Geometric Biplane Graphs

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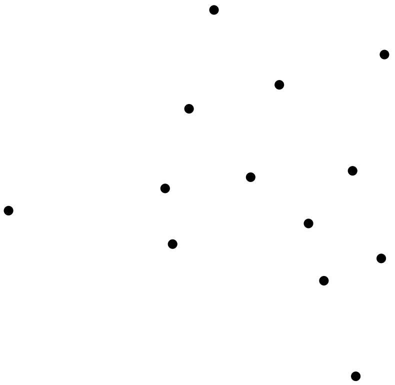
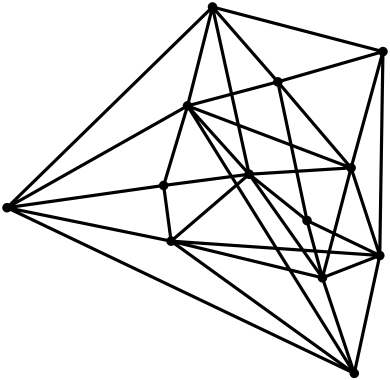
What are biplane graphs?



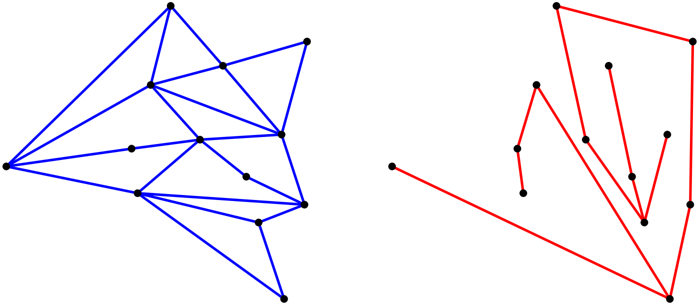
A set S of points on the plane.

Geometric graphs.



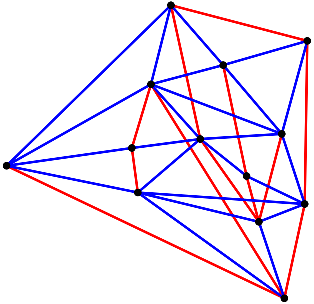
A *geometric graph* on S: Vertices are points of S and edges are straight line segments between pairs of points.

Plane graphs.

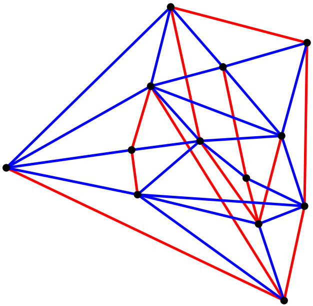


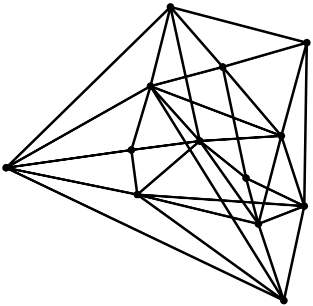
A *plane graph* on S: A geometric graph in which no two edges cross.

Biplane graphs.



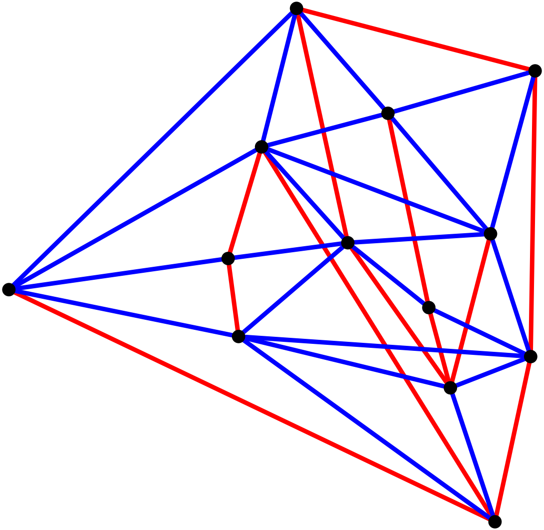
Biplane graphs.





A *biplane graph* on S: A geometric graph $G = (V, E)$ admitting a partition of its edges $E = E_b \cup E_r$ such that $G_b = (V, E_b)$ and $G_r = (V, E_r)$ are each plane graphs.

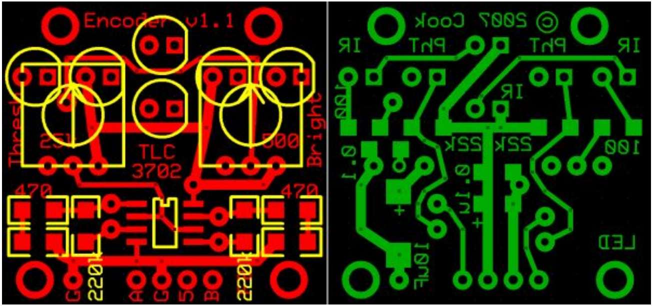
Why are we interested in biplane graphs?



Biplane graphs are a natural extension of plane graphs.

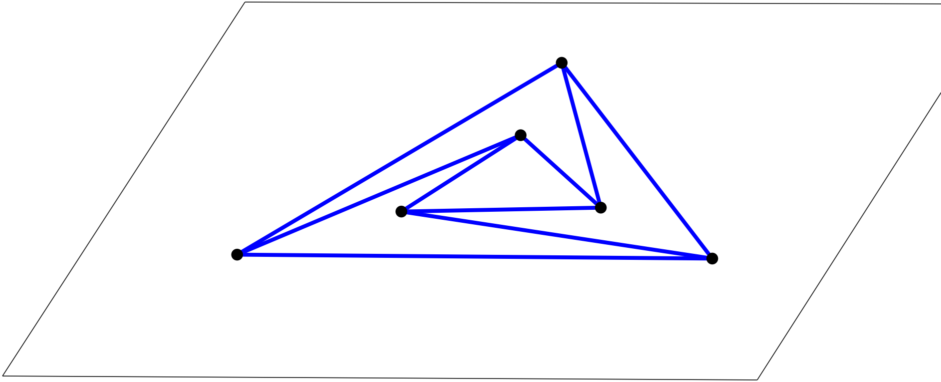
Why are we interested in biplane graphs?

Printed circuit boards:
We can use the two sides of a board to connect components.



Why are we interested in biplane graphs?

We can use the two sides of a plane to connect the points.



