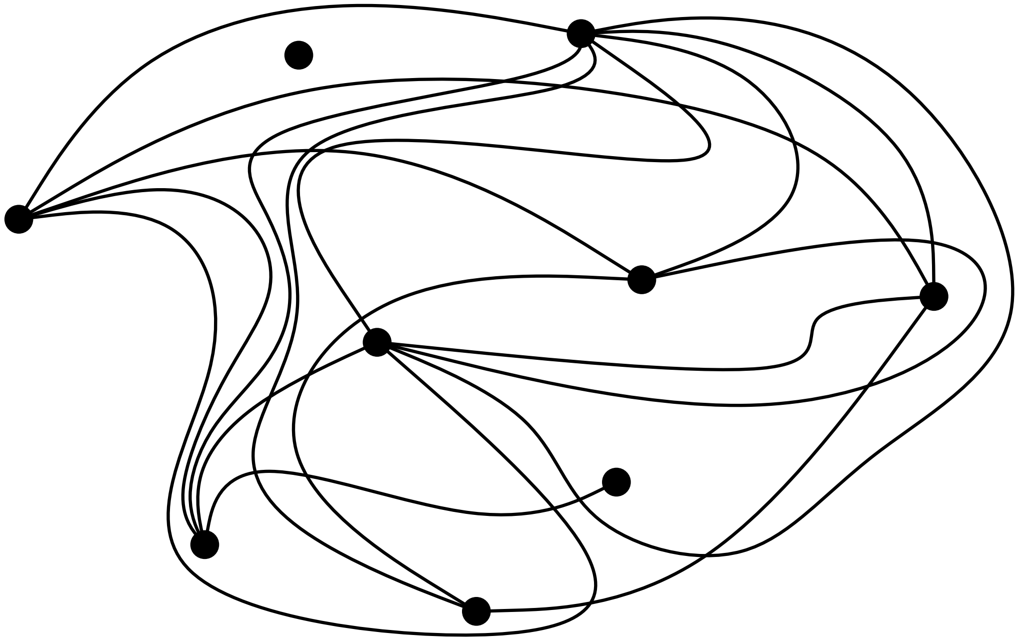

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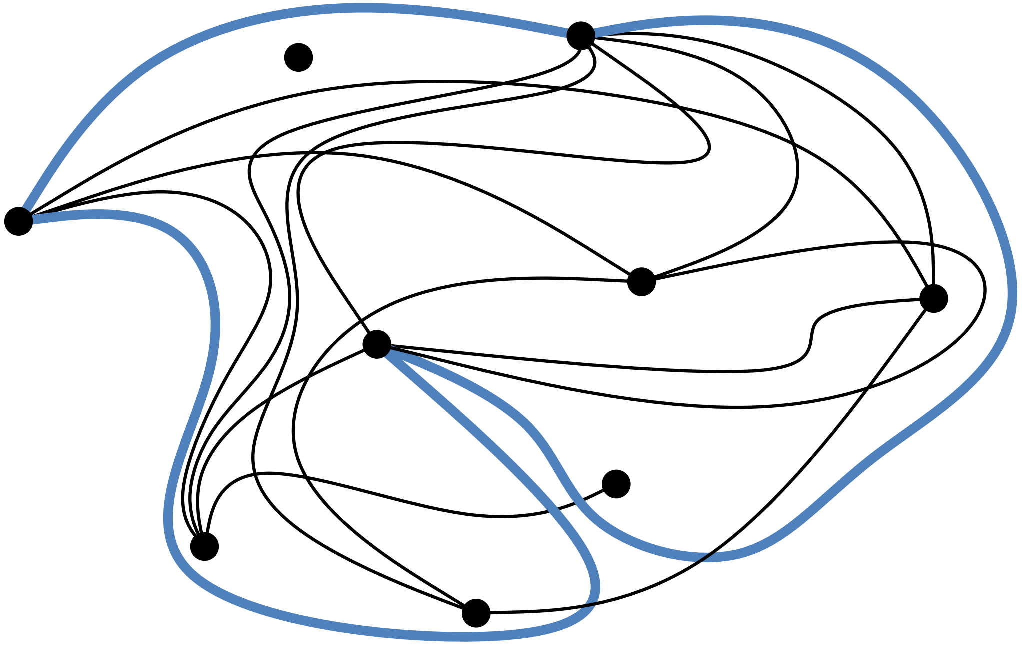

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Background

- crossing number $cr(n)$:
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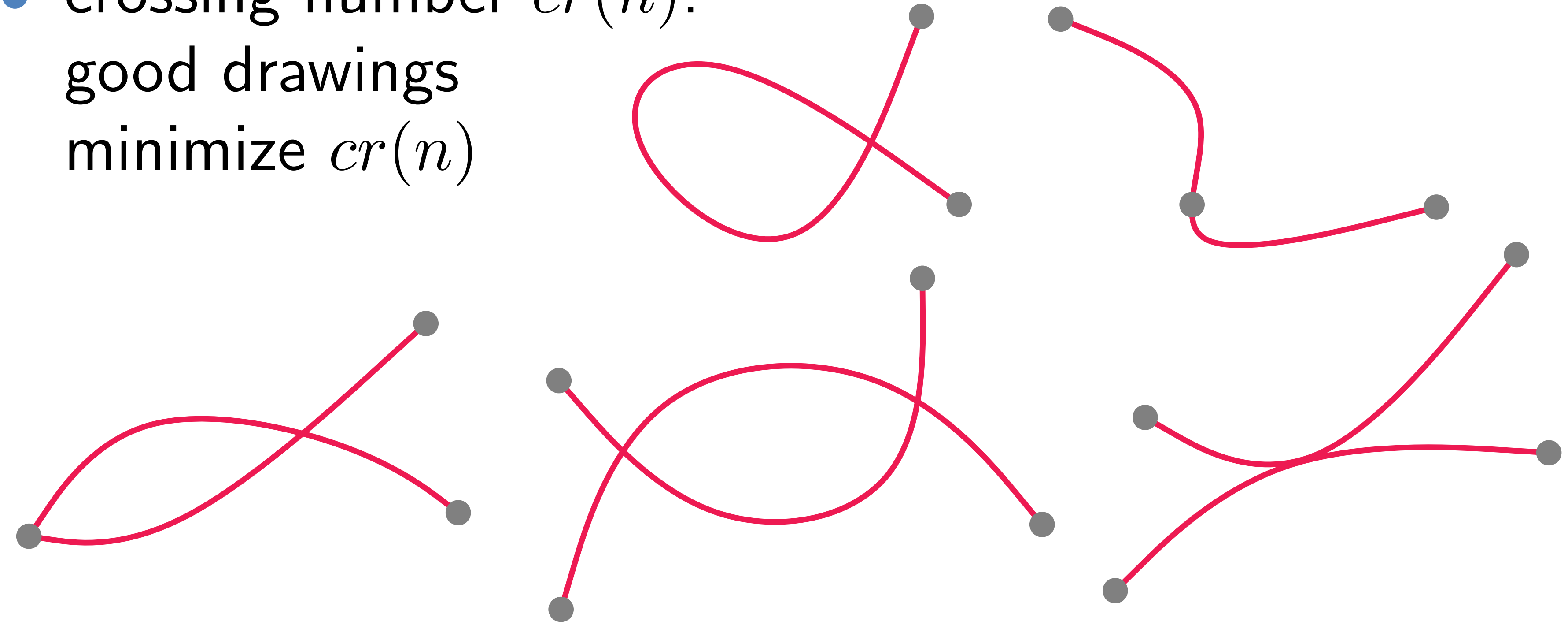
  minimize $cr(n)$



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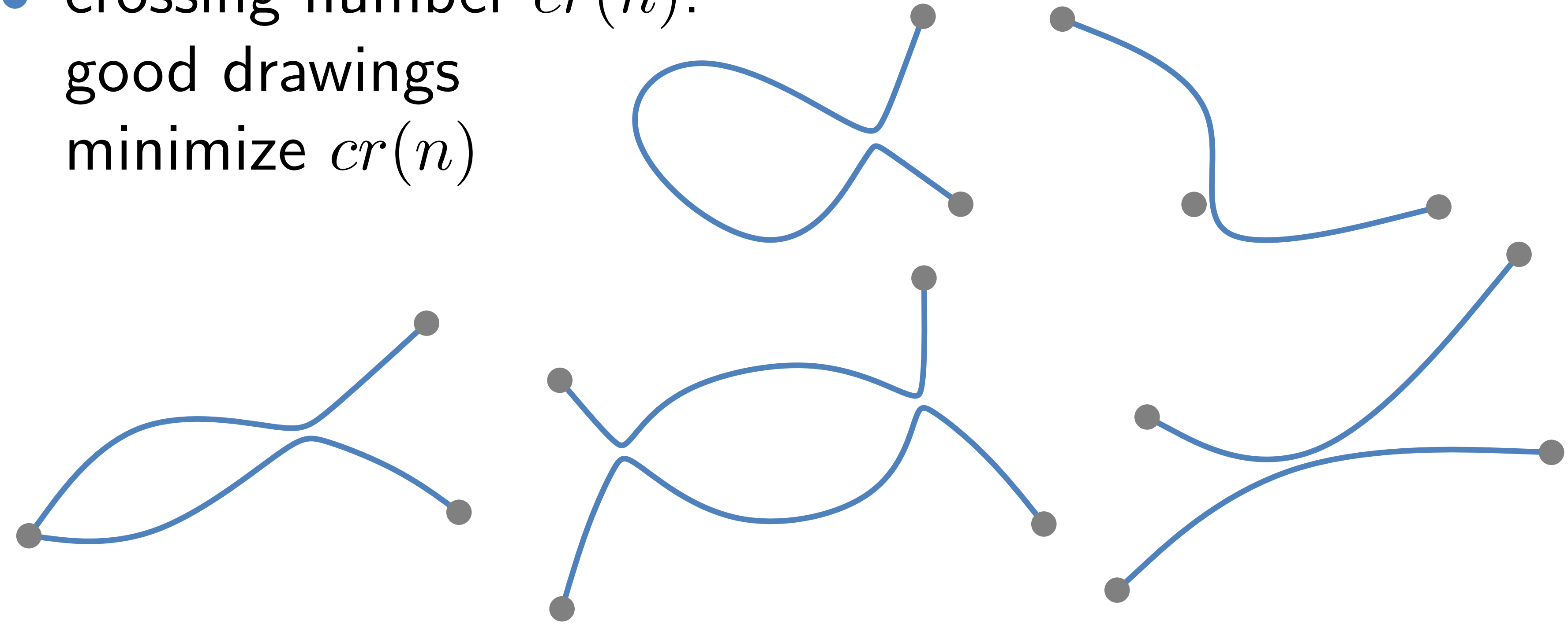
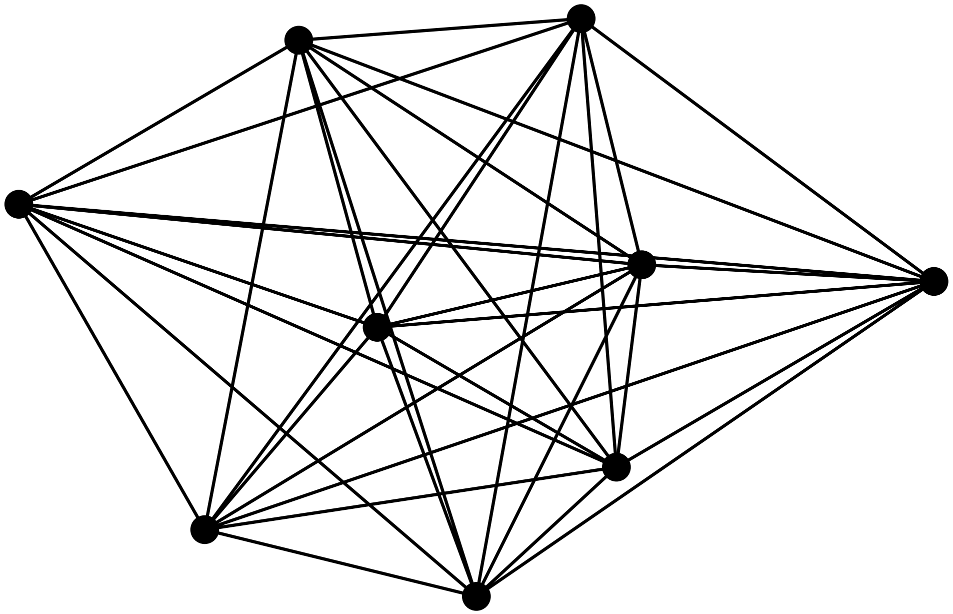


$\Rightarrow$ **Wednesday at 9:30**

Background

- minimum number $h_3(n)$ of empty triangles in straight-line drawings of K_n:
    - at least one empty triangle per edge



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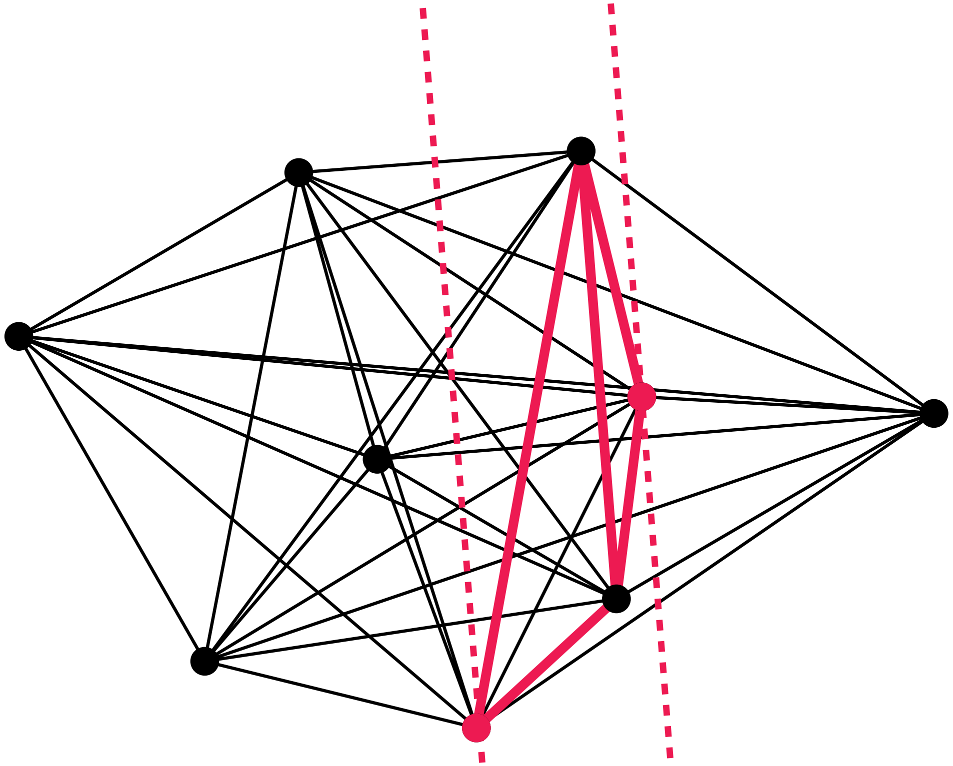

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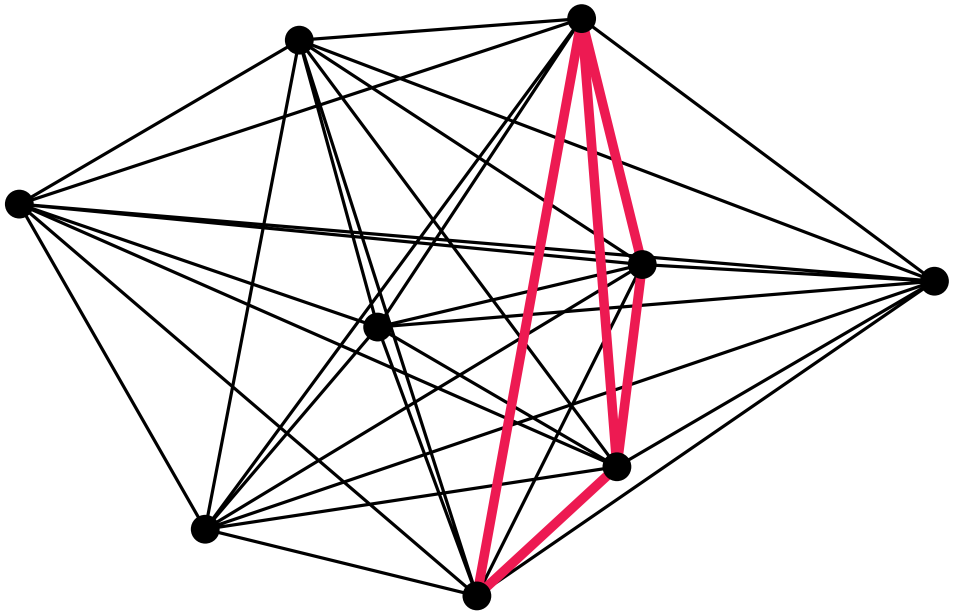
  - currently best bounds: [BV 2004, AFHHPV 2012]: $n^2 - \frac{32}{7}n + \frac{22}{7} \leq h_3(n) \leq 1.6196n^2 + o(n^2)$



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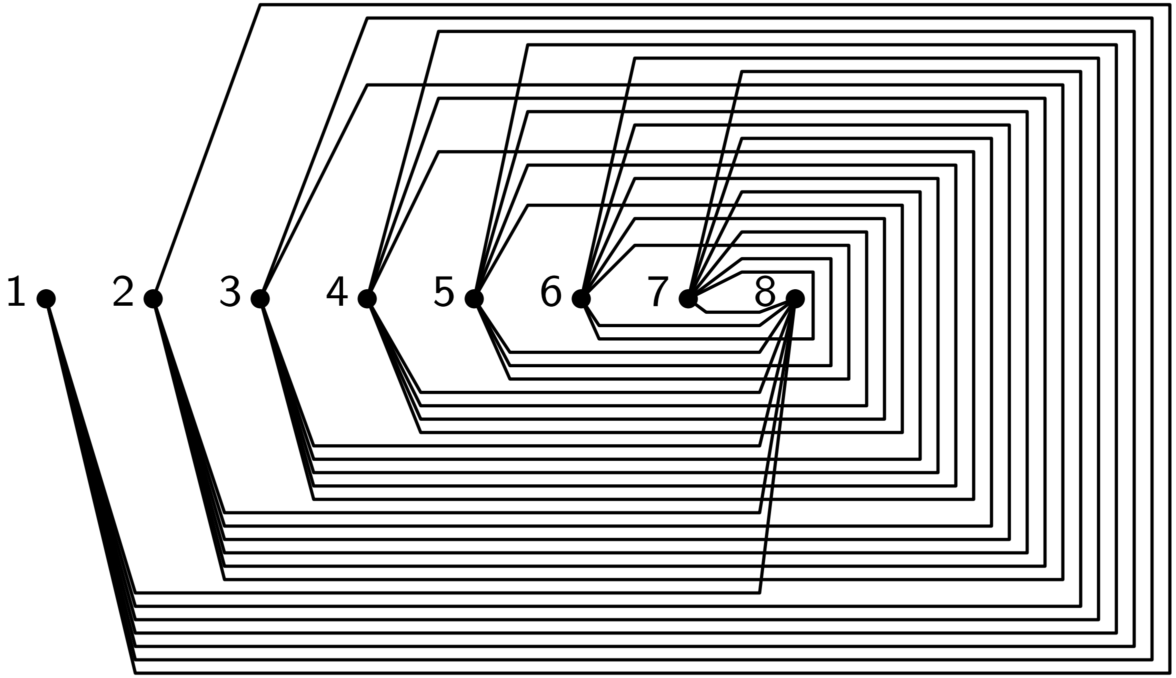

