Geometric Biplane Graphs

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What are biplane graphs?

A set $S$ of points on the plane.
A geometric graph on $S$: Vertices are points of $S$ and edges are straight line segments between pairs of points.
Plane graphs.

A plane graph on $S$: A geometric graph in which no two edges cross.
A biplane graph on $S$: A geometric graph $G = (V, E)$ admitting a partition of its edges $E = E_b \cup E_r$ such that $G_b = (V, E_b)$ and $G_r = (V, E_r)$ are each plane graphs.
A biplane graph on $S$: A geometric graph $G = (V, E)$ admitting a partition of its edges $E = E_b \cup E_r$ such that $G_b = (V, E_b)$ and $G_r = (V, E_r)$ are each plane graphs.
Why are we interested in biplane graphs?

Biplane graphs are a natural extension of plane graphs.
Why are we interested in biplane graphs?

Printed circuit boards:
We can use the two sides of a board to connect components.
Why are we interested in biplane graphs?

We can use the two sides of a plane to connect the points.
Why are we interested in biplane graphs?

We can use the two sides of a plane to connect the points.
Given a geometric graph $G$, is it biplane?
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$G$ is biplane, if and only if, its intersection graph $G_X$ is bipartite.
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$G$ is biplane, if and only if, its intersection graph $G_X$ is bipartite.
Given a geometric graph $G$, is it $k$-plane?

- NP-Complete for $k \geq 3$. (Eppstein, 2009).
- Algorithm $O(n \lg(n))$ if $k = 2$. (Eppstein, 2009).
Maximal Plane Graphs

Where can I place a new edge?
Is this plane graph maximal (a triangulation)?
Maximal Biplane Graphs

Where can I place a new edge?
Is this biplane graph maximal?
A maximal biplane graph is a natural extension of a triangulation.
Maximal Biplane Graphs

Where can I place a new edge?
Obtaining Maximal Biplane Graphs

Where can I place new edges?
Obtaining Maximal Biplane Graphs

Triangulate red and blue subgraphs.

Geometric Biplane Graphs
Obtaining Maximal Biplane Graphs

Complete the blue and red subgraphs to triangulations. A maximal biplane graph on $S$ consists of the edges of two plane triangulations of $S$. 

Geometric Biplane Graphs
Adding edges to a Biplane Graph.

**Step 1:** Complete the blue and red subgraphs to triangulations. A maximal biplane graph on $S$ consists of the edges of two plane triangulations of $S$. 
The purple graph.

**Step 2:** Compute the faces of the purple graph. The edges inside each face of the purple graph can be colored in exactly two ways.
Adding edges to a Biplane Graph.

Step 3: Adding new edges by flipping purple edges. If all the purple edges are not flippable then the graph is maximal.
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Faces of the purple graph.
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Constructing Maximal Biplane Graphs.

Algorithm:

1. Complete the graph $G = (S, E)$ to a blue triangulation $T_b$, a red triangulation $T_r$ and compute the purple edges $E_p = T_b \cap T_r$.

2. Compute the faces of the purple plane graph $E_p$.

3. For each edge $e \in E_p$, check if it is flippable in $T_b$ or $T_r$, and, if adjacent to two different faces of $E_p$, check if it is colour-blind flippable. If flippable insert a flipped counterpart of $e$ as a new edge and change the colours of one of the adjacent faces if necessary.

The complexity is the same as the Union-Find problem ($O(n\alpha(n))$, but $O(n \lg n)$ in practice).
Number of edges of a Maximal Biplane Graph

- Maximal Biplane Graphs on the same set $S$ can have a different number of edges.

  For points in convex position they always have $3n - 6$ edges.

- Maximal biplane graphs have at least $\max(\frac{7n}{2} - h - 5, 3n - 6)$ edges.
- We can build a maximal biplane graph with at least $4n - h - 6$ edges.
- There are maximal biplane graphs with $6n - 20$ edges.
Number of edges of a Maximal Biplane Graph

Given a set of points $S$,

What is the maximum size of a biplane graph on $S$?

How can we build a biplane graph of maximum size?

What is the minimum size of a maximal biplane graph on $S$?

How can we build a maximal biplane graph of minimum size?
Maximum Connectivity of Biplane Graphs

- Maximal Biplane Graphs are 3-connected.

- The maximum connectivity of a biplane graph could be 11.

- There are configurations of points admitting 11-connected biplane graphs.
Maximum Connectivity of Biplane Graphs.

An almost 12-regular biplane graph (Hutchinson[99]).
Maximum Connectivity of Biplane Graphs.

The red grid.
Maximum Connectivity of Biplane Graphs.

The blue grid.

The blue grid.
Maximum Connectivity of Biplane Graphs.
Maximum Connectivity of Biplane Graphs.

5-connected graph
Points in convex position.

All the planar 5-connected graphs can be drawn as biplane graphs for points in convex position.
Maximum Connectivity of Biplane Graphs.

This extended grid is 10-connected.
Maximum Connectivity of Biplane Graphs.

Flipping edges to increase the degree of some vertices.

The modified grid is 11-connected.
Maximum Connectivity of Biplane Graphs.

This extended and modified grid is 11-connected.
How to build 5-connected biplane graphs: Points in convex position.

All the planar 5-connected graphs can be drawn as biplane graphs for points in convex position.
How to build 5-connected biplane graphs: Points in general position.
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How to build 5-connected biplane graphs: Points in general position.
How to build 5-connected biplane graphs: Inserting interior points.
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How to build 5-connected biplane graphs: Inserting interior points.
How to build 5-connected biplane graphs: Inserting exterior points.
How to build 5-connected biplane graphs: 3 main steps.
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Alternative: If $|S| \geq 27$ either $S$ contains 14 points in convex position or contains a point $p$ such that $S - \{p\}$ admits a 4-connected plane graph.
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Beyond 5-connectivity

The biplane graph $K_8$

- 6-connected biplane graphs.
- Augmentation problems.
Augmentation problems
Augmentation problems
Augmentation problems

- There are 4-connected plane graphs that cannot be transformed in 5-connected biplane graphs by adding edges.
That is all.

Thanks for your kind attention!!