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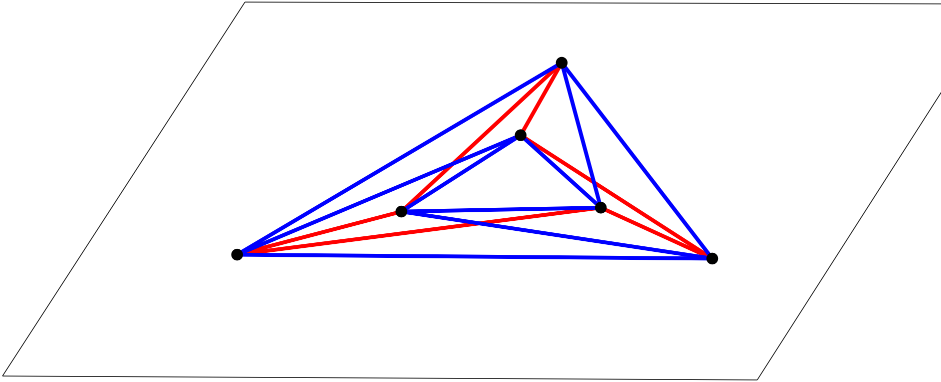
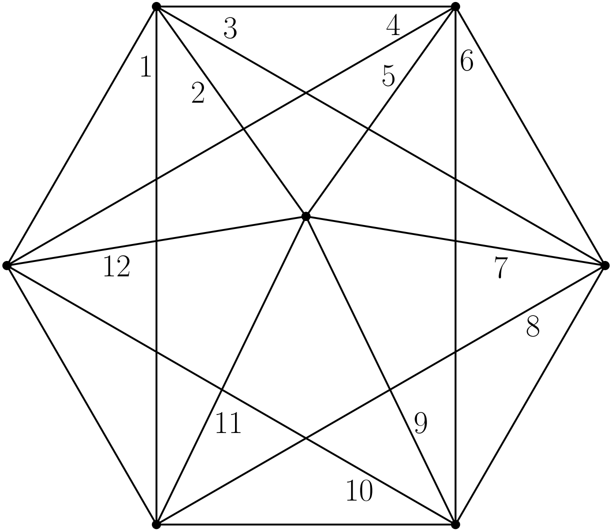


Given a geometric graph G, is it biplane?



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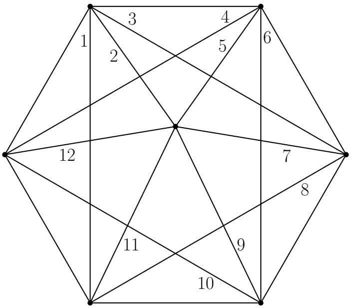

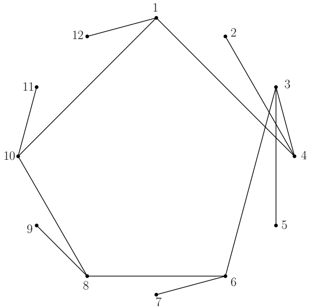

G is biplane, if and only if, its intersection graph G_X is bipartite.

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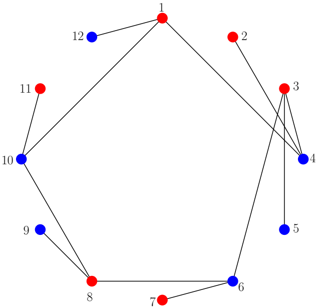
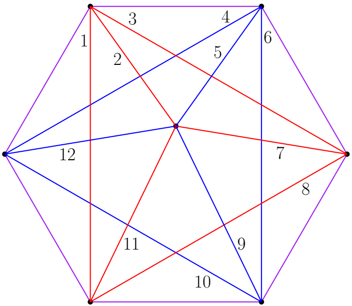
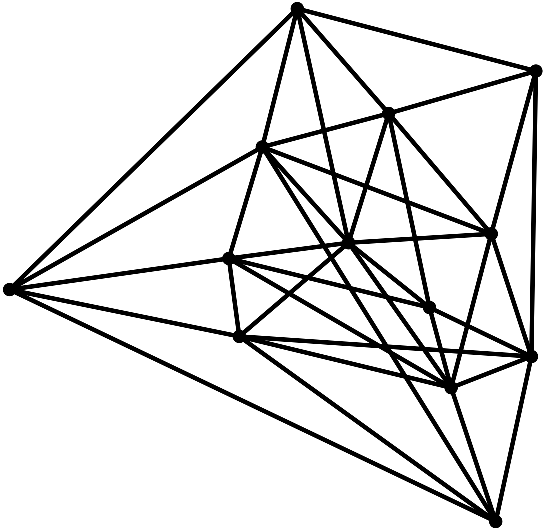




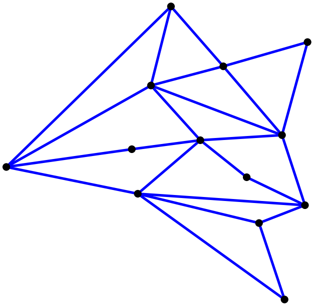
G is biplane, if and only if, its intersection graph G_X is bipartite.

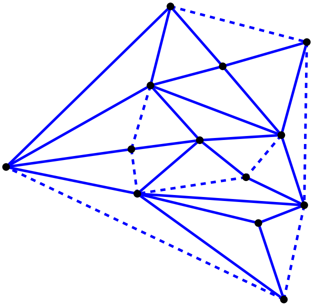
Given a geometric graph G, is it k-plane?



- NP-Complete for $k \geq 3$. (Eppstein,2009).
- Algorithm $O(n \lg(n))$ if $k = 2$. (Eppstein,2009).

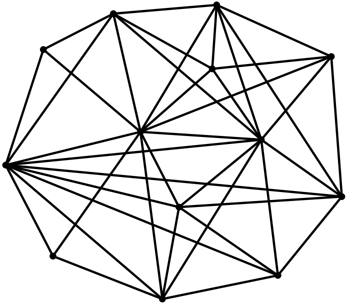
Maximal Plane Graphs

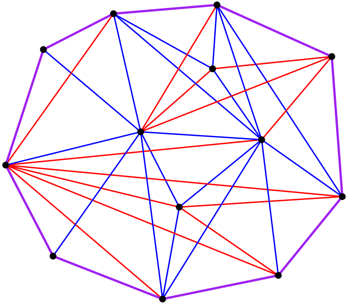




Where can I place a new edge?
Is this plane graph maximal (a triangulation)?