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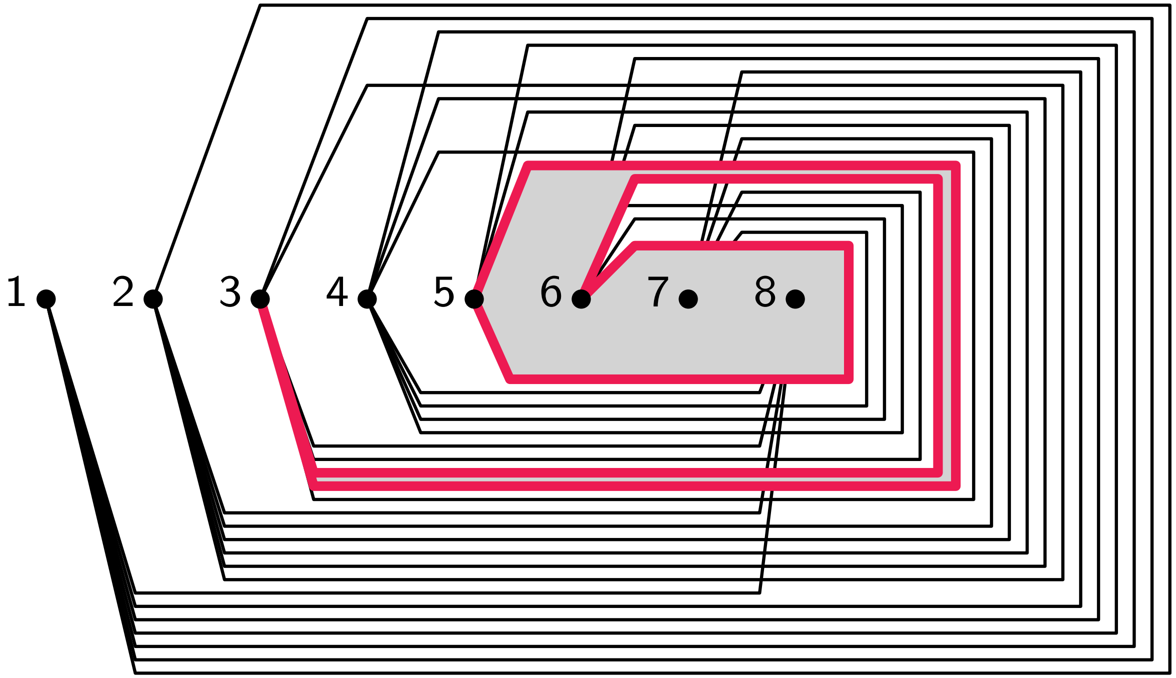

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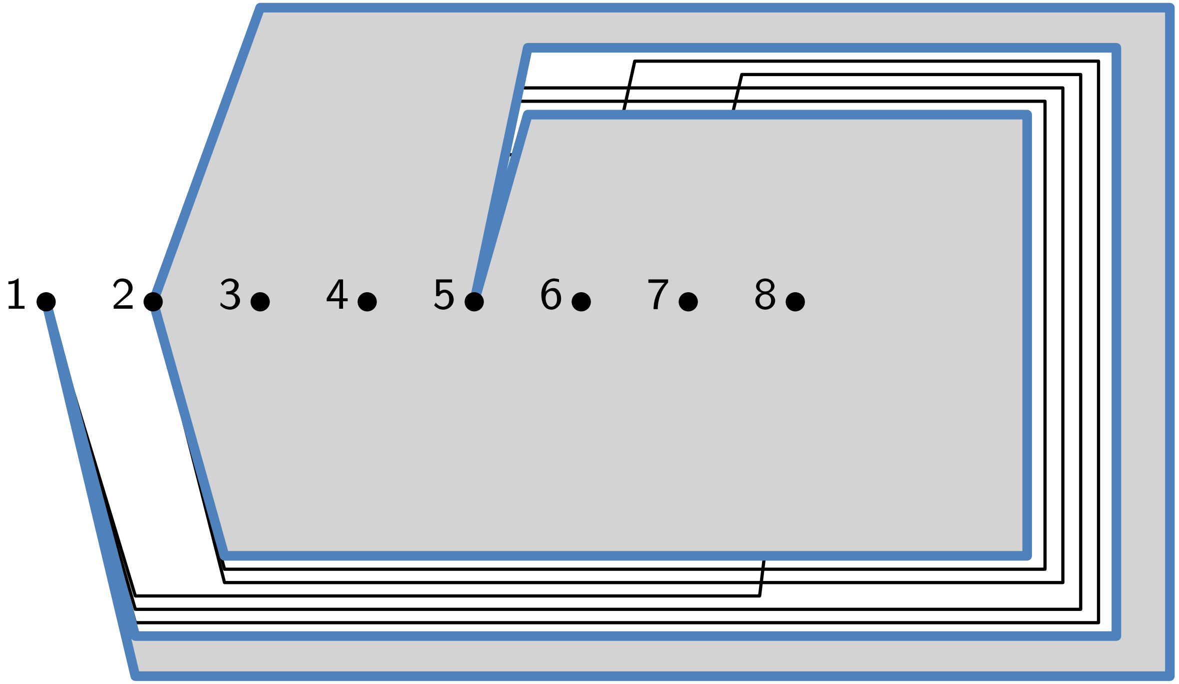

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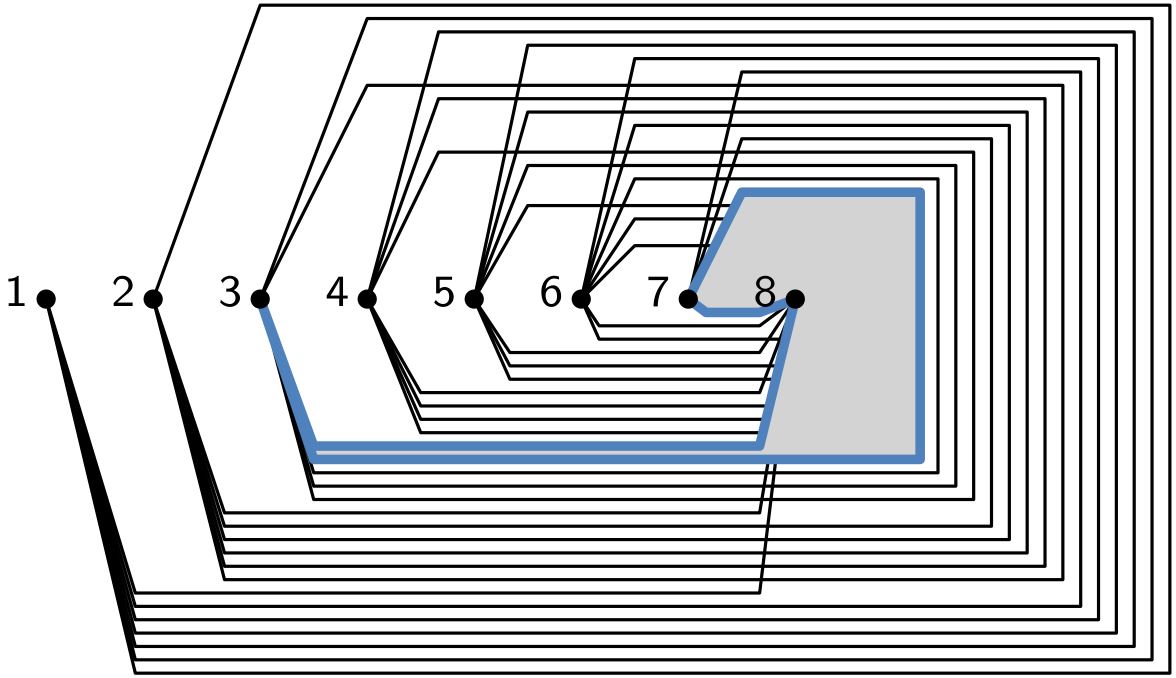

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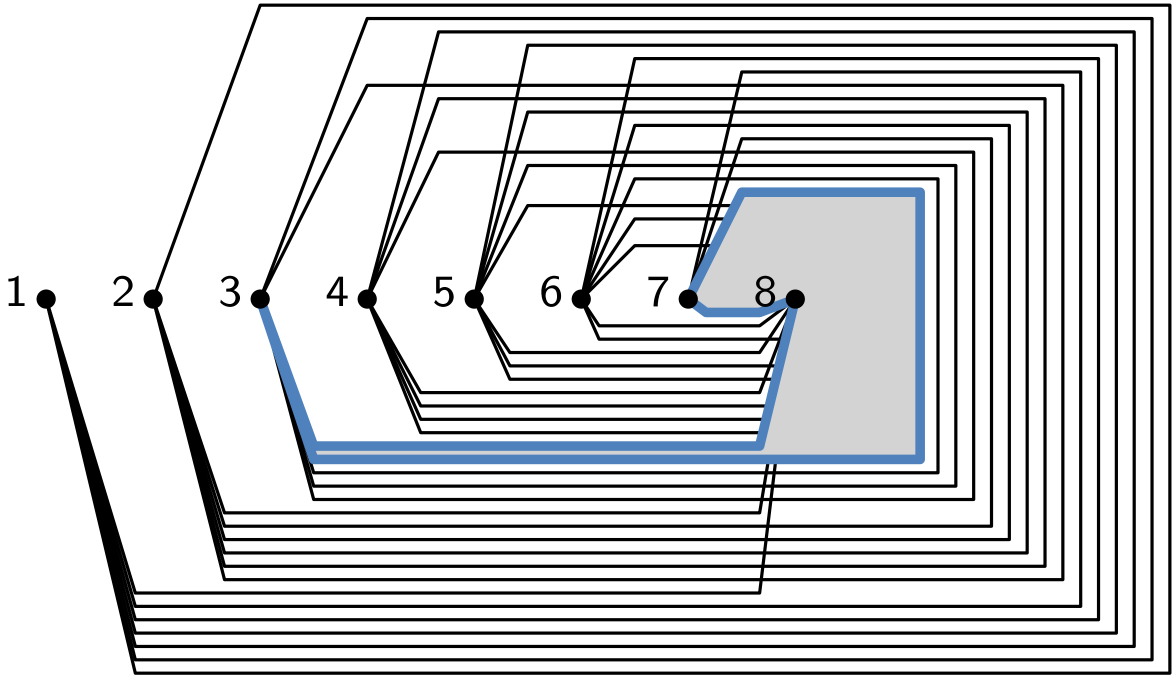
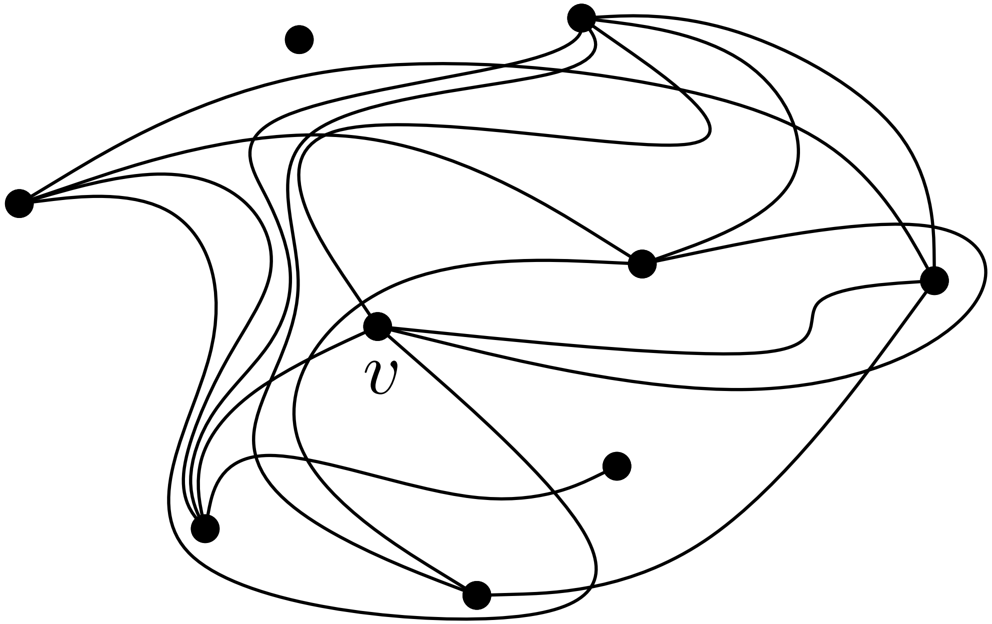


Star Graphs

- star graph of v: all vertices of G, all incident edges of v



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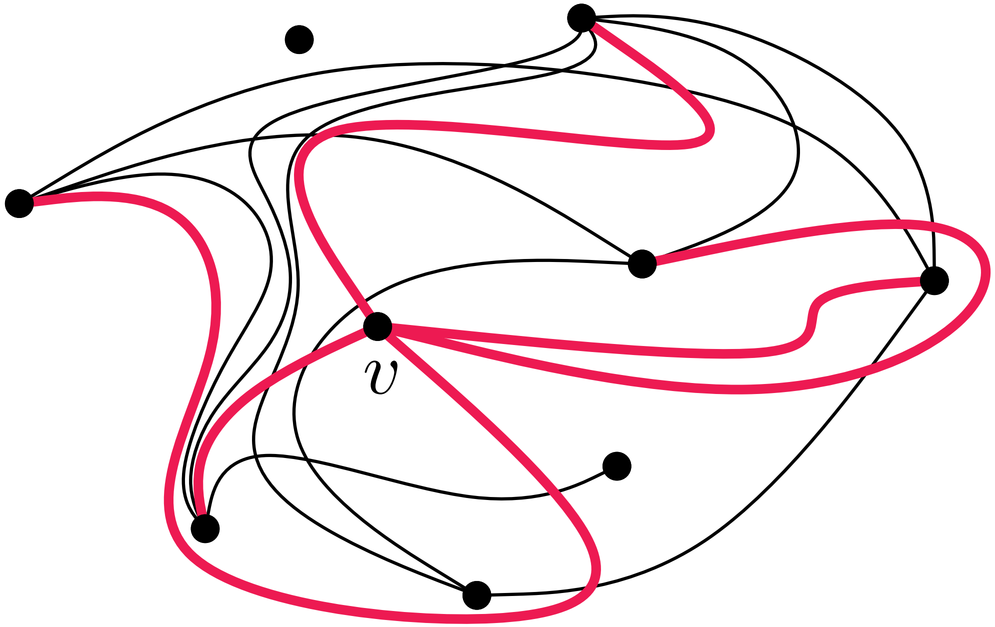

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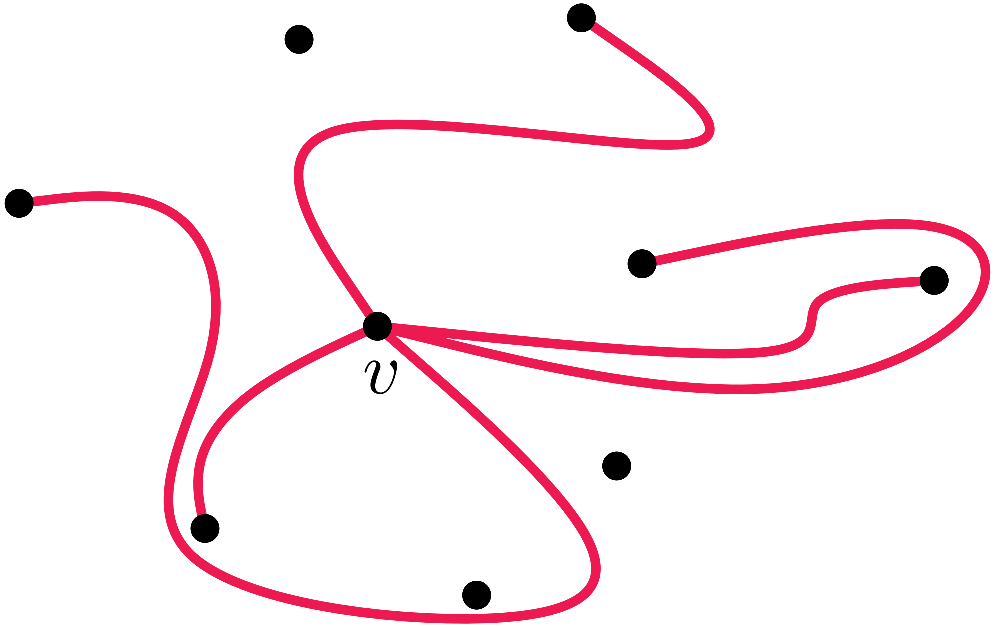

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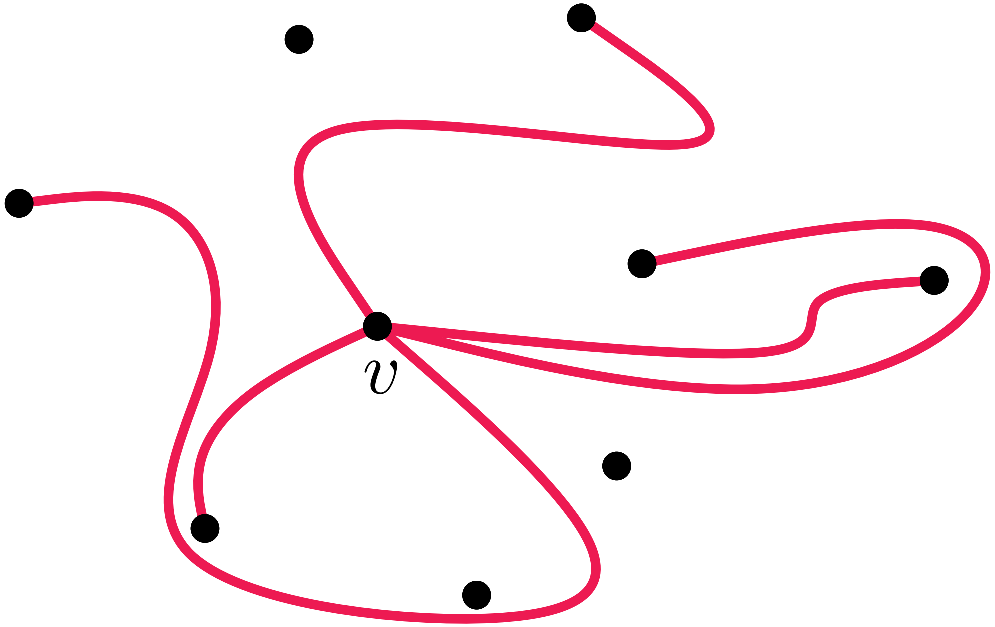