Maximal Biplane Graphs



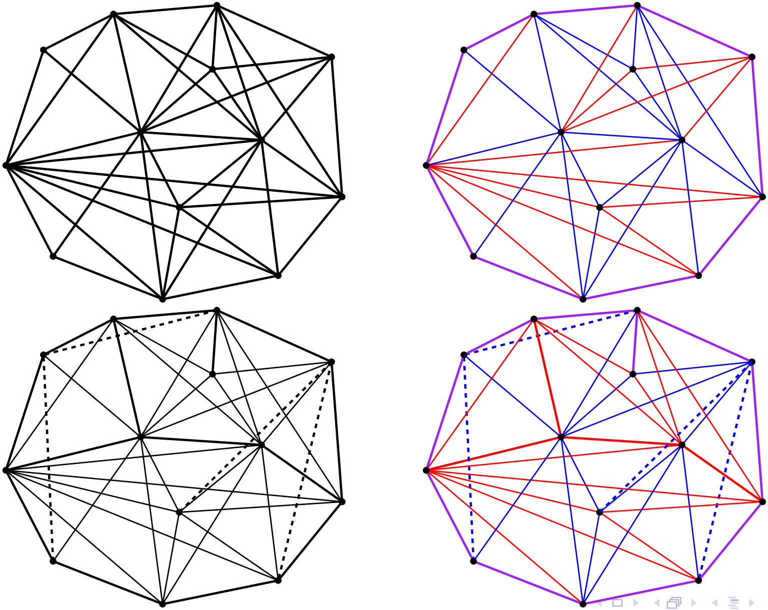


Where can I place a new edge?

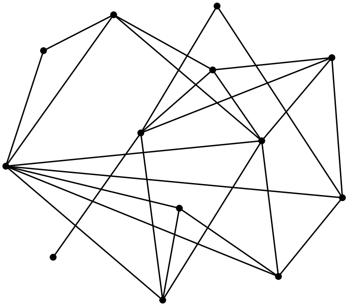
Is this biplane graph maximal?

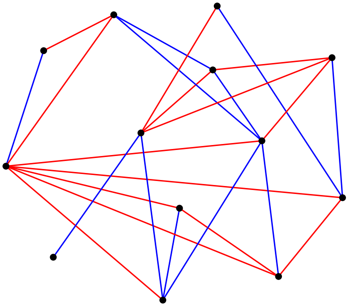
A maximal biplane graph is a natural extension of a triangulation.

Maximal Biplane Graphs



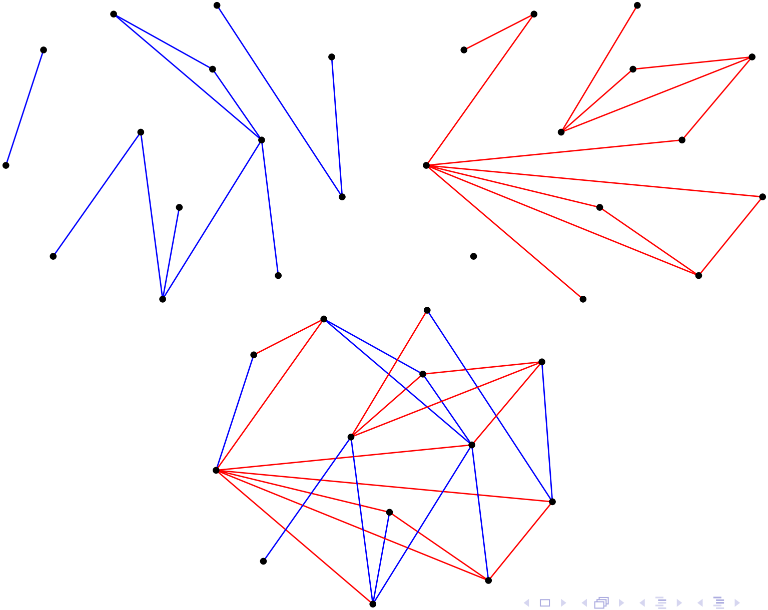
Obtaining Maximal Biplane Graphs



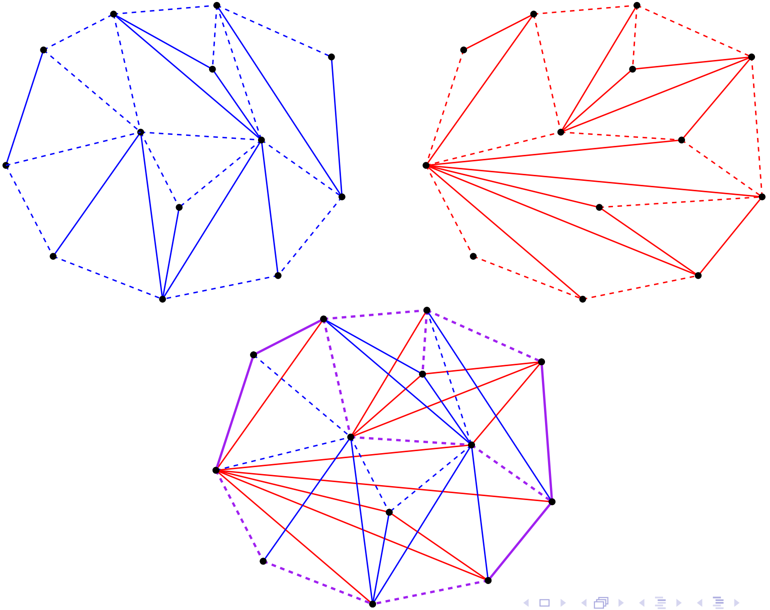


Where can I place new edges?

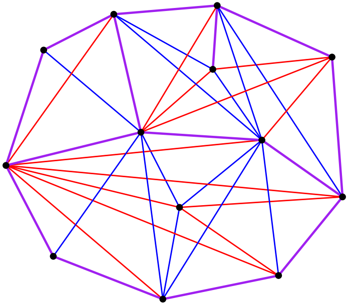
Obtaining Maximal Biplane Graphs



Obtaining Maximal Biplane Graphs

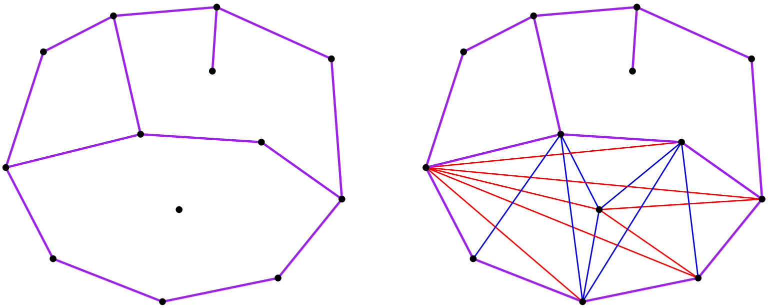


Adding edges to a Biplane Graph.



Step 1: Complete the blue and red subgraphs to triangulations.
A maximal biplane graph on S consists of the edges of two plane triangulations of S.

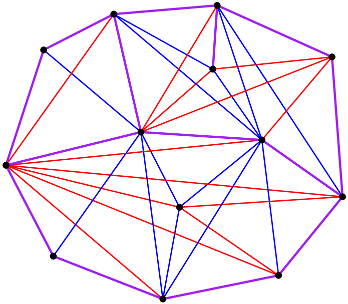
The purple graph.



Step 2: Compute the faces of the purple graph.

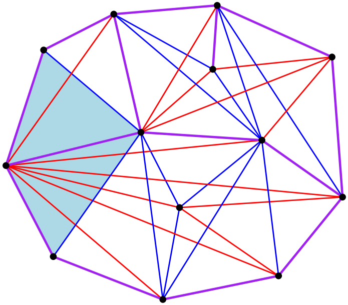
The edges inside each face of the purple graph can be colored in exactly two ways.

Adding edges to a Biplane Graph.



Step 3: Adding new edges by flipping purple edges.
If all the purple edges are not flippable then the graph is maximal.

Adding edges to a Biplane Graph.



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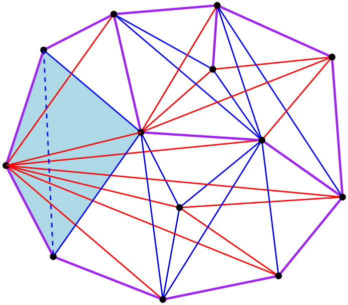

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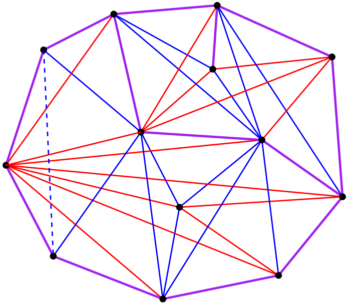

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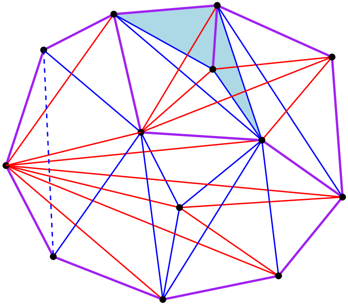

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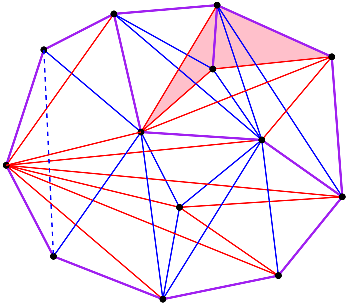

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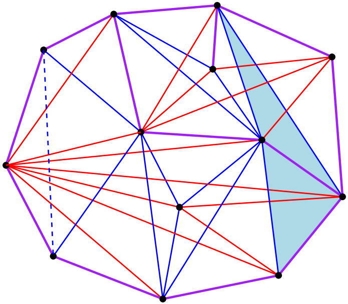

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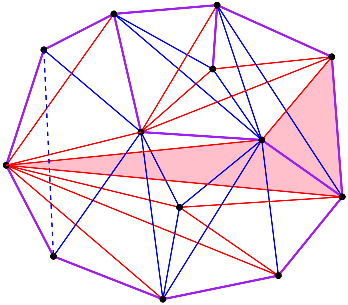

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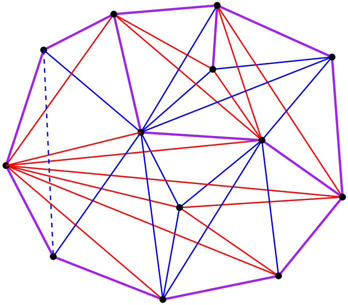

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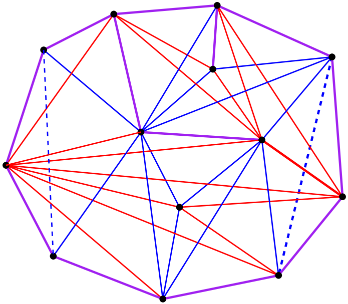

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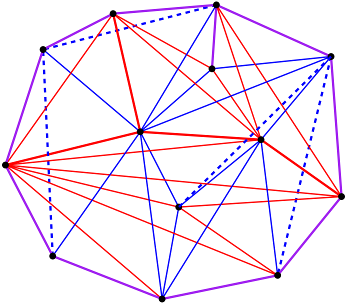

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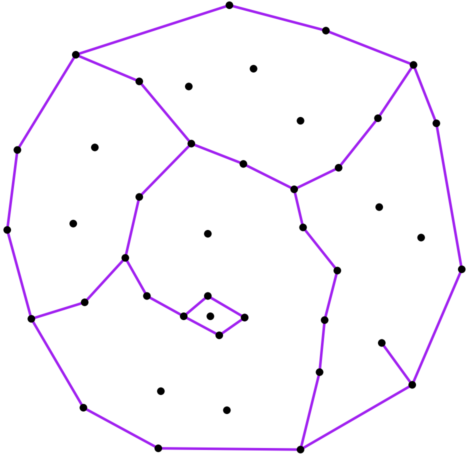
Faces of the purple graph.



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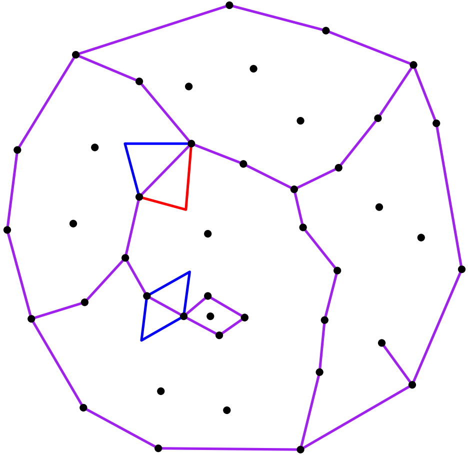


Constructing Maximal Biplane Graphs.

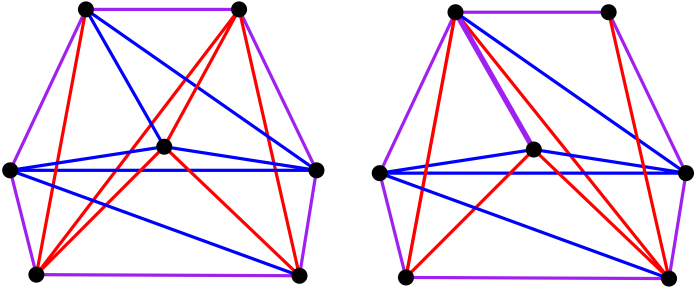
Algorithm:

1. Complete the graph $G = (S, E)$ to a blue triangulation T_b, a red triangulation T_r and compute the purple edges $E_p = T_b \cap T_r$.
2. Compute the faces of the purple plane graph E_p.
3. For each edge $e \in E_p$, check if it is flippable in T_b or T_r, and, if adjacent to two different faces of E_p, check if it is colour-blind flippable.
   If flippable insert a flipped counterpart of e as a new edge and change the colours of one of the adjacent faces if necessary.

The complexity is the same as the Union-Find problem ($O(n\alpha(n))$, but $O(n \lg n)$ in practice).