- crossing-free

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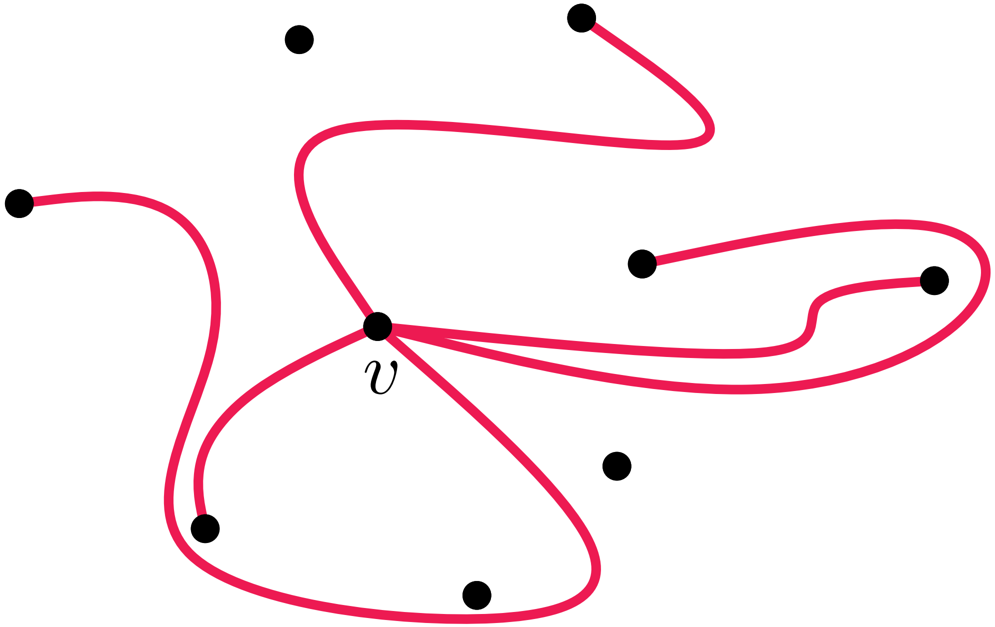

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- circular order of incident edges around v

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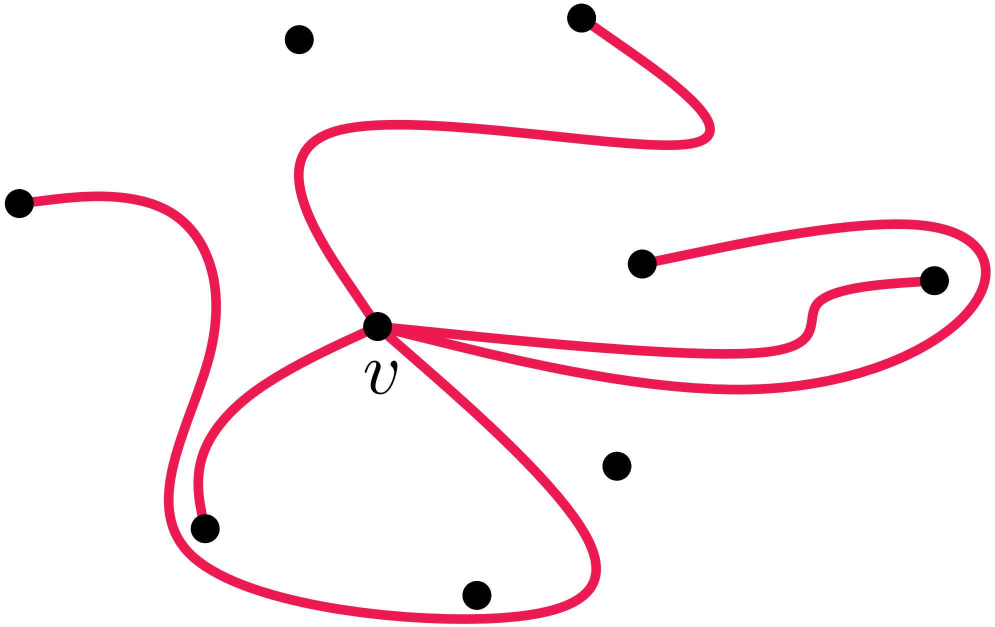

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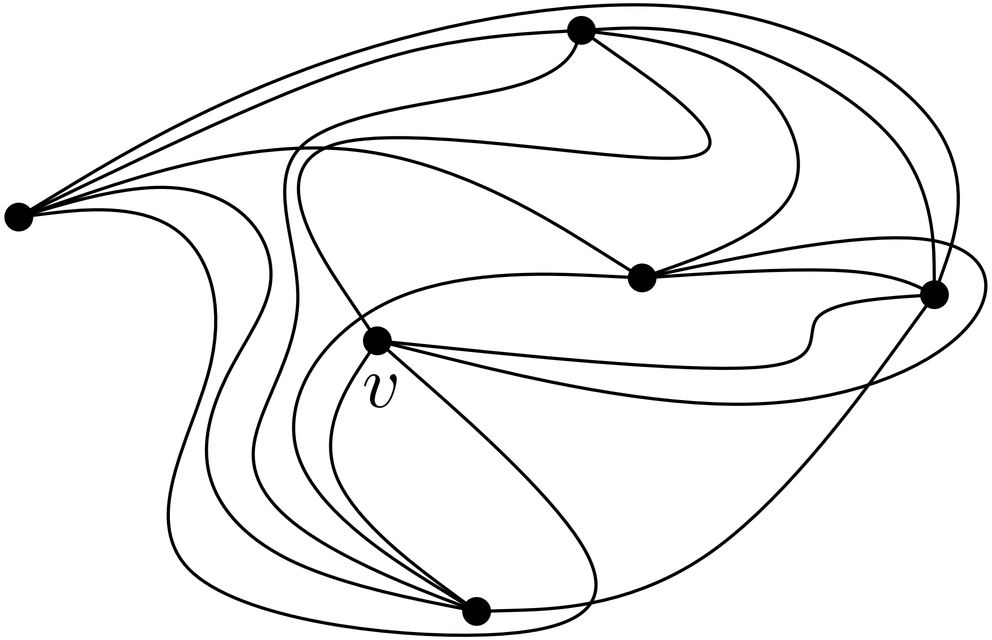

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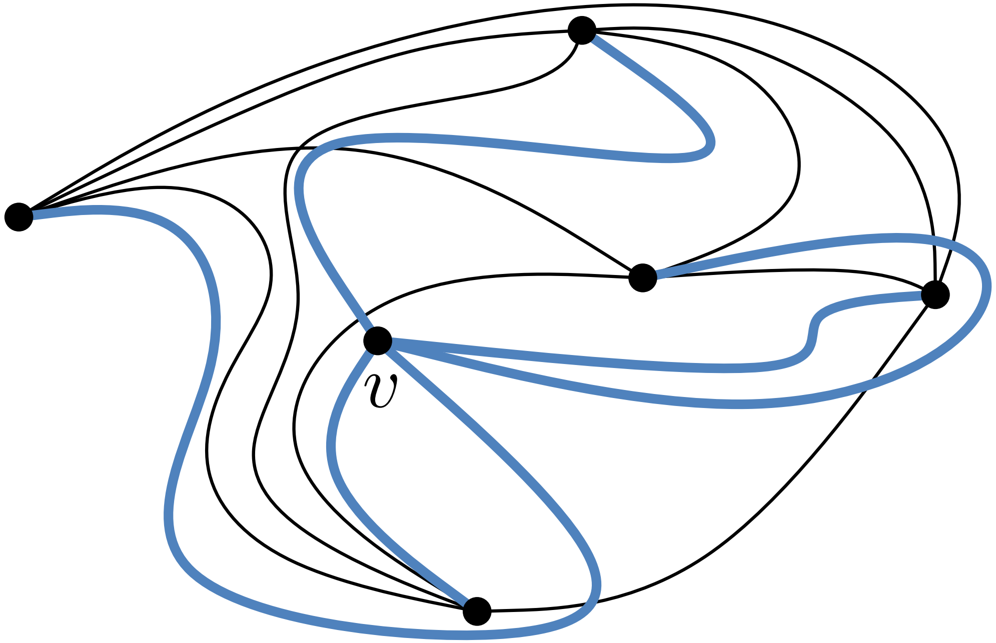

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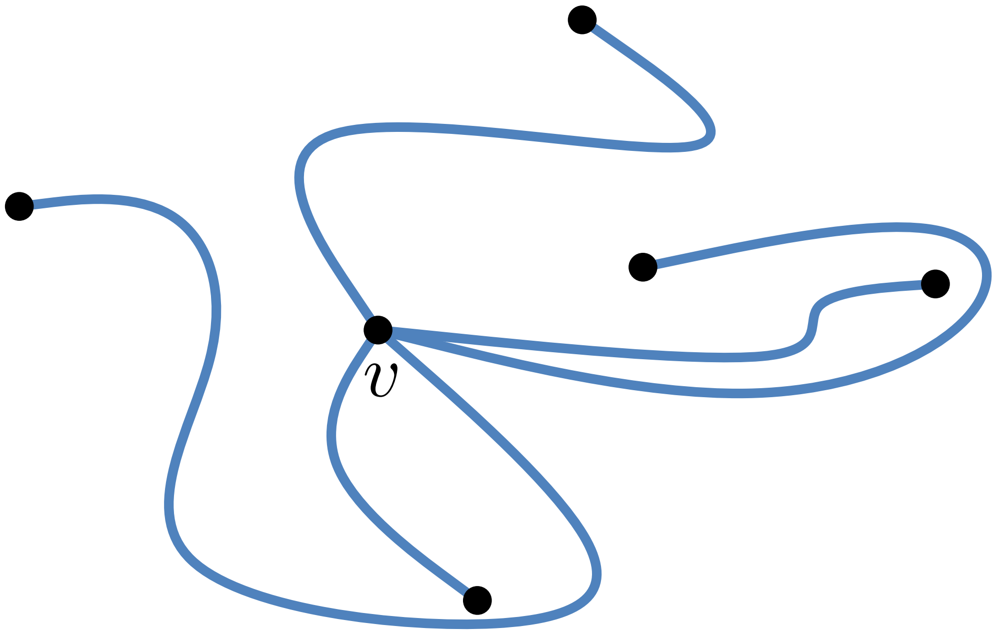

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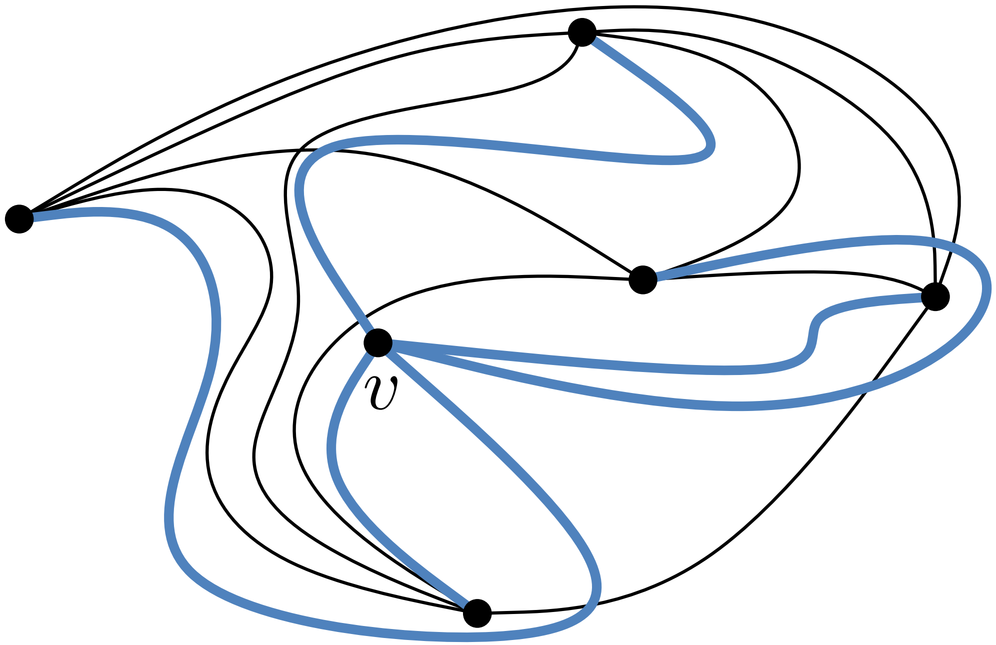

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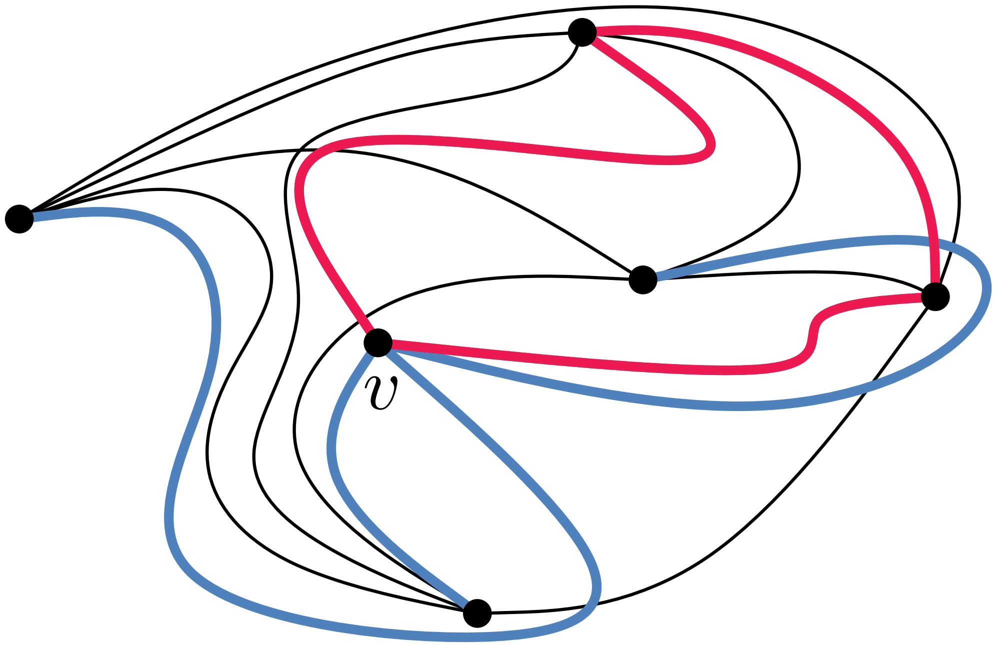

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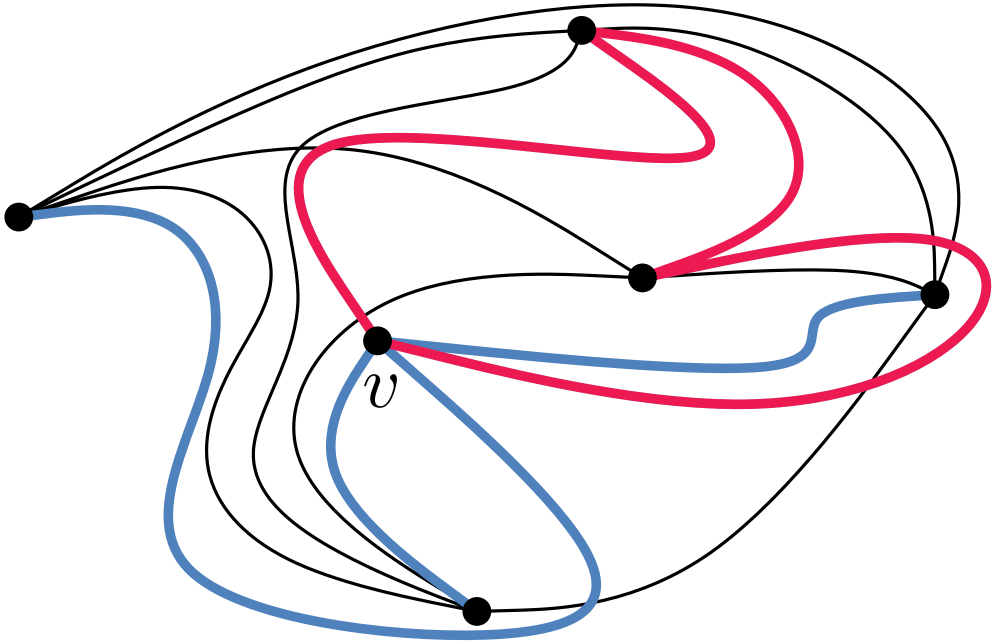

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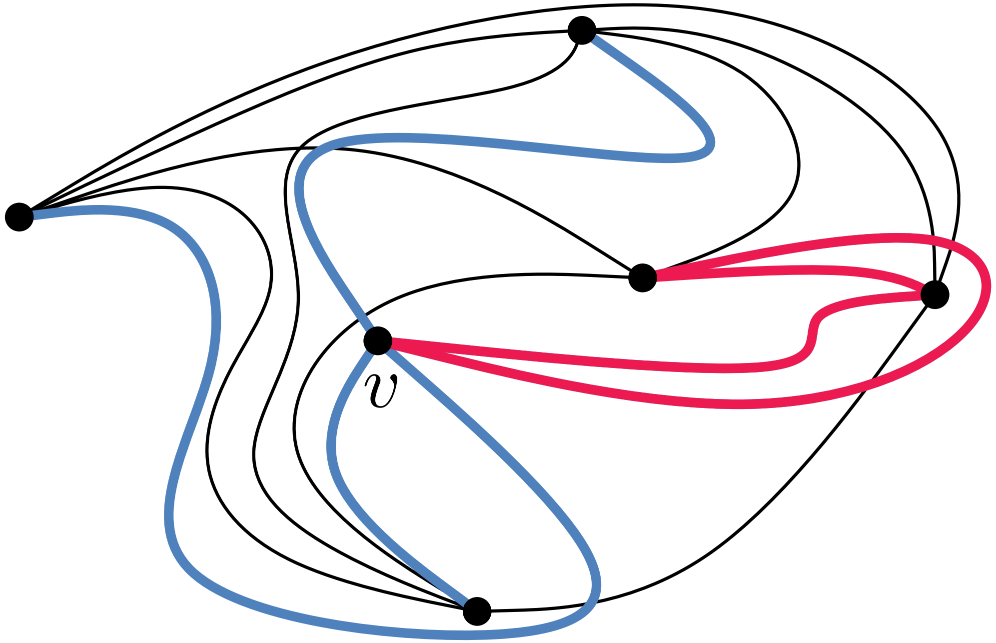

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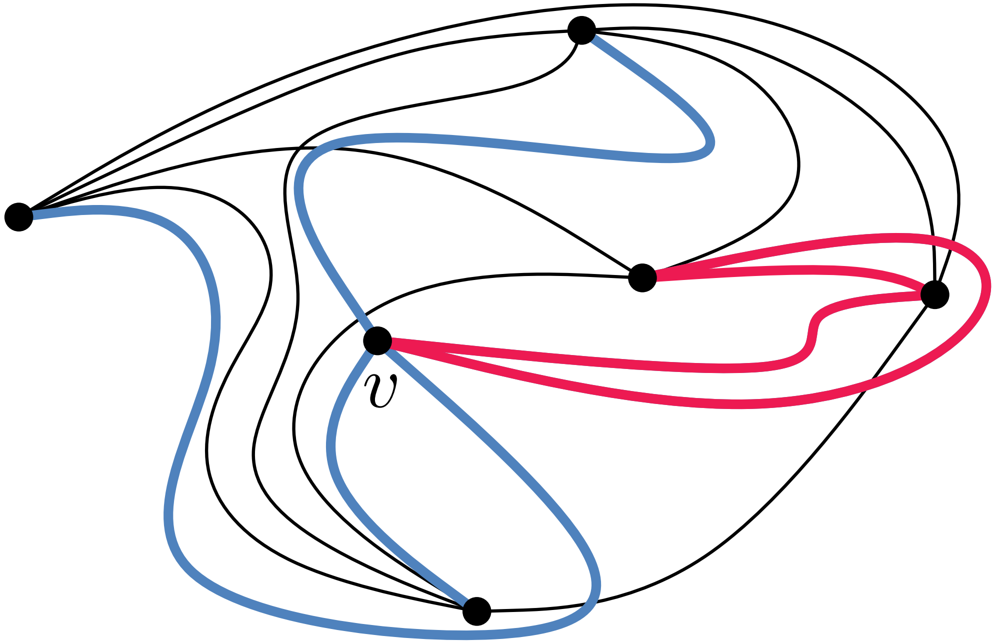

Proposition. $G = K_n$:

Δ empty star triangle at v $\Leftrightarrow$
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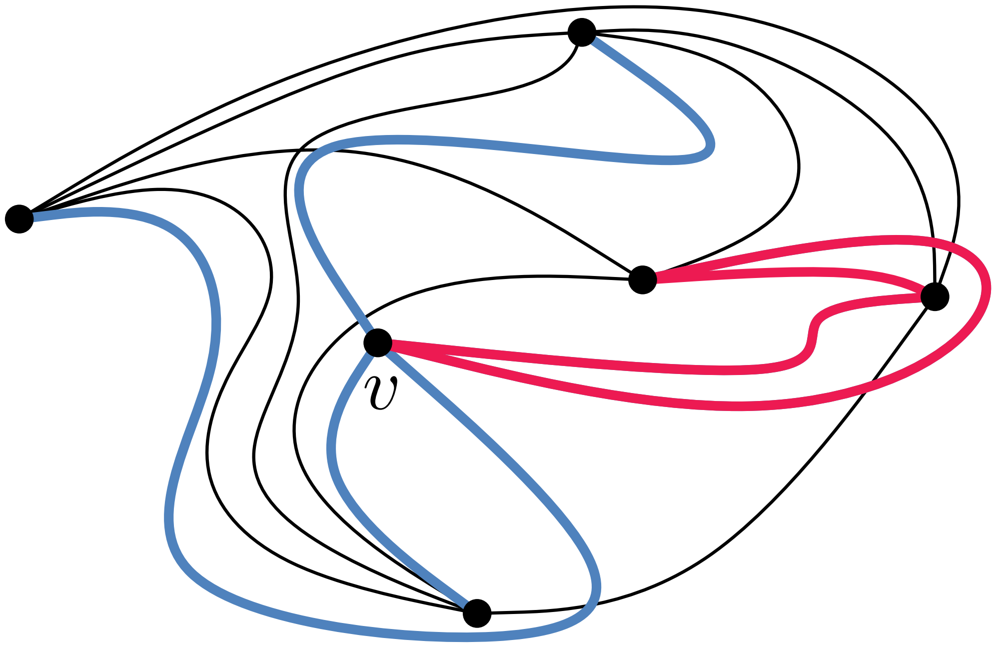

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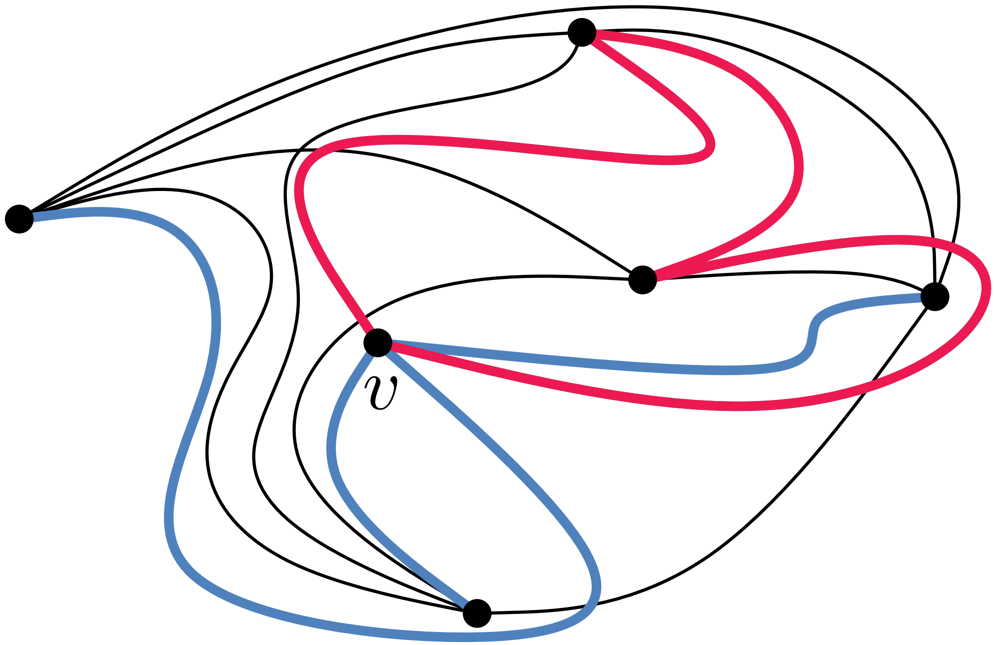

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Background Cont.

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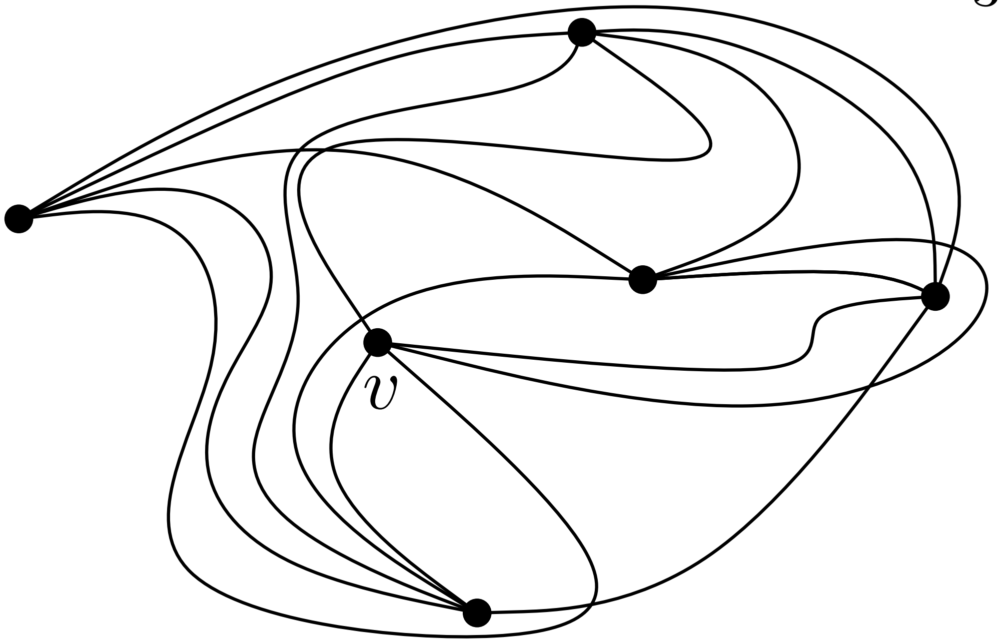

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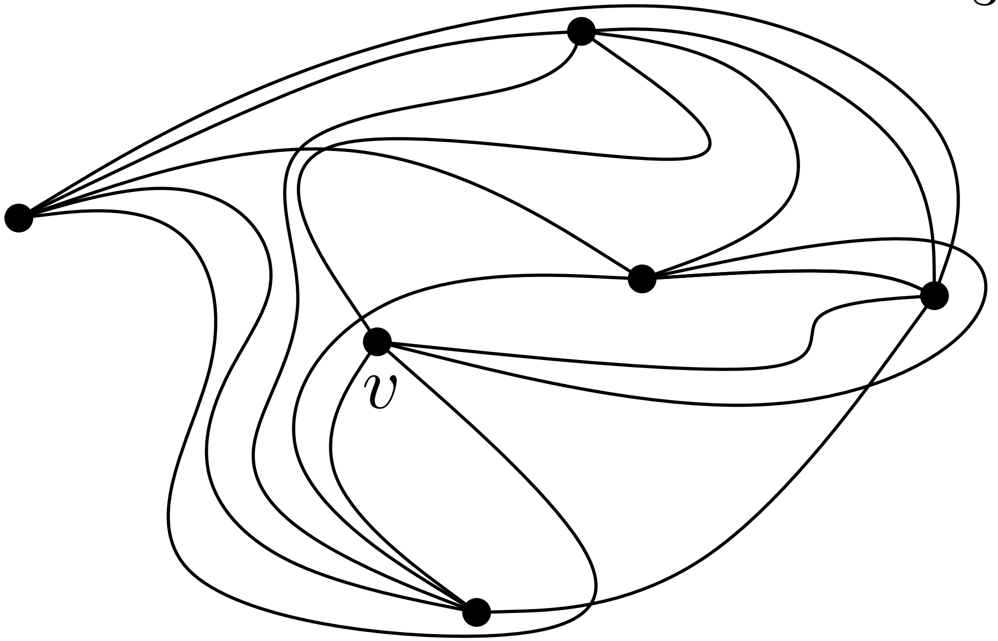

- $D(K_n)$, plane subdrawing H, vertex $v \notin H$:
  2 edges completely in same face of H as v

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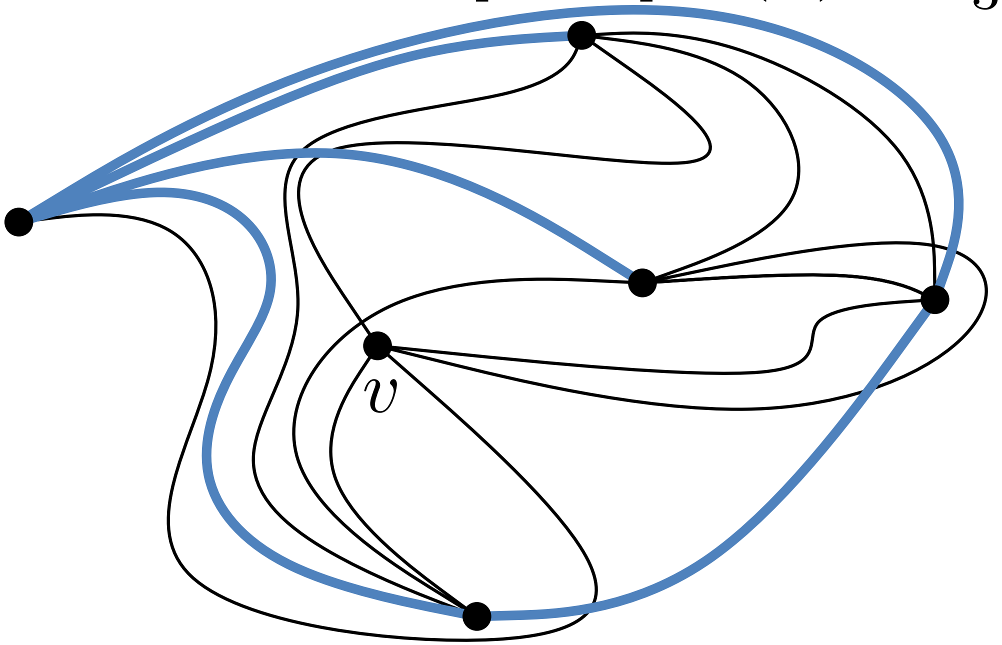

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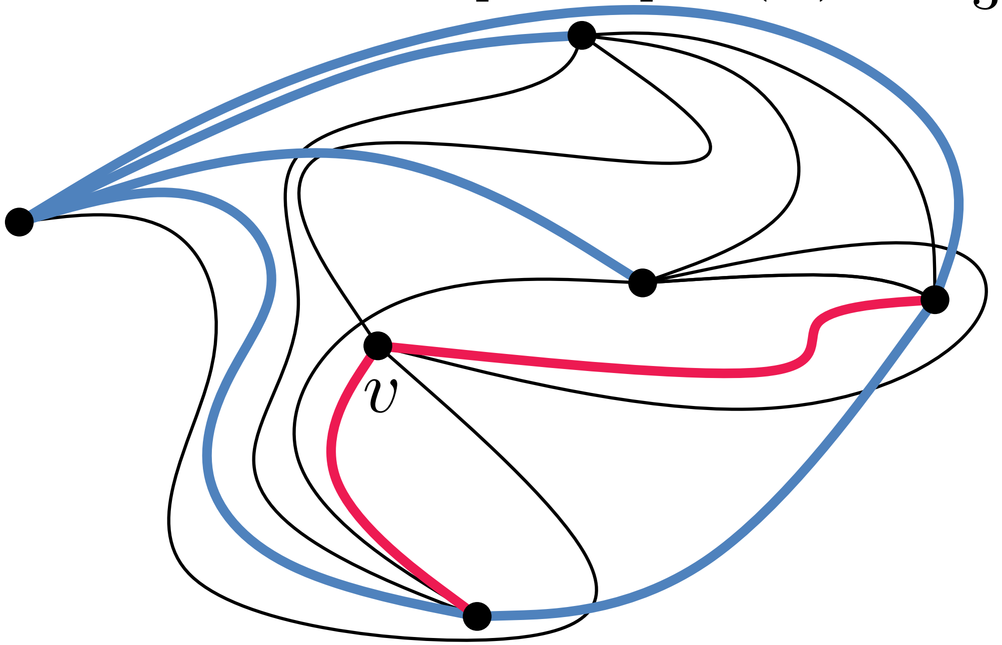

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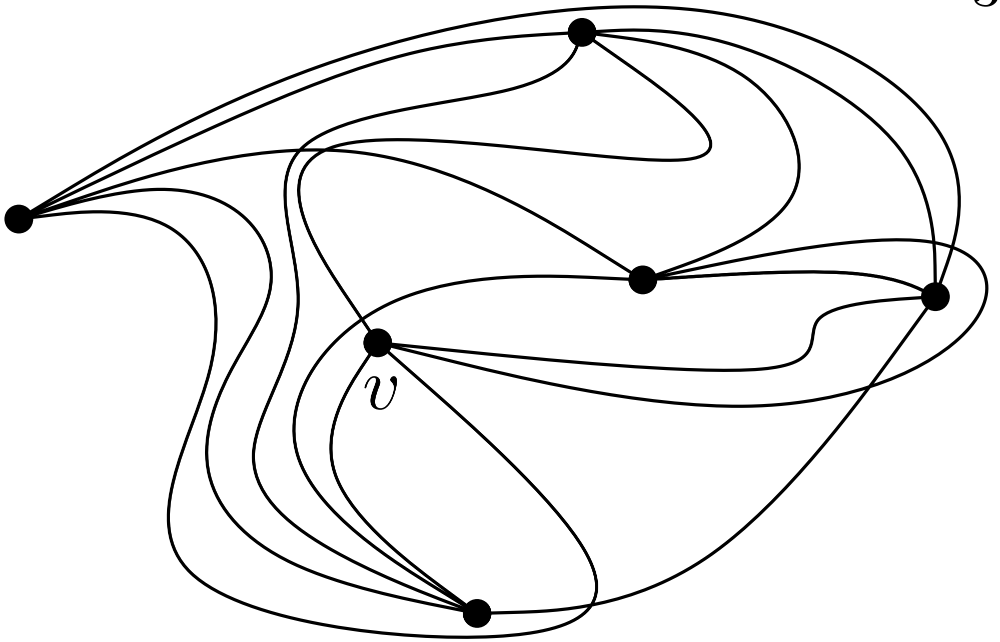

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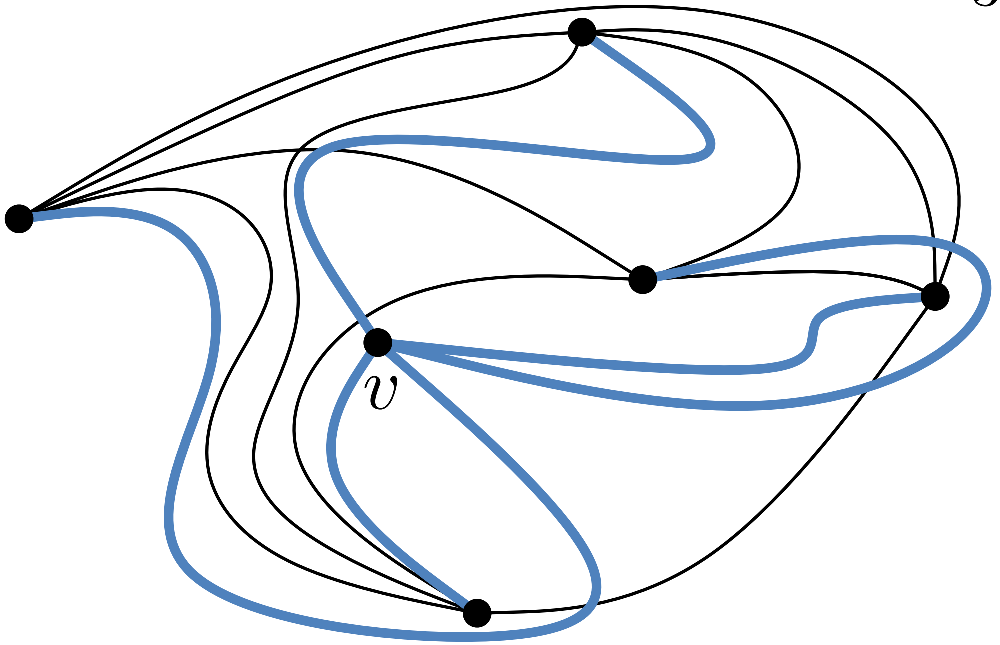
- every vertex v is incident to an interior-empty triangle

Background Cont.