Number of edges of a Maximal Biplane Graph

- Maximal Biplane Graphs on the same set S can have a different number of edges.



- For points in convex position they always have $3n - 6$ edges.
- Maximal biplane graphs have at least $\max(\frac{7n}{2} - h - 5, 3n - 6)$ edges.
- We can build a maximal biplane graph with at least $4n - h - 6$ edges.
- There are maximal biplane graphs with $6n - 20$ edges.

Number of edges of a Maximal Biplane Graph

Given a set of points S,

What is the maximum size of a biplane graph on S?

How can we build a biplane graph of maximum size?

What is the minimum size of a maximal biplane graph on S?

How can we build a maximal biplane graph of minimum size?

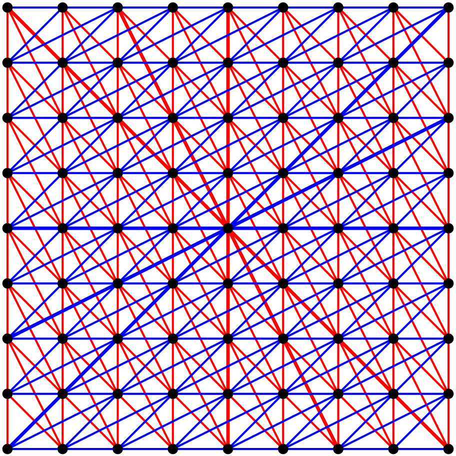
Maximum Connectivity of Biplane Graphs

- Maximal Biplane Graphs are 3-connected.

- The maximum connectivity of a biplane graph could be 11.

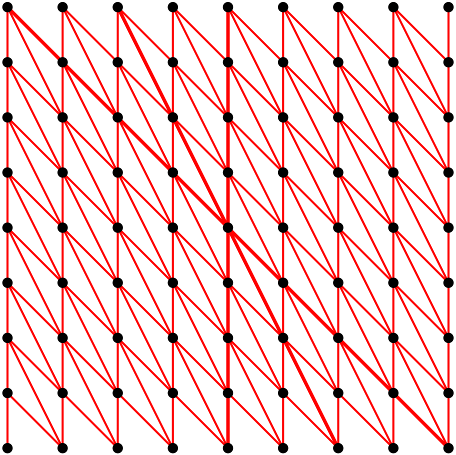
- There are configurations of points admitting 11-connected biplane graphs.

Maximum Connectivity of Biplane Graphs.



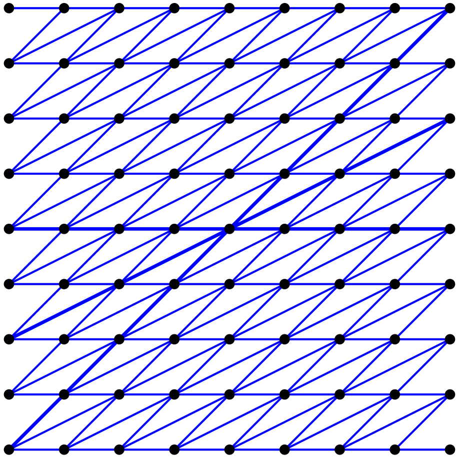
An almost 12-regular biplane graph (Hutchinson[99]).

Maximum Connectivity of Biplane Graphs.



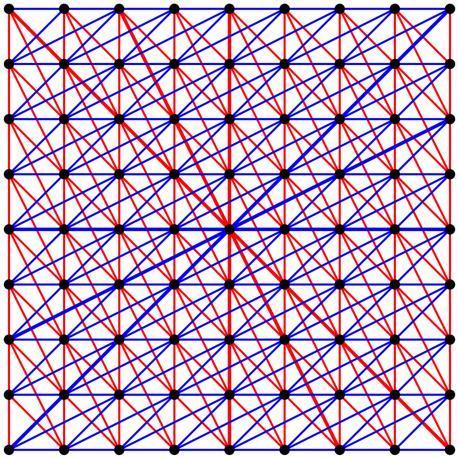
The red grid.

Maximum Connectivity of Biplane Graphs.

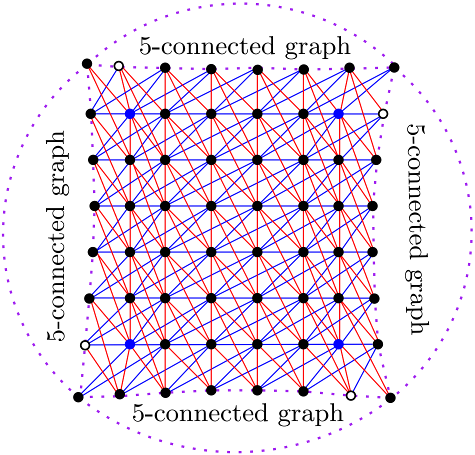


The blue grid.

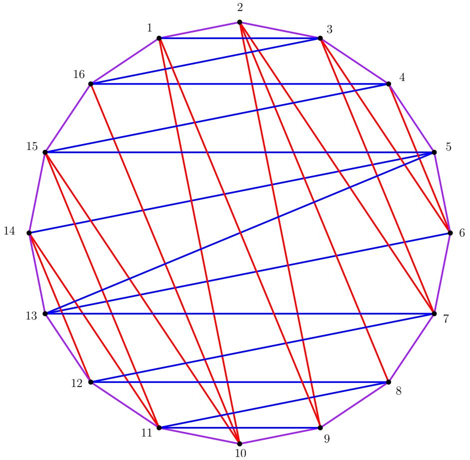
Maximum Connectivity of Biplane Graphs.



Maximum Connectivity of Biplane Graphs.

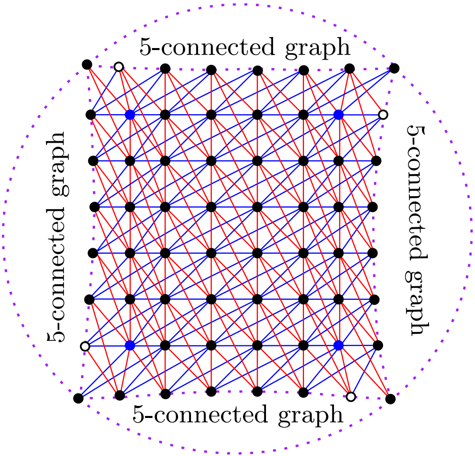


Points in convex position.



All the planar 5-connected graphs can be drawn as biplane graphs for points in convex position.

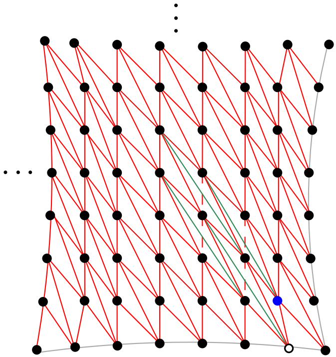
Maximum Connectivity of Biplane Graphs.