R. Fulek, A. Ruiz-Vargas [2013]: $t(n) \geq \frac{2n}{3}$



- $D(K_n)$, plane subdrawing H, vertex $v \notin H$:
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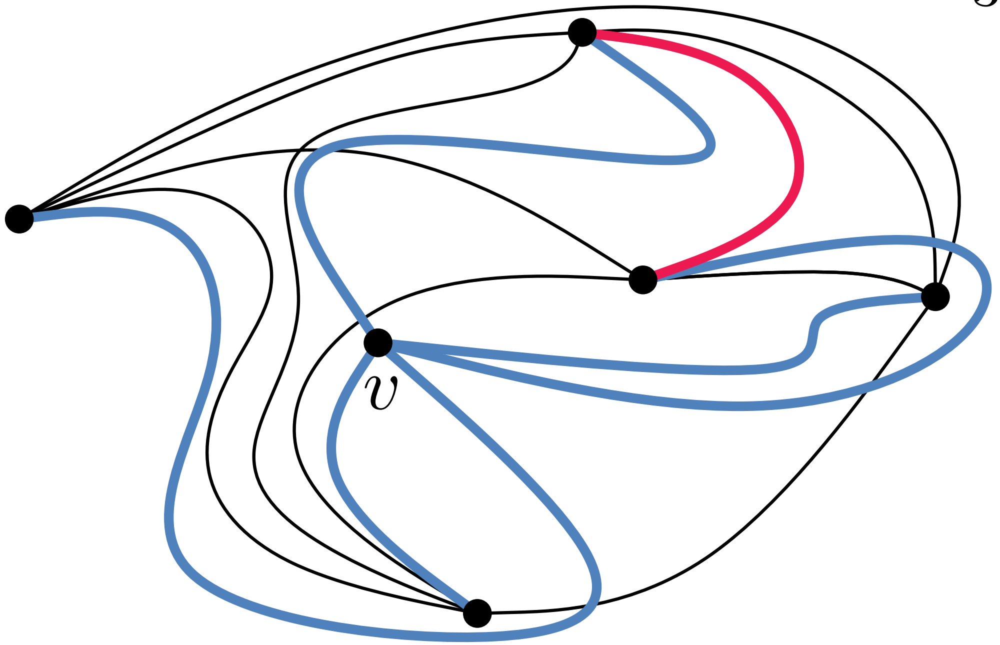

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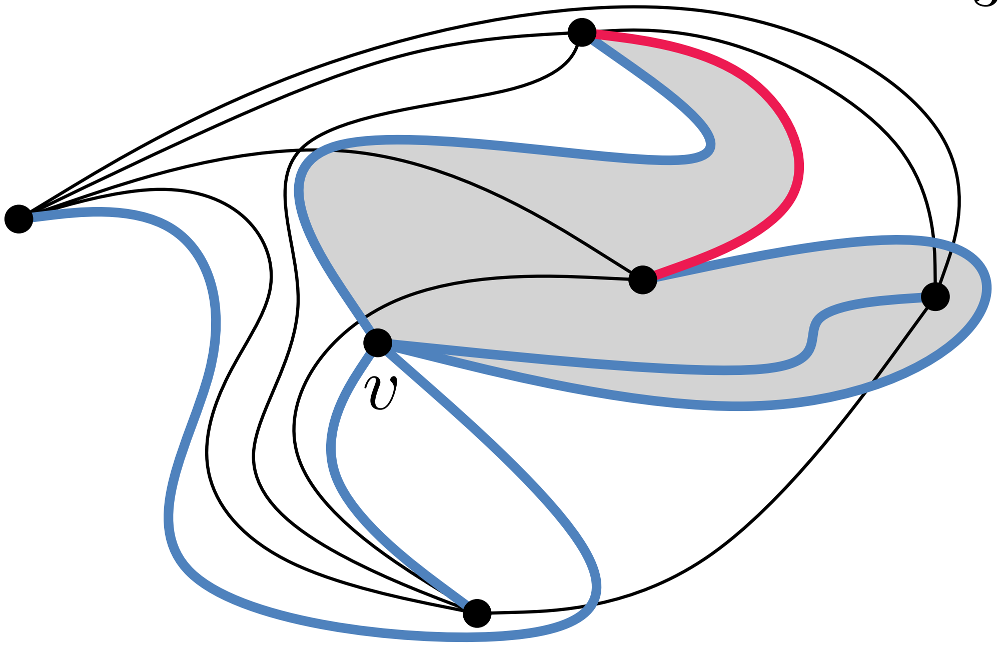

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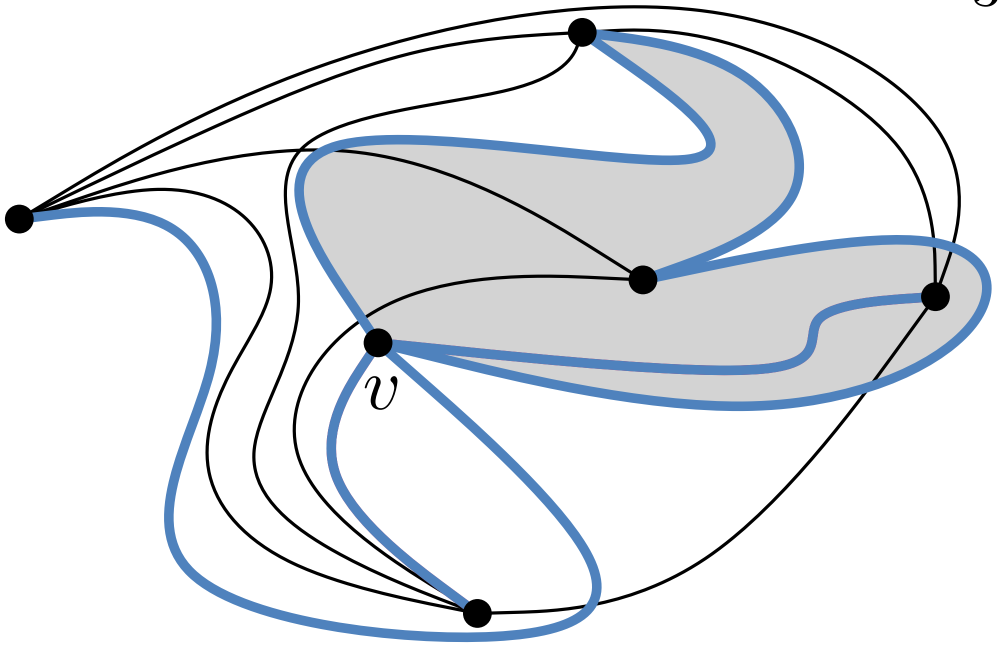

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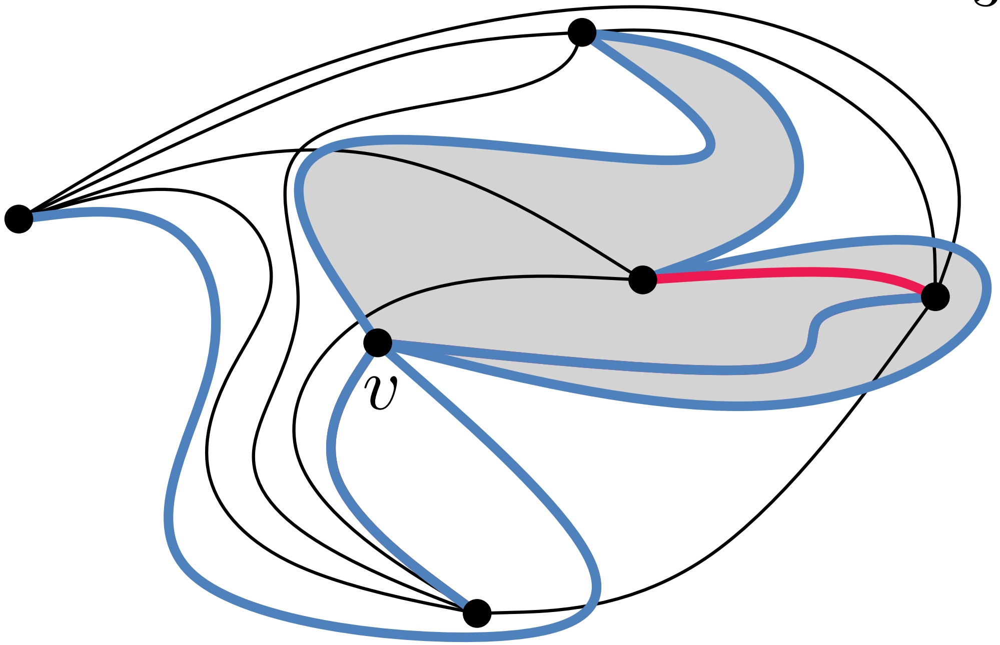

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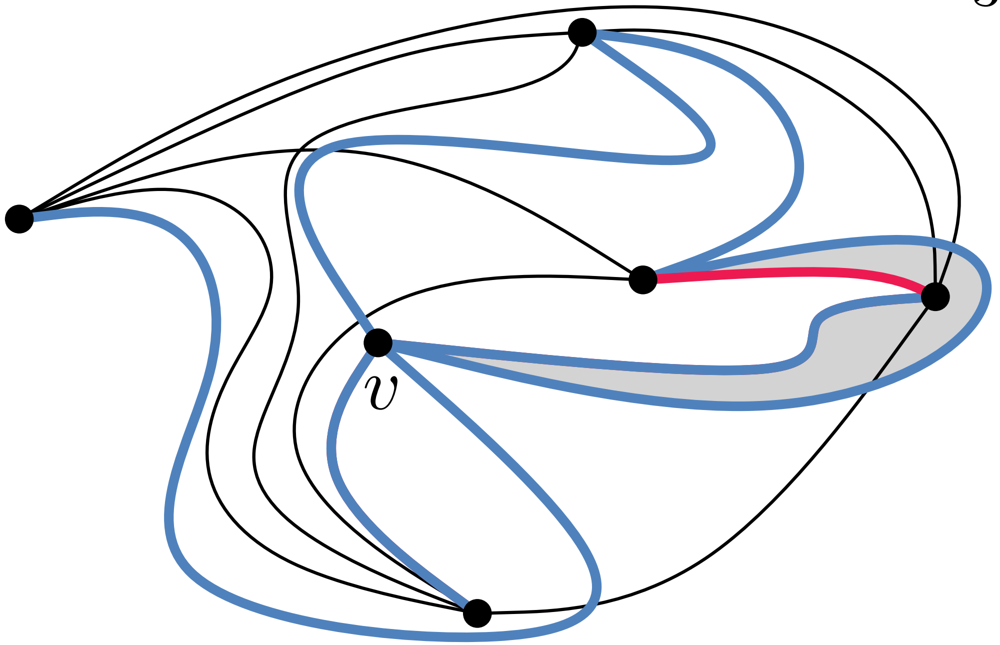

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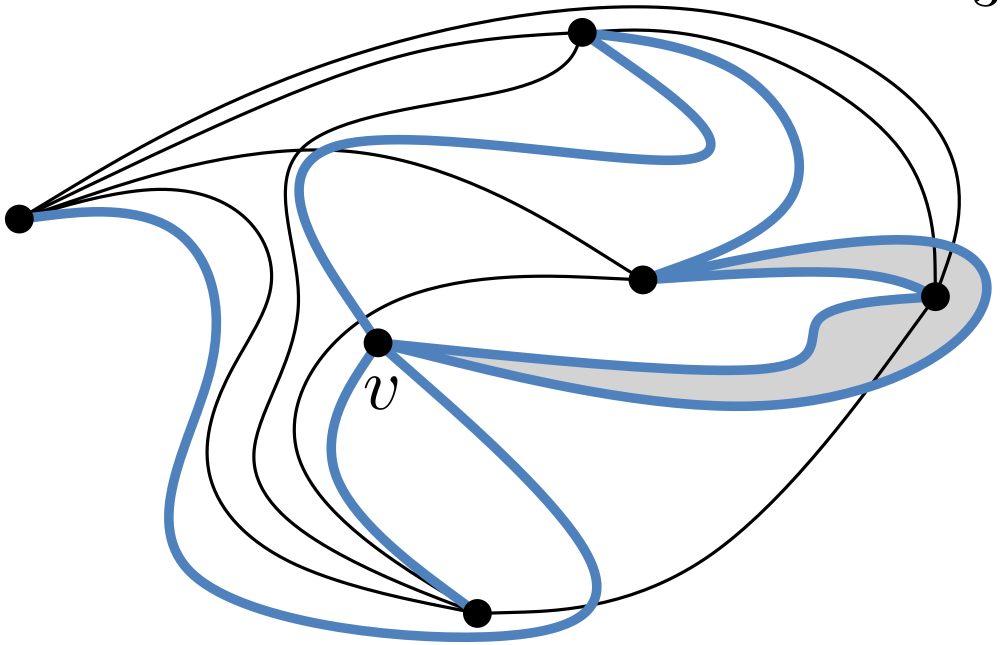

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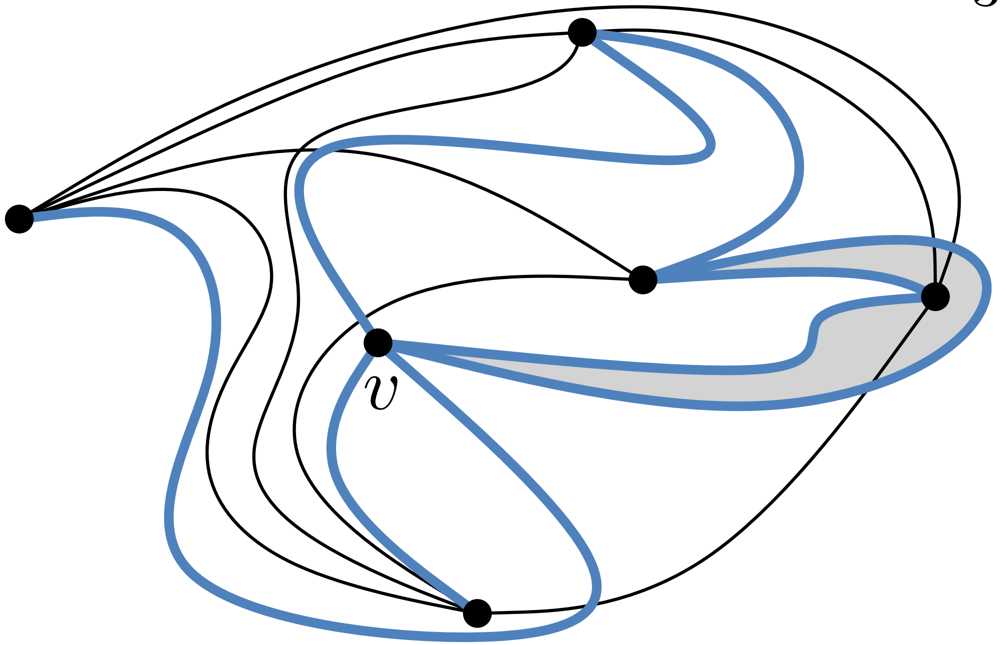

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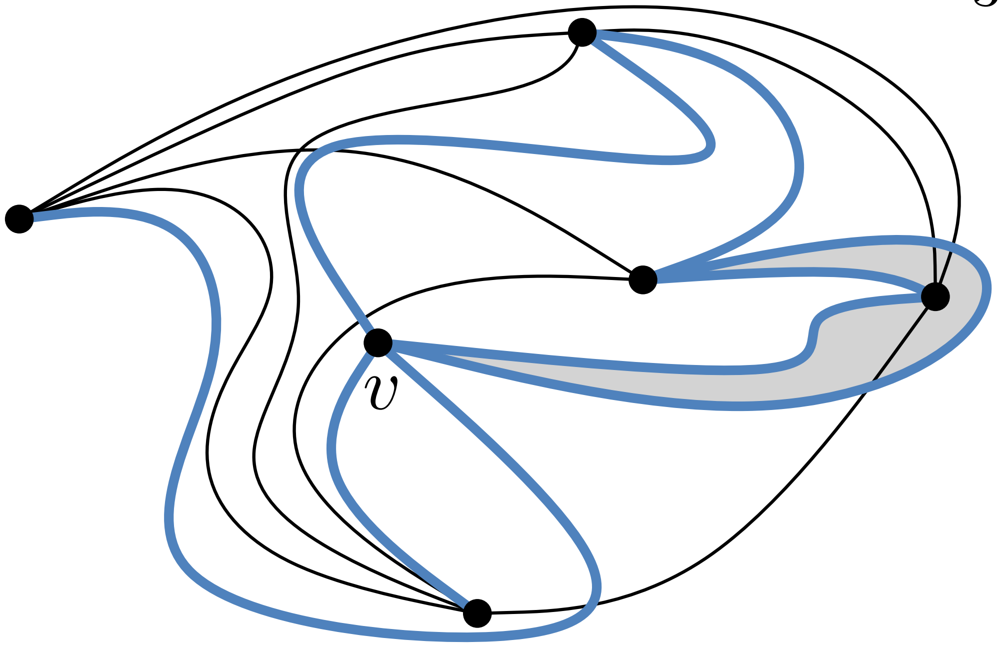

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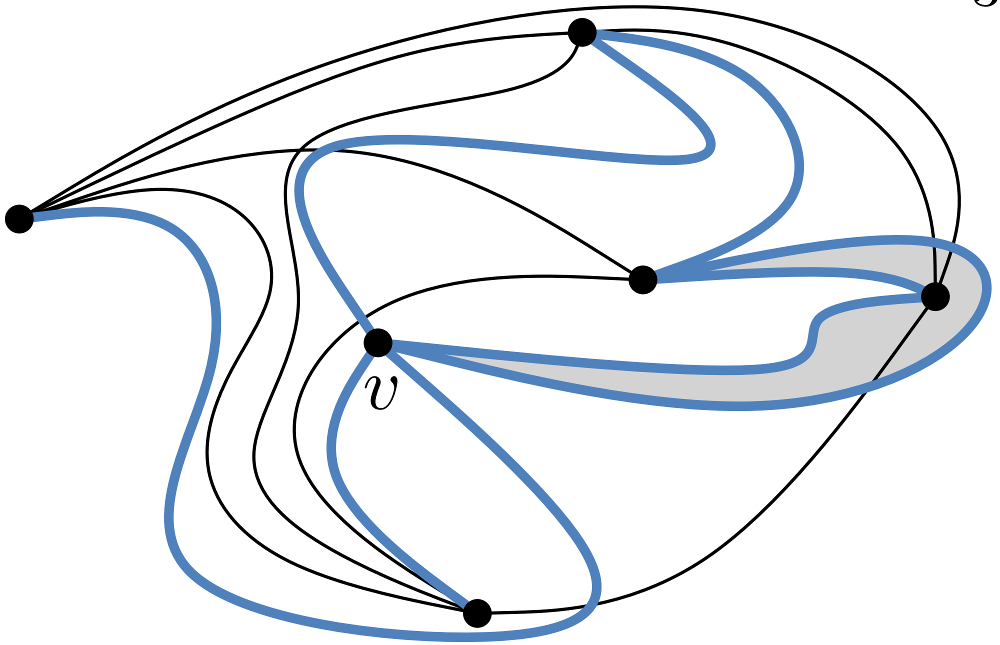

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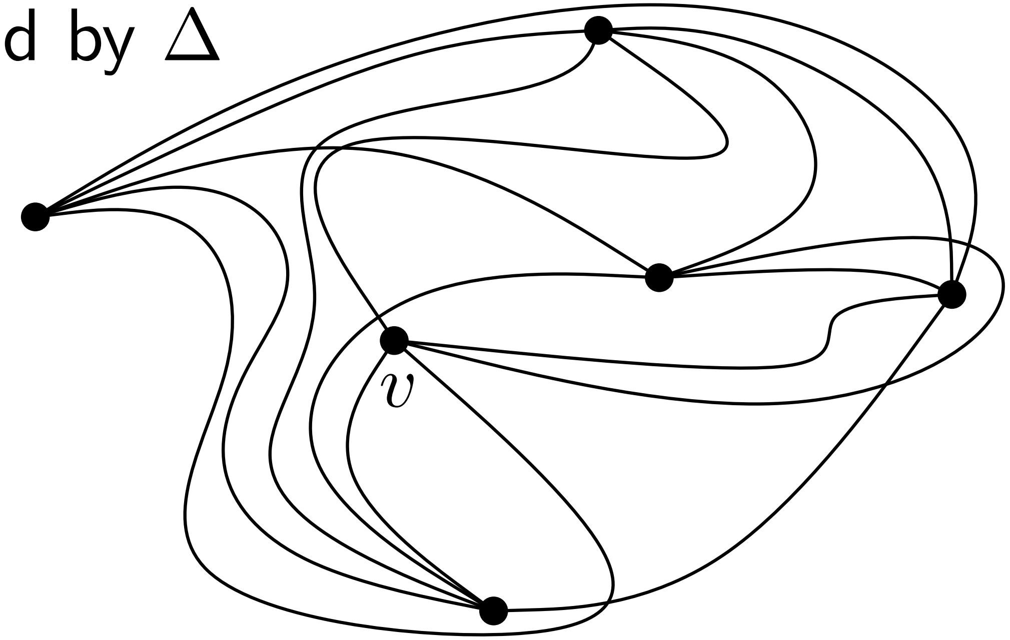
Loneliness and Luckiness

- v lonely (for Δ): v is the only vertex in a component induced by Δ



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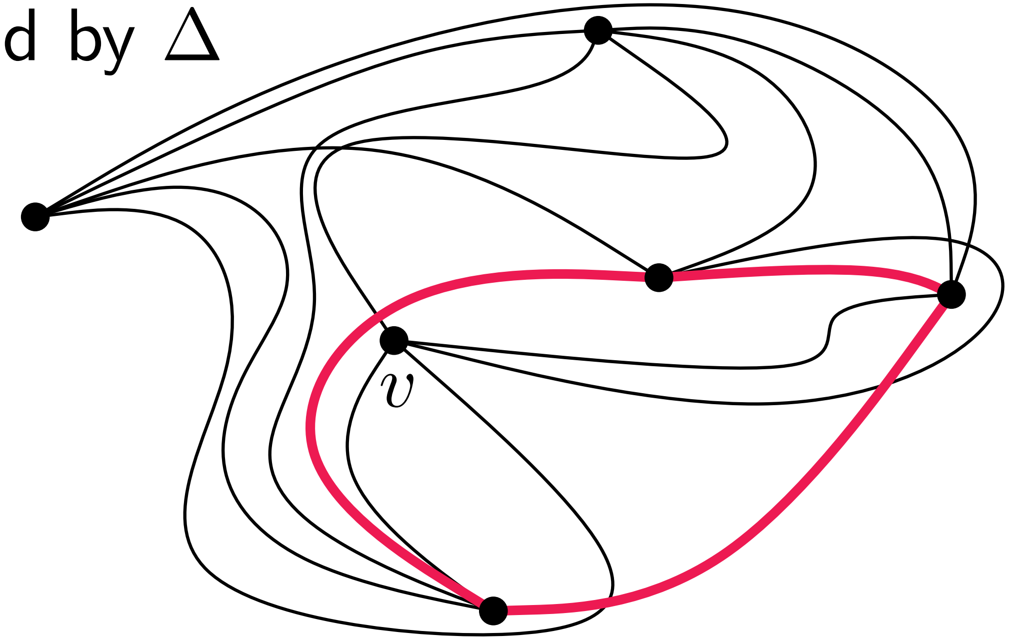

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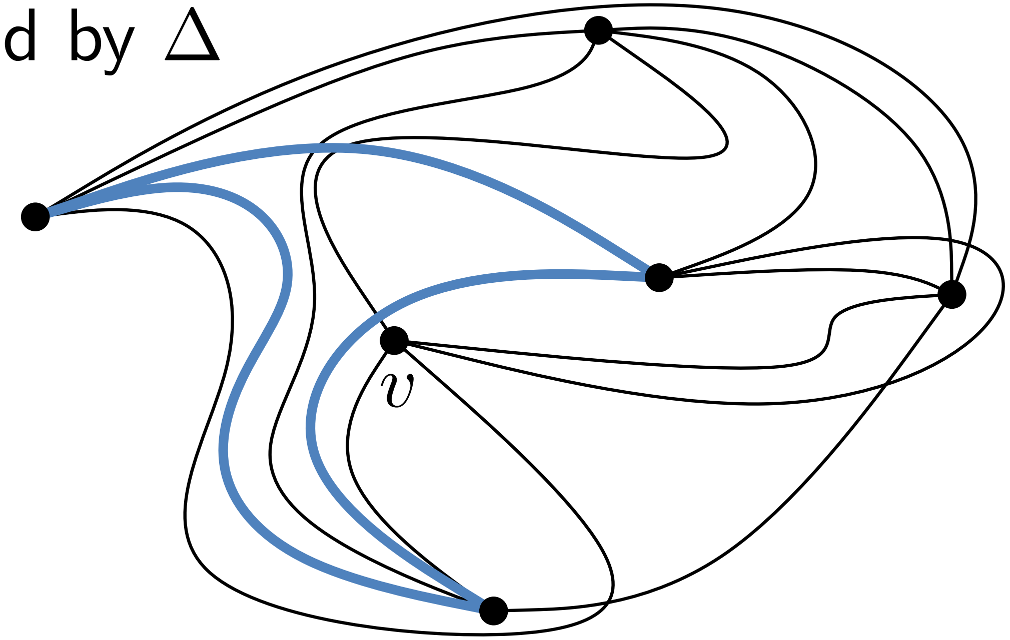

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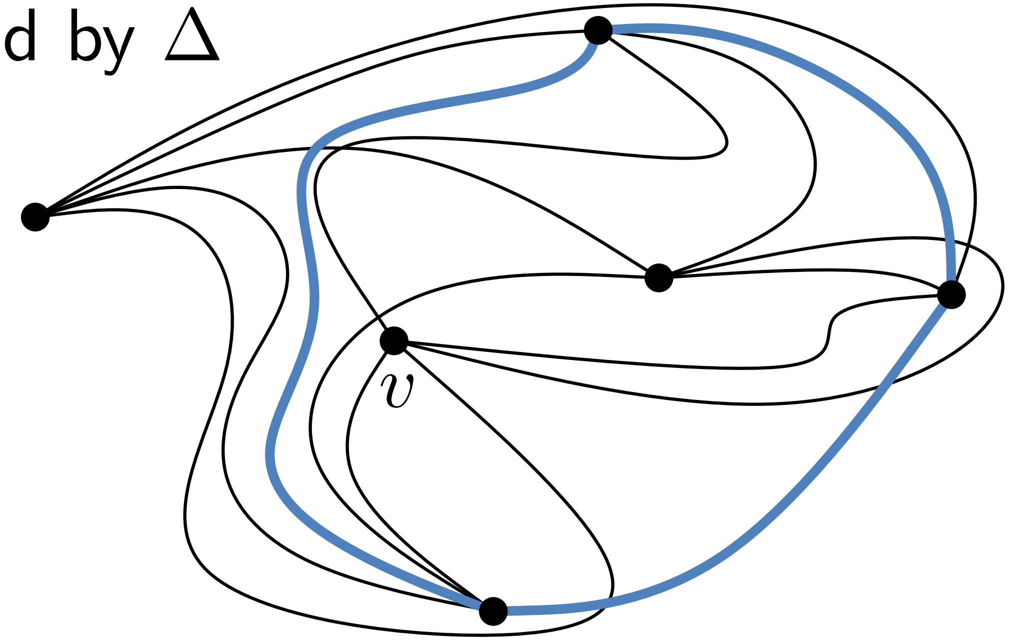

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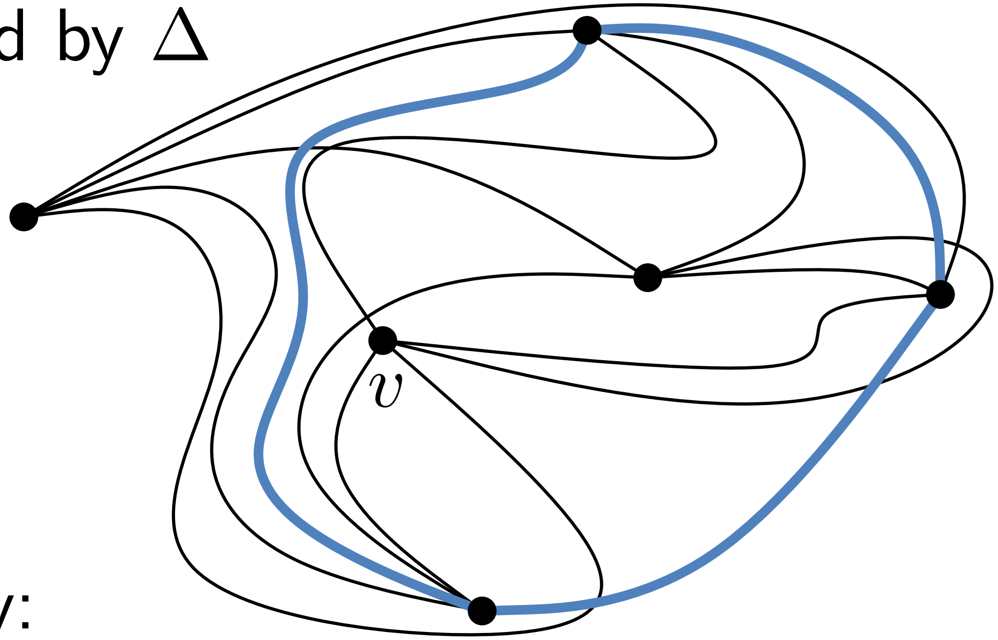

- v lucky:
  - $\ell(v) \ldots$ number of triangles for which v is lonely
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$\Rightarrow$ v lucky if $t(v) - \ell(v) \geq 2$

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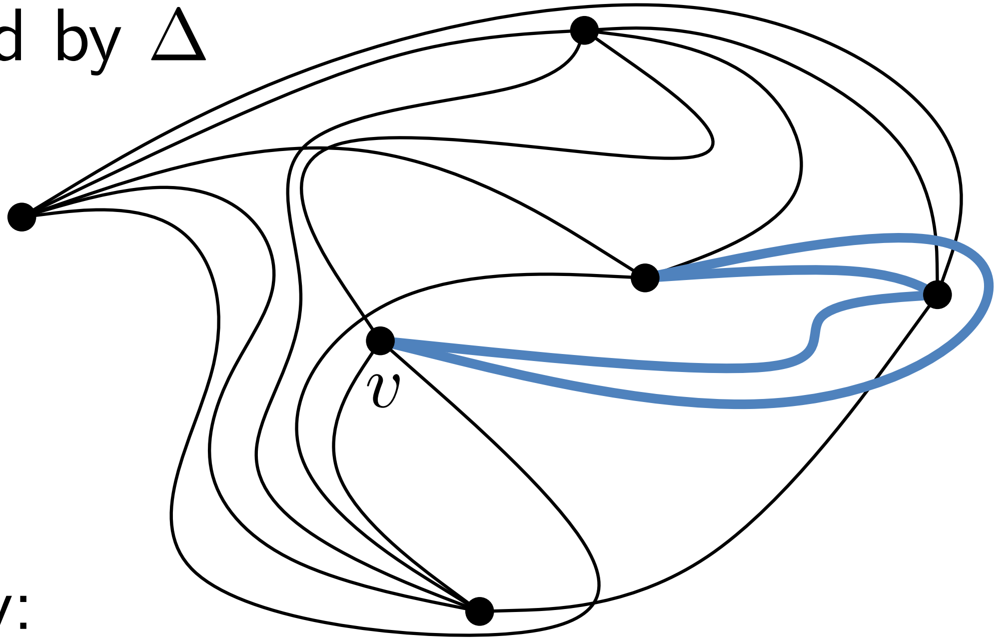

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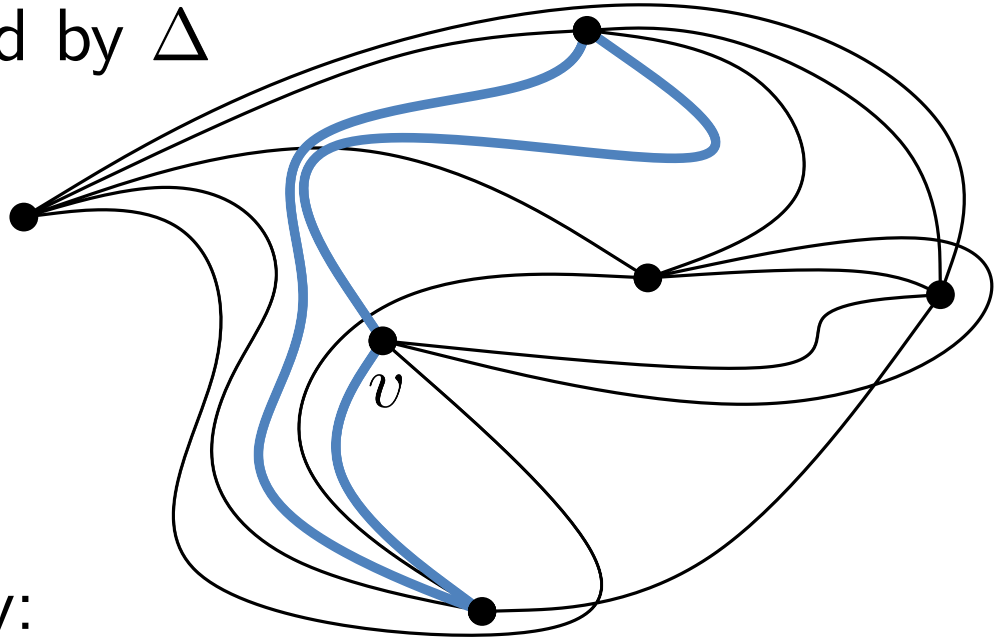

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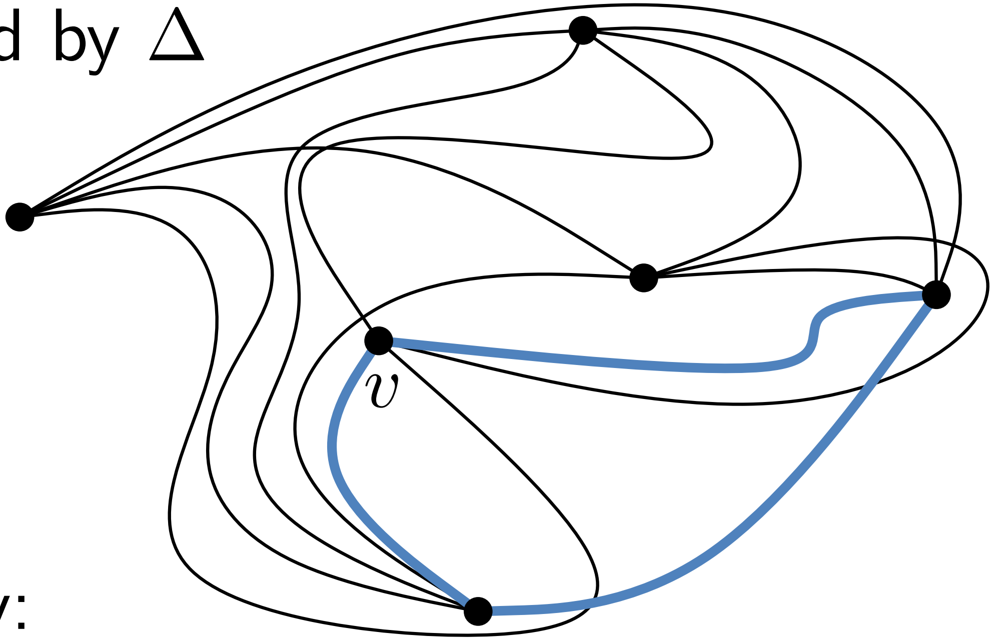

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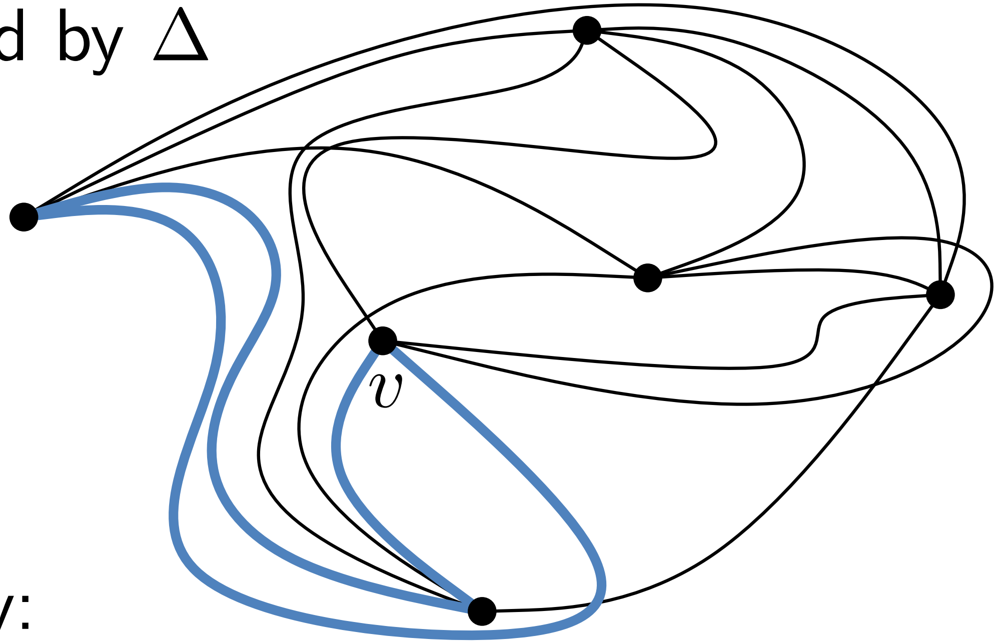

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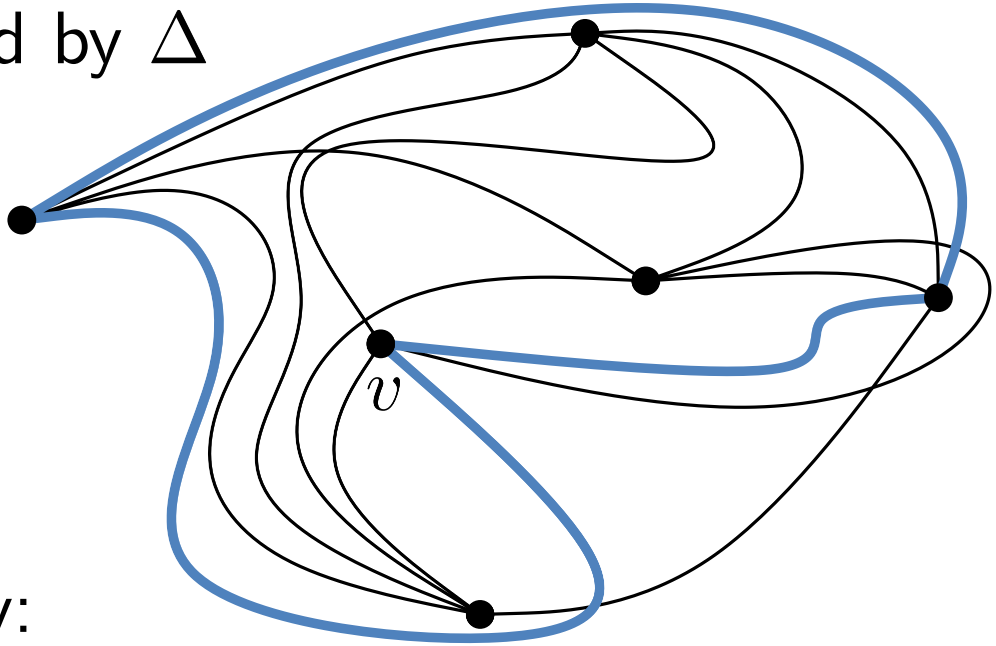