This extended grid is 10-connected.

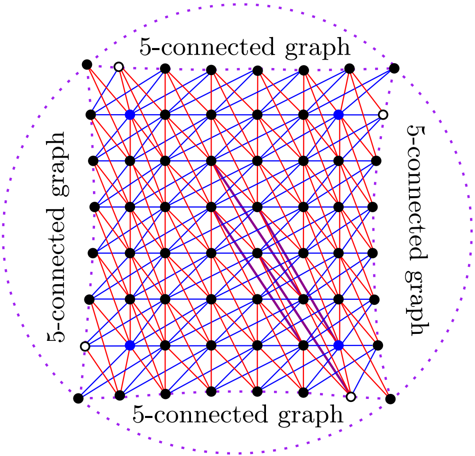
Maximum Connectivity of Biplane Graphs.

Flipping edges to increase the degree of some vertices.



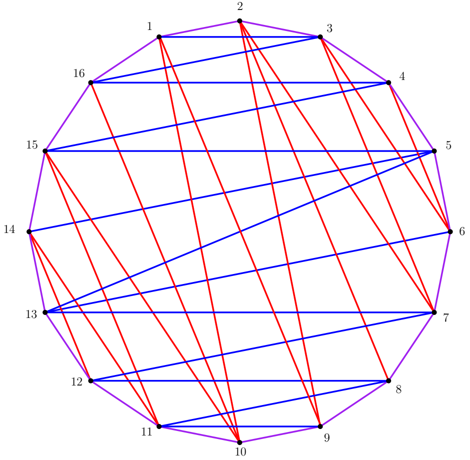
The modified grid is 11-connected.

Maximum Connectivity of Biplane Graphs.



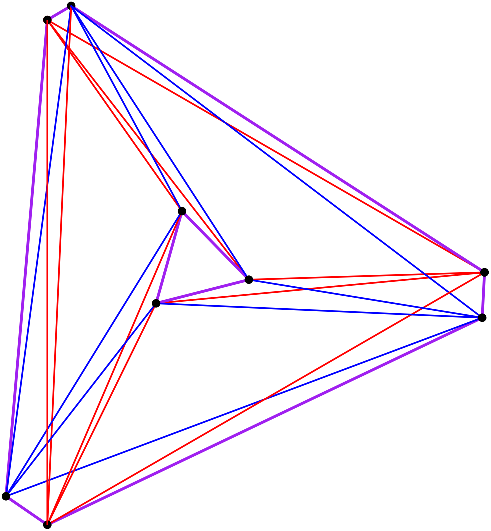
This extended and modified grid is 11-connected.

How to build 5-connected biplane graphs: Points in convex position.



All the planar 5-connected graphs can be drawn as biplane graphs for points in convex position.

How to build 5-connected biplane graphs: Points in general position.



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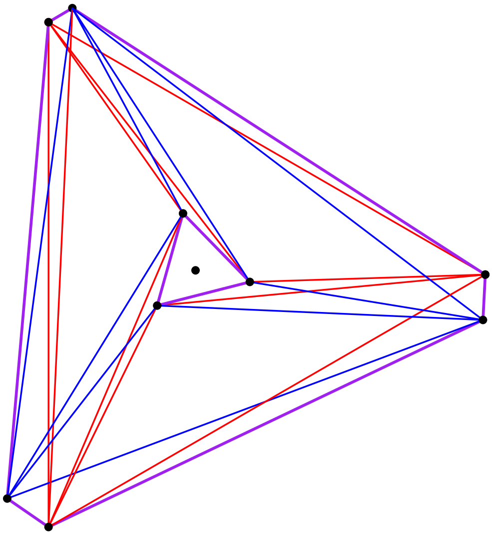

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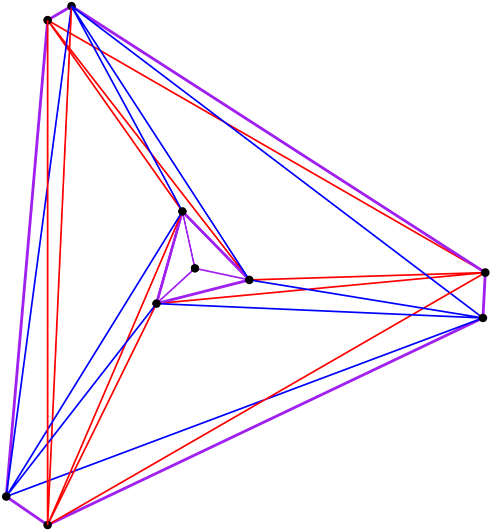

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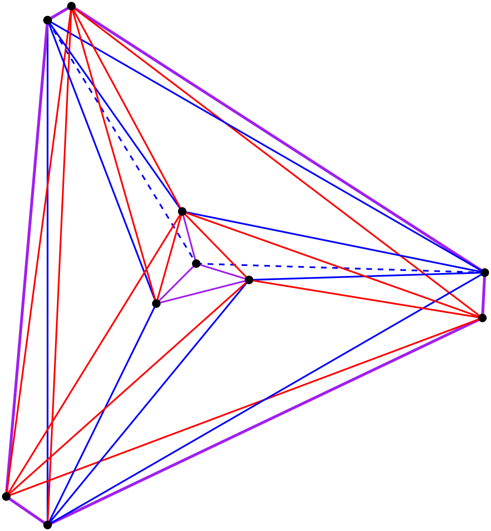

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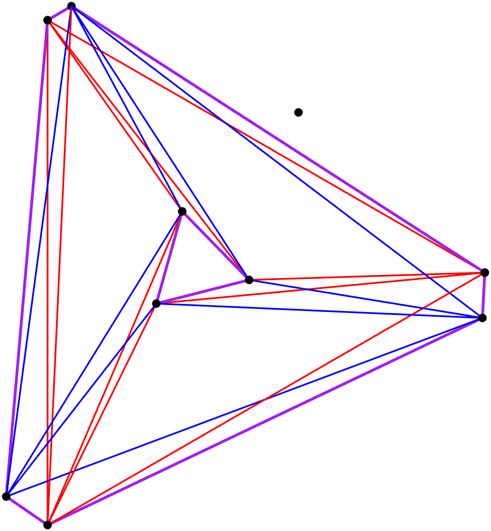

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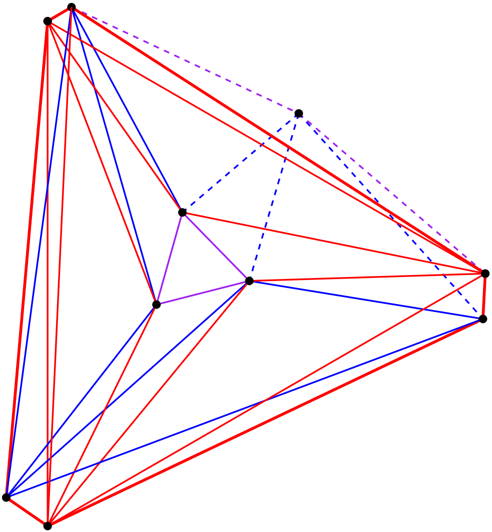
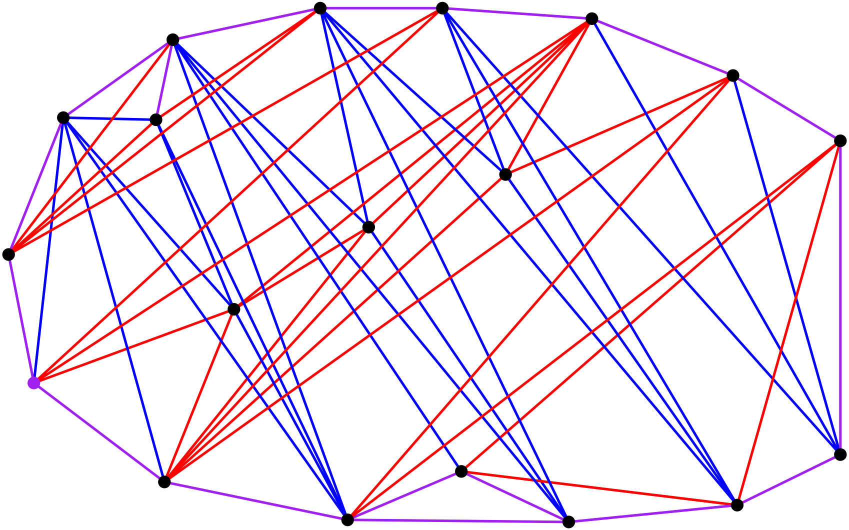
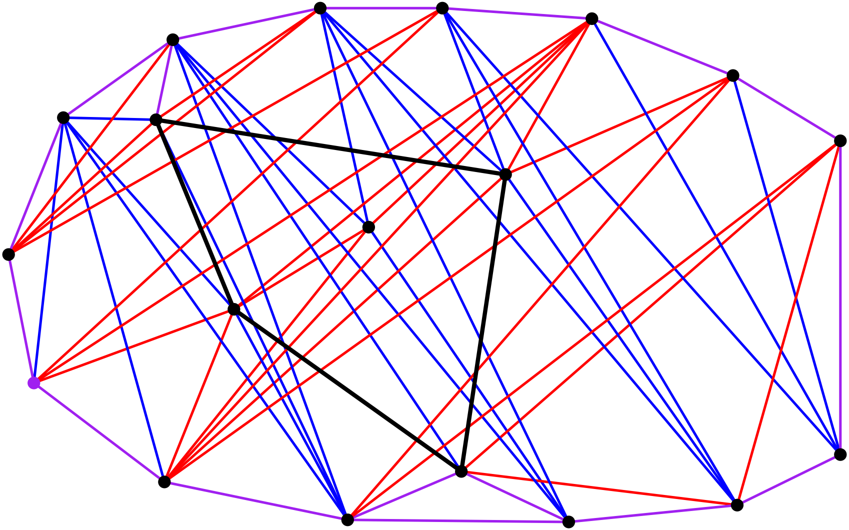


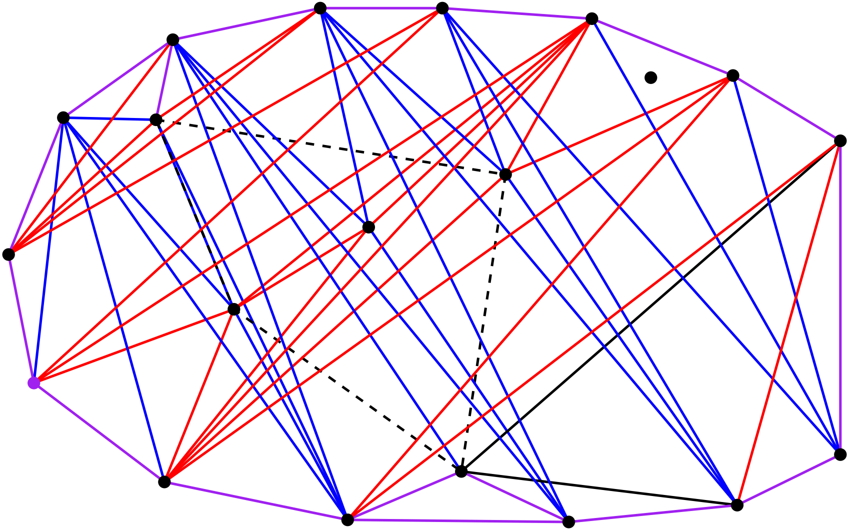
How to build 5-connected biplane graphs: Inserting interior points.



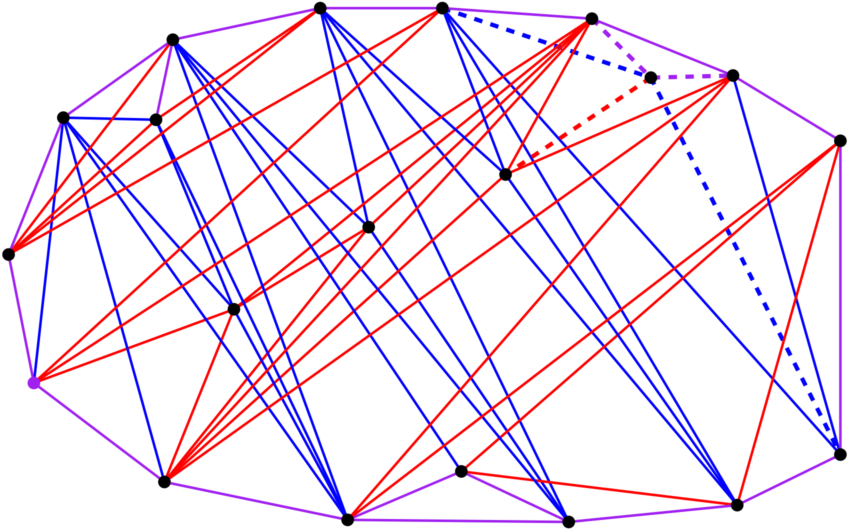
How to build 5-connected biplane graphs: Inserting interior points.



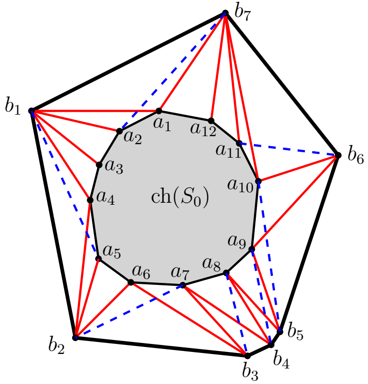
How to build 5-connected biplane graphs: Inserting interior points.