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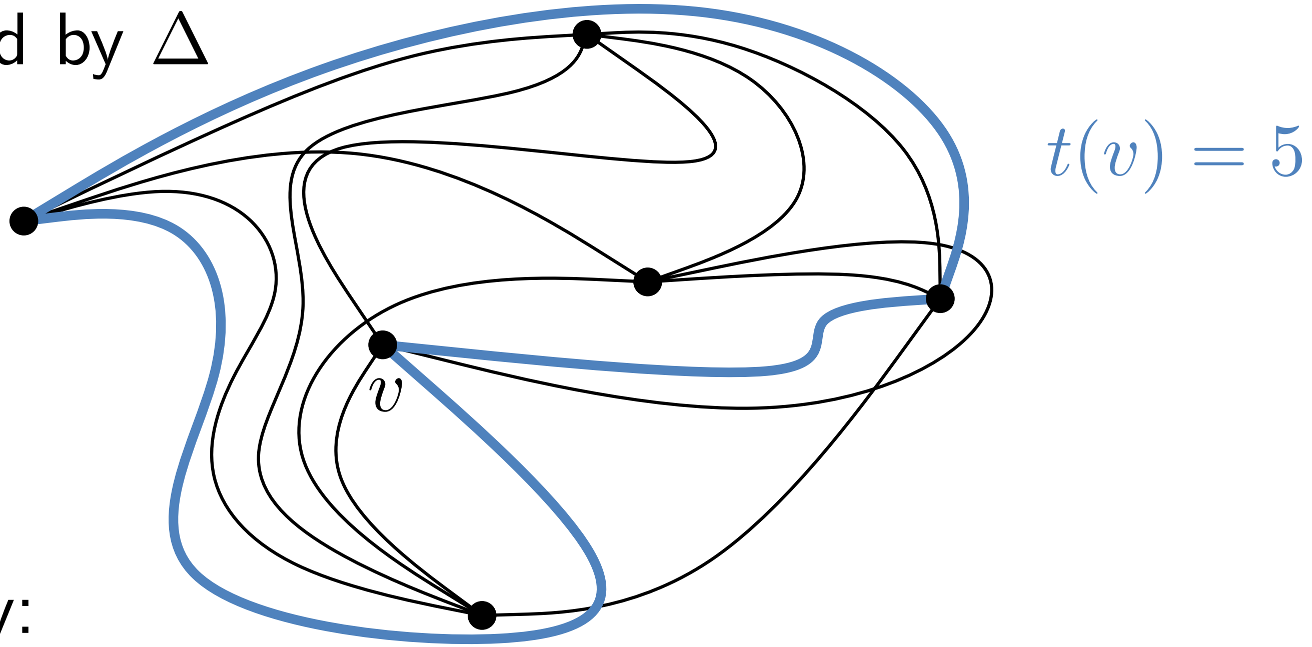

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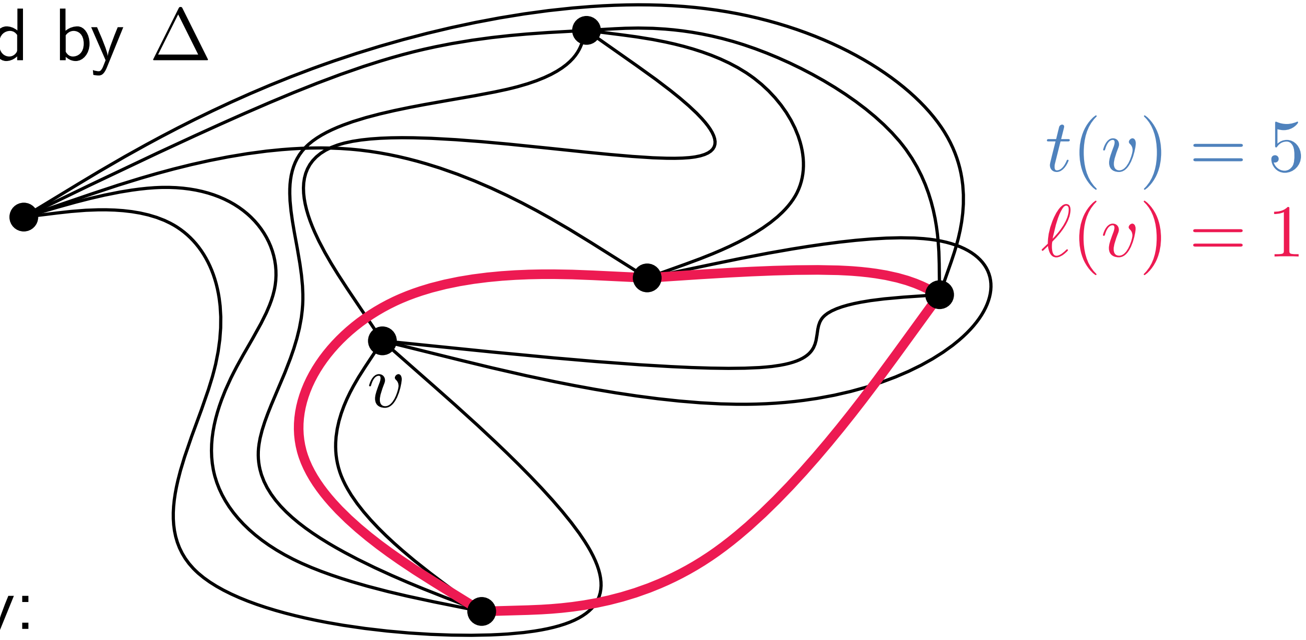

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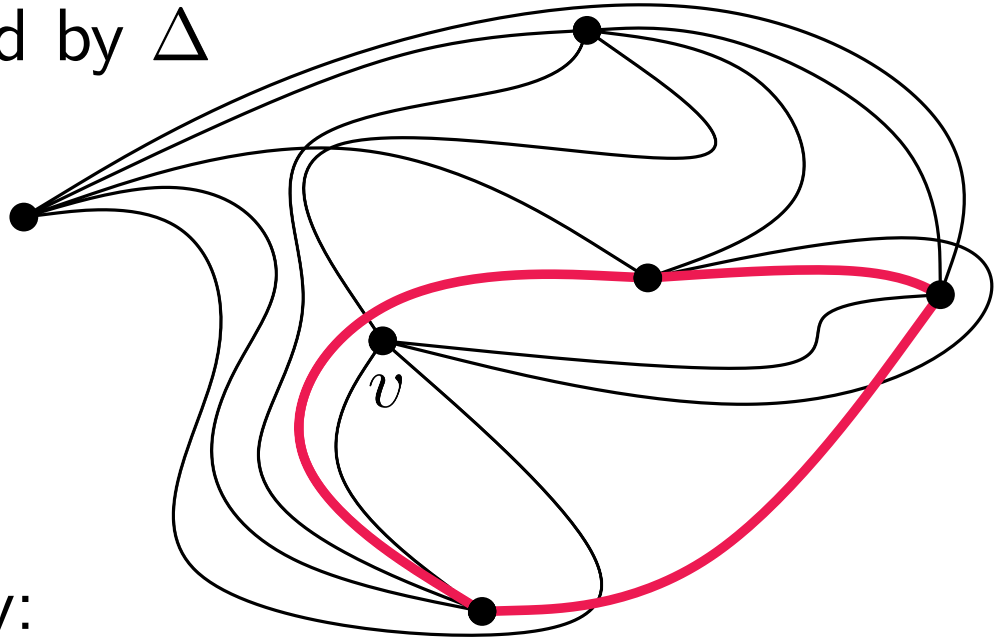

$t(v) = 5$

$\ell(v) = 1$

$\Rightarrow v$ lucky

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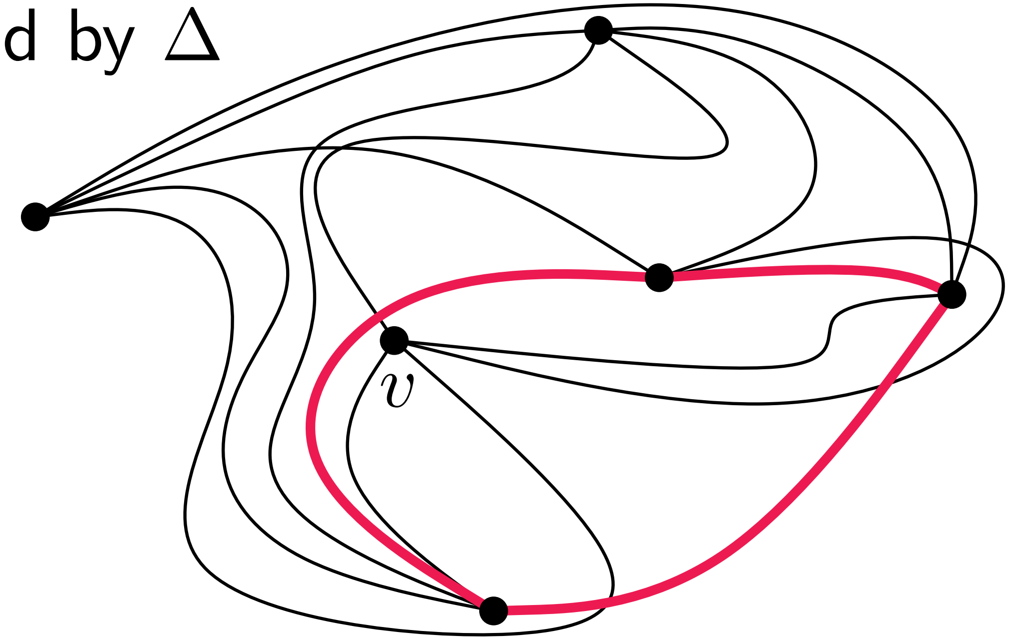

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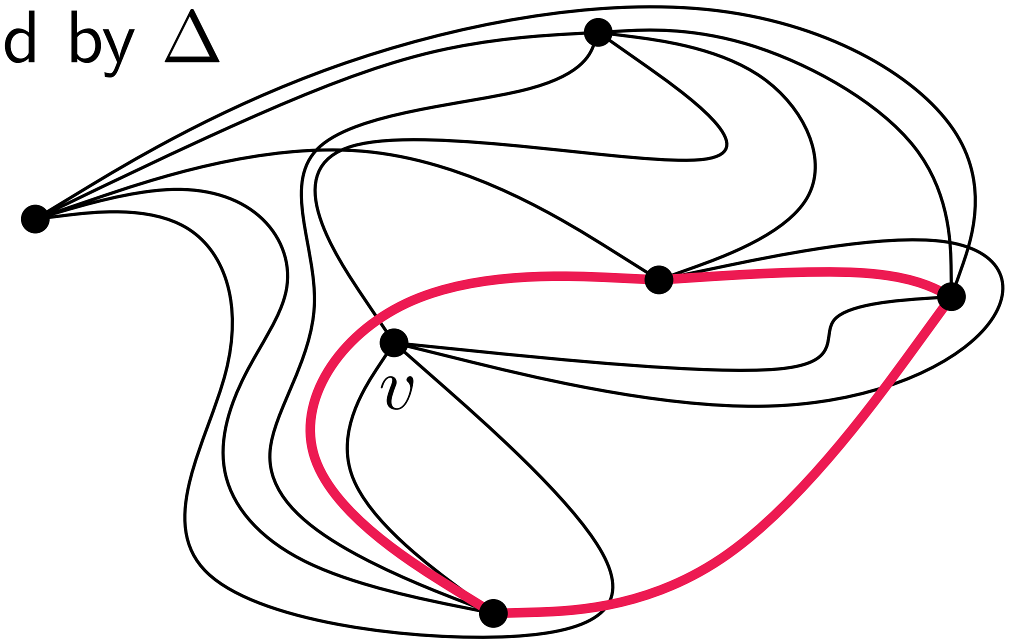

Proposition. v lonely (for some Δ) $\Rightarrow$ $t(v) \geq 3$

Loneliness and Luckiness

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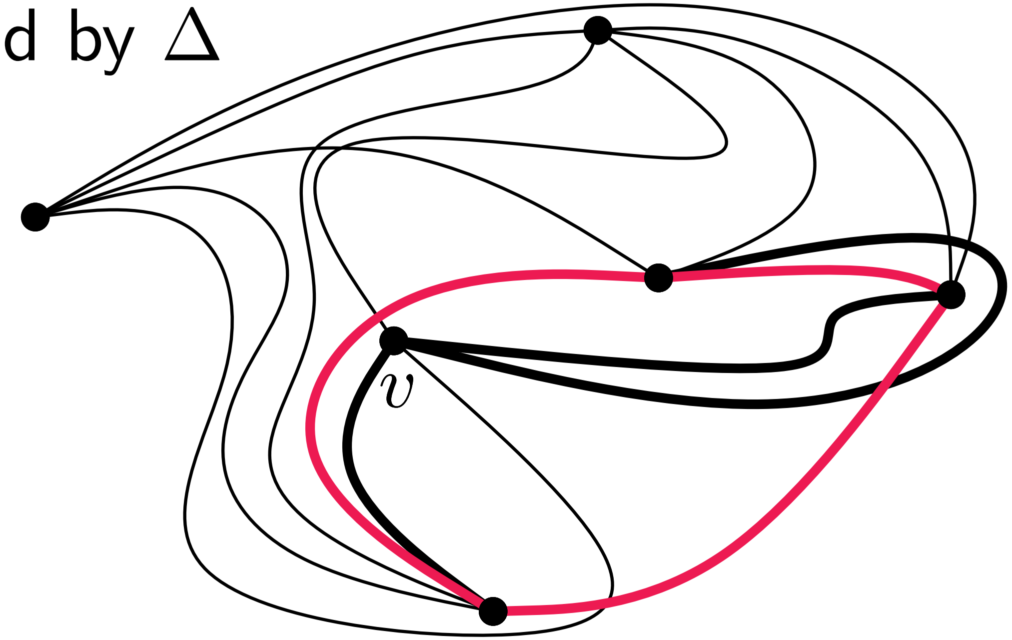

Proposition. v lonely (for some Δ) $\Rightarrow$ $t(v) \geq 3$

Proof. $D(v, \Delta)$ and star $S(v)$ of v in $D(v, \Delta)$:

Loneliness and Luckiness

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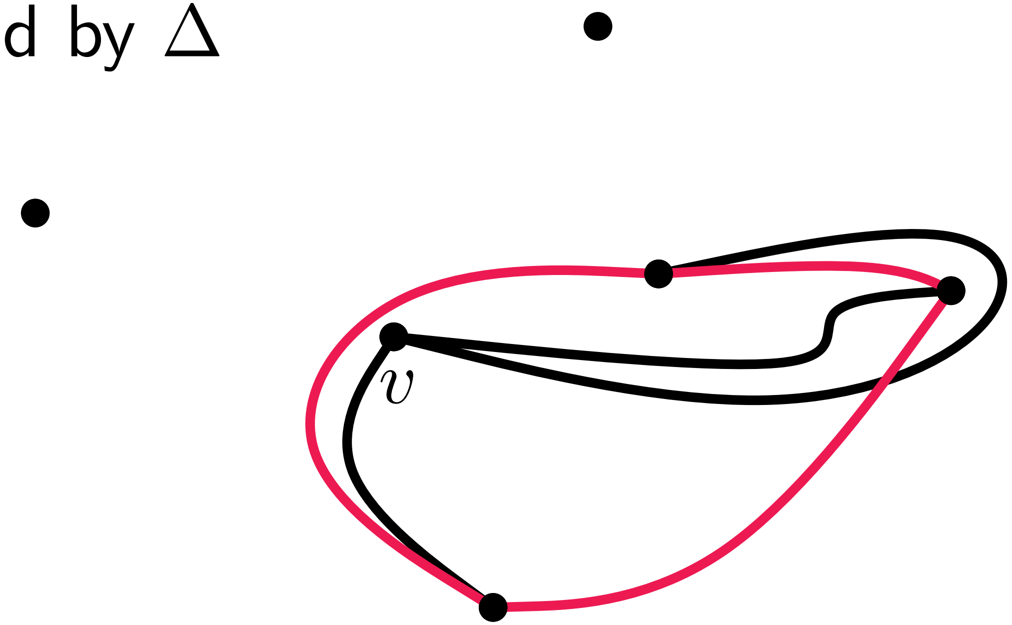

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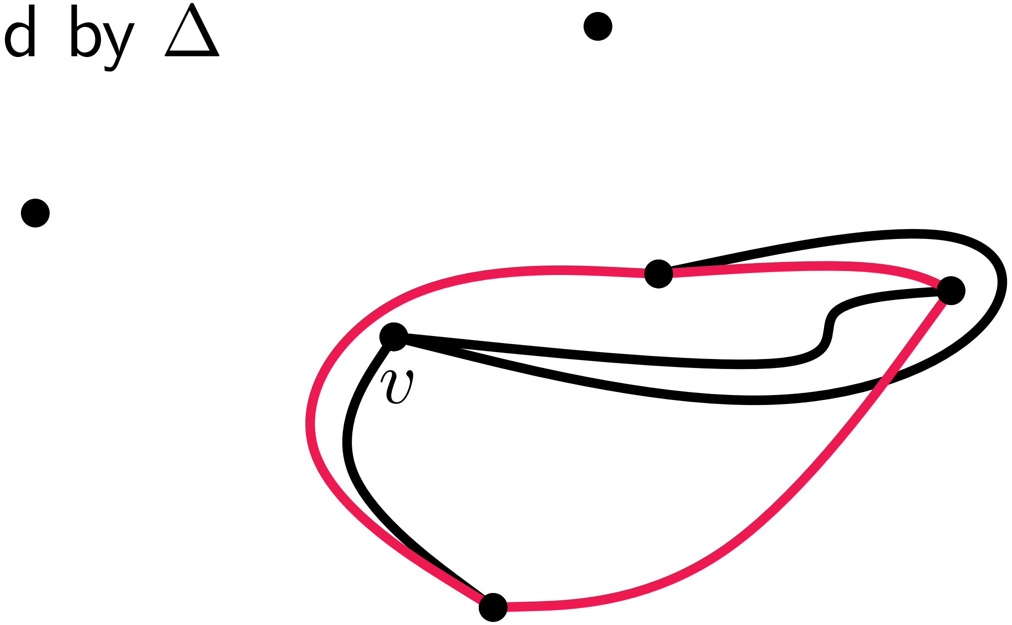

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$S(v)$ intersects Δ $\Rightarrow$ empty non-star triangle at v