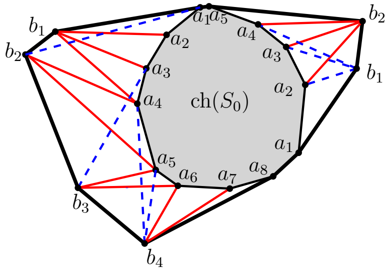


How to build 5-connected biplane graphs: Inserting interior points.

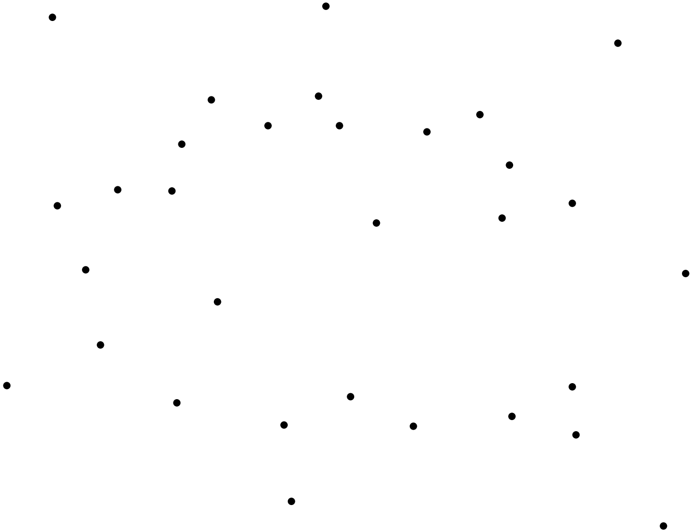


How to build 5-connected biplane graphs: Inserting exterior points.





How to build 5-connected biplane graphs: 3 main steps.



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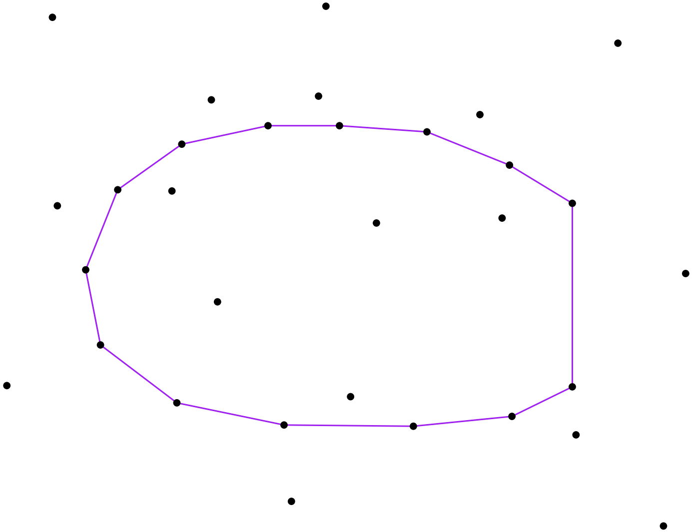

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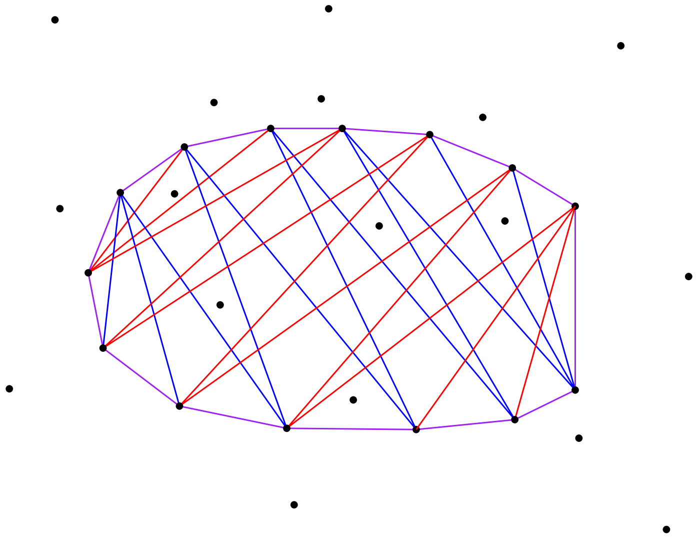

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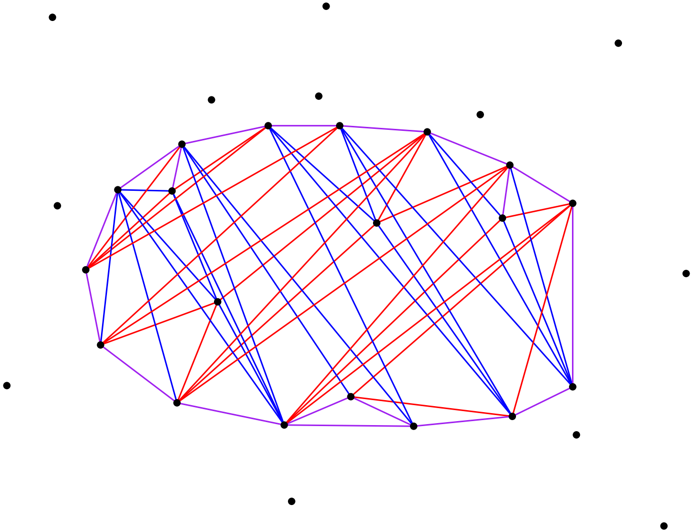

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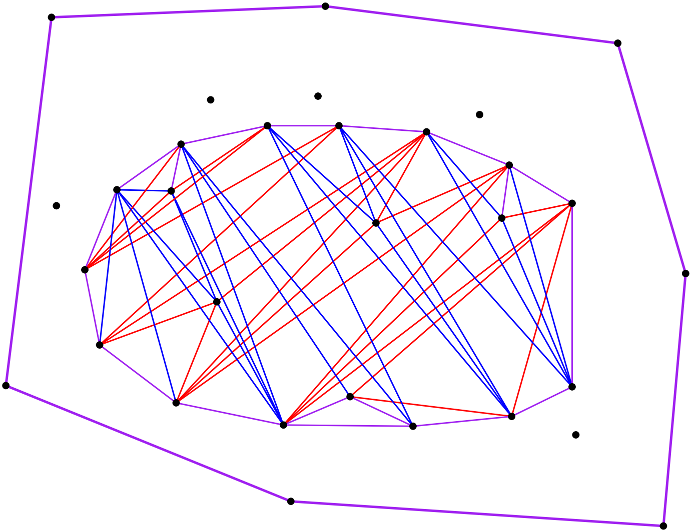

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