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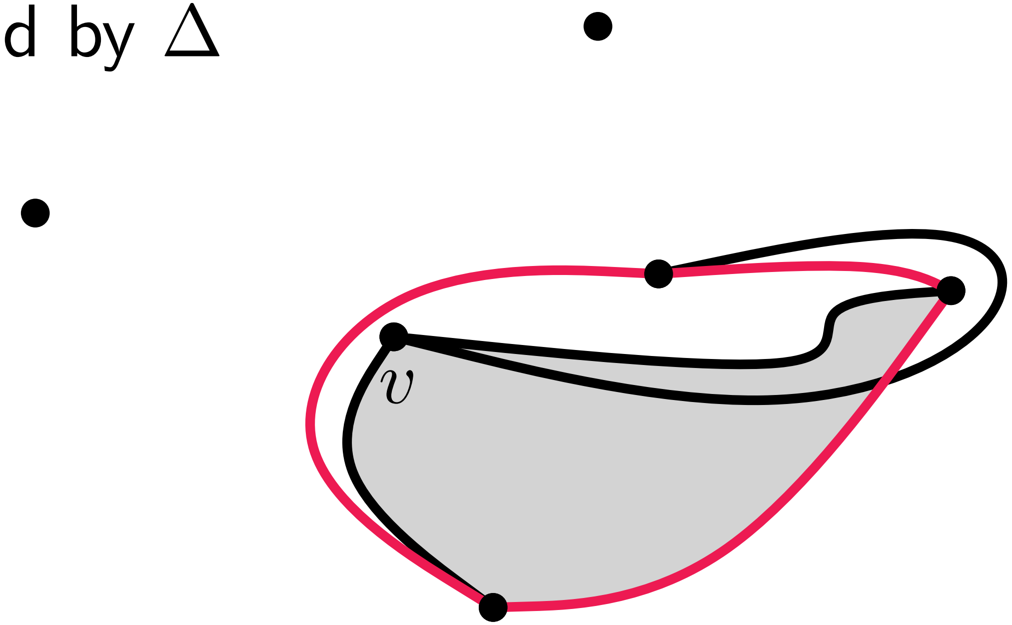

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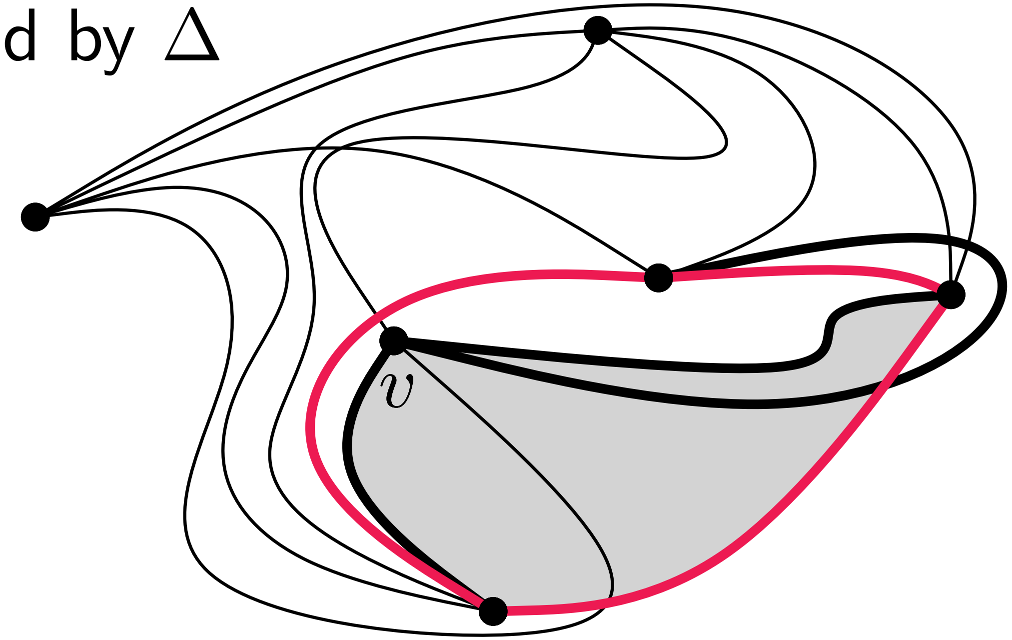

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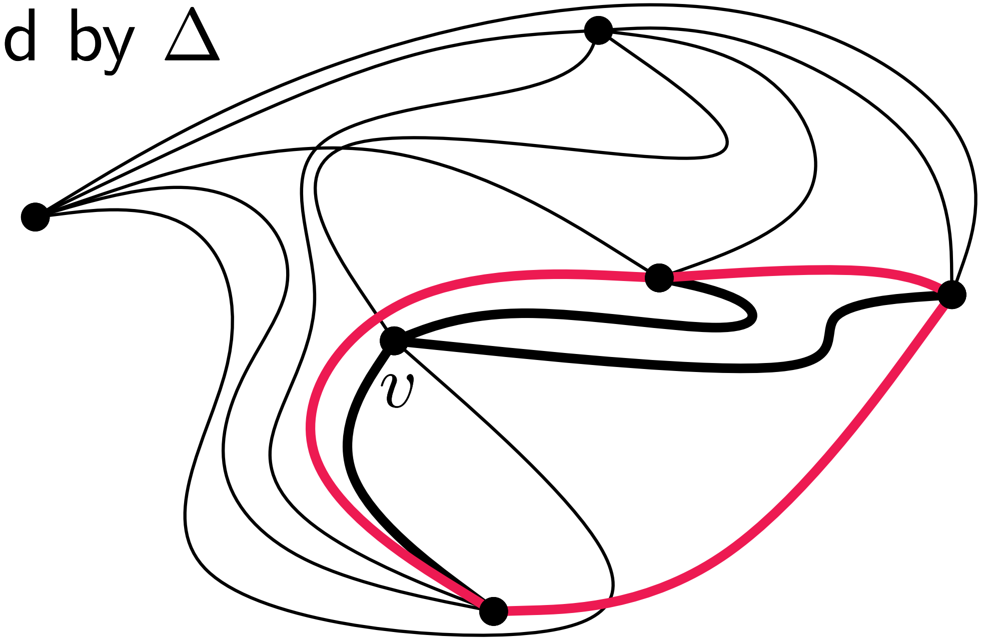

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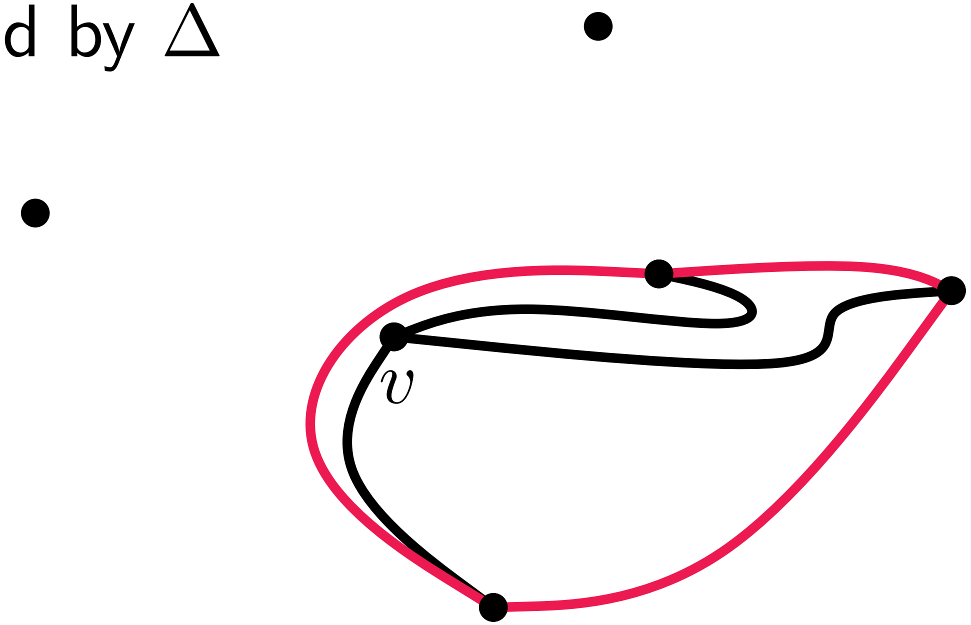

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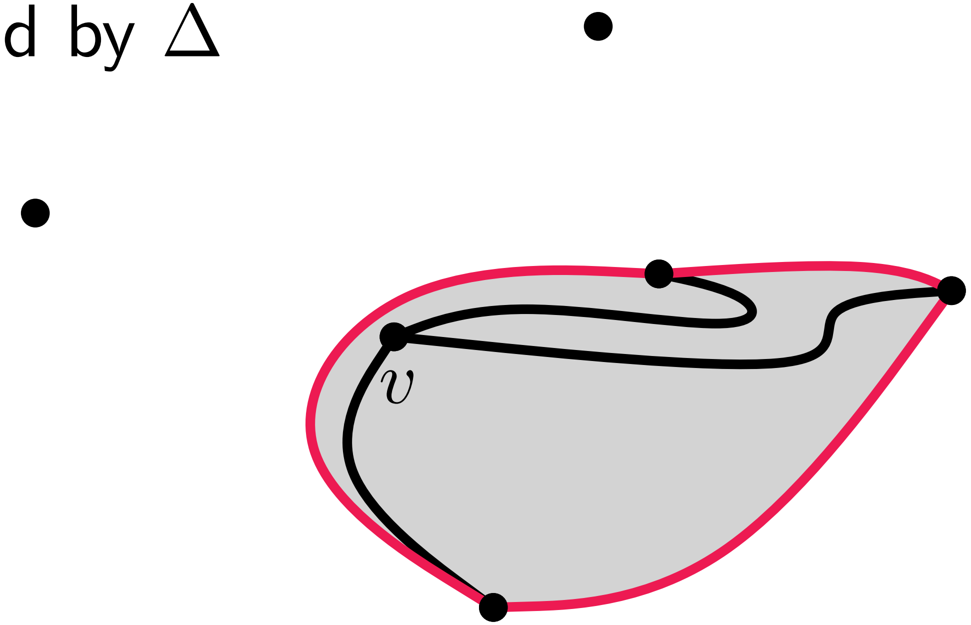

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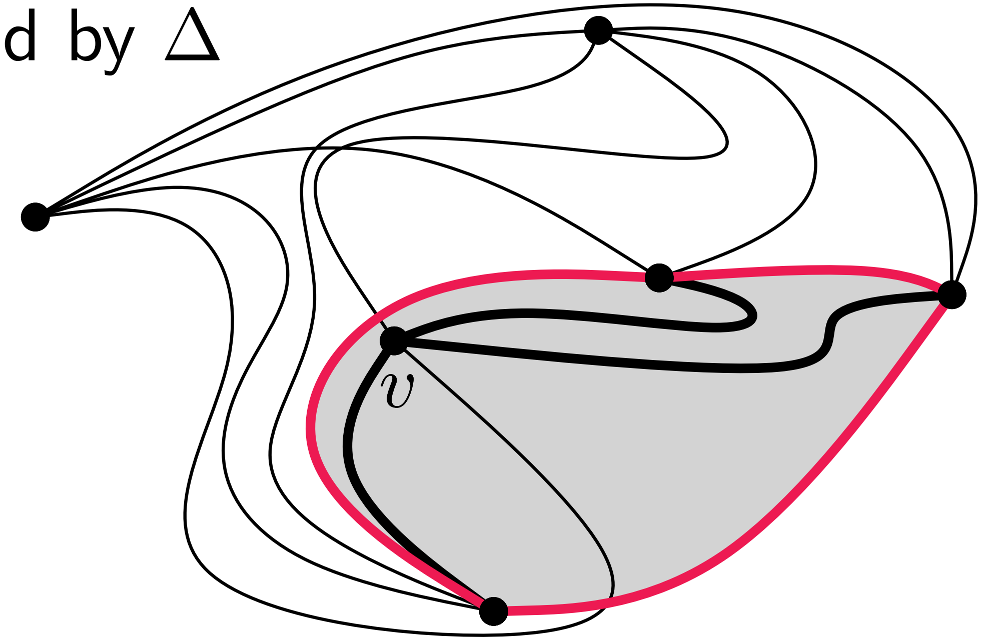

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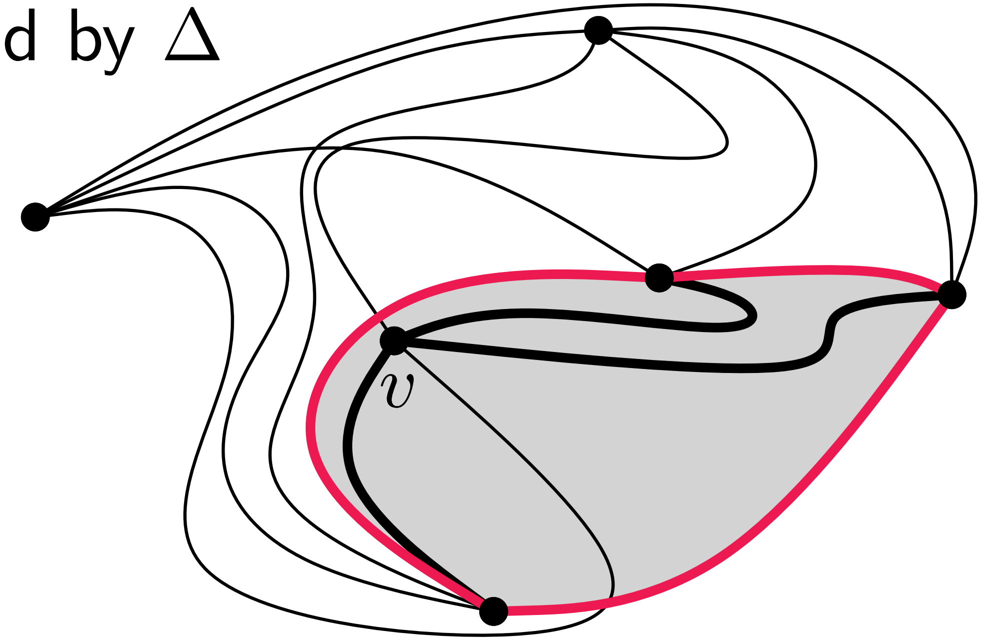

Proposition. v lonely (for some Δ) $\Rightarrow$ $t(v) \geq 3$

Corollary. v not lucky $\Rightarrow$ $t(v) \geq 3$ and $n \geq 4$

Putting Things Together

- $n > 4$, no lucky vertices in $D(K_n)$
  $\Rightarrow$ $t(v) \geq 3$ for all vertices v
  $\Rightarrow$ at least $\frac{3n}{3} = n$ empty triangles in $D(K_n)$

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  $\Rightarrow$ t(4) $=$ 4 empty triangles in any $D(K_4)$

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- $n > 4$, v lucky in $D(K_n)$
  $\Rightarrow$ t empty triangles in $D(K_n \backslash v)$
  $\Rightarrow$ $t + t(v) - l(v) \geq t + 2 \geq t(n-1) + 2$
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$\Rightarrow$ **Theorem.** For $n \geq 4$, every drawing $D(K_n)$ contains at least n empty triangles.

Conclusion

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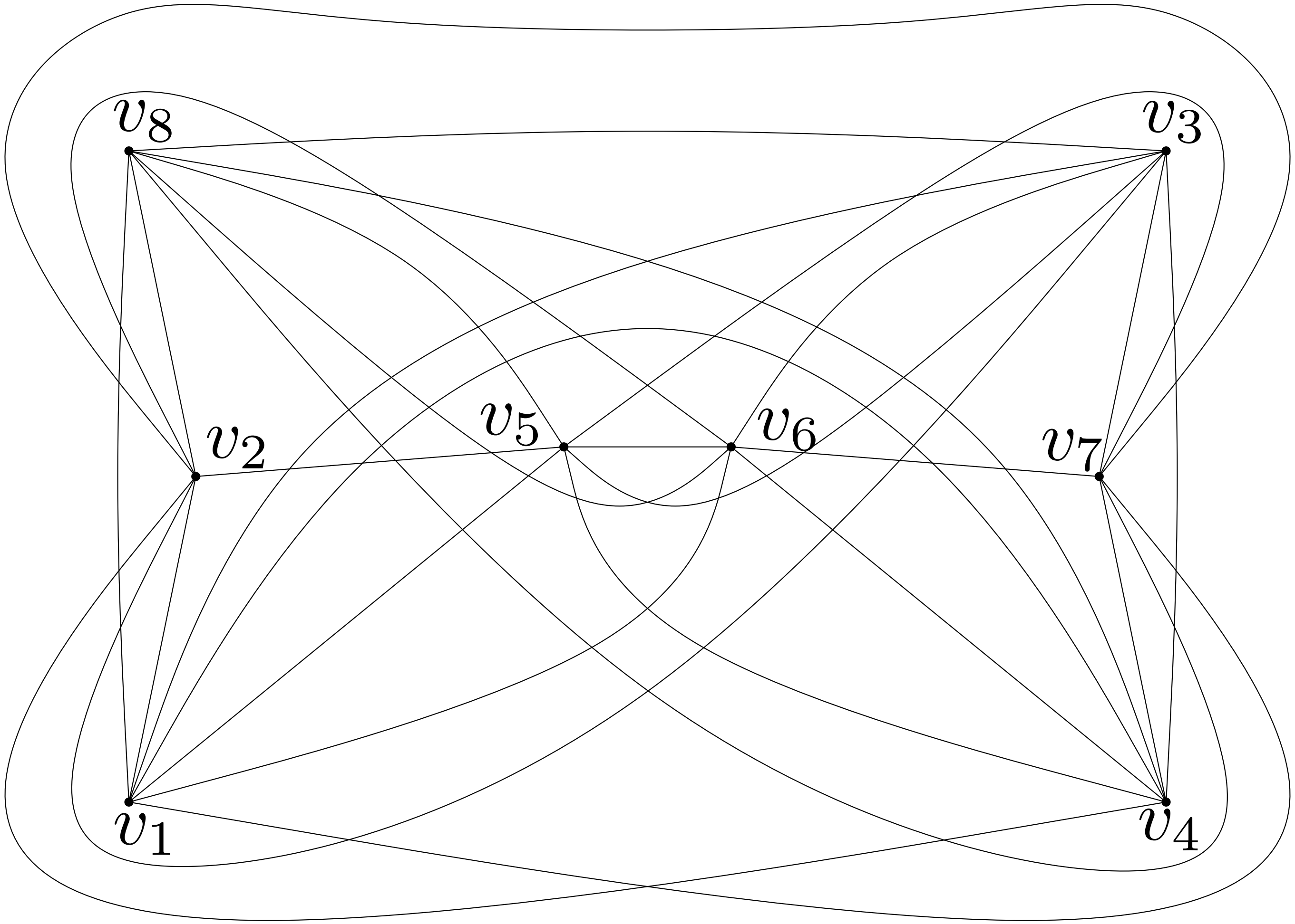
- For $3 \leq n \leq 8$, every drawing $D(K_n)$ contains at least $2n - 4$ empty triangles.

$\Rightarrow$ **Conjecture.** Every drawing $D(K_n)$ contains at least $2n - 4$ empty triangles.

Thank you for your attention!

No lucky vertices



v	$t(v)$	$\ell(v)$
v_1	4	3
v_2	5	5
v_3	3	2
v_4	4	3
v_5	6	5
v_6	6	5
v_7	5	5
v_8	3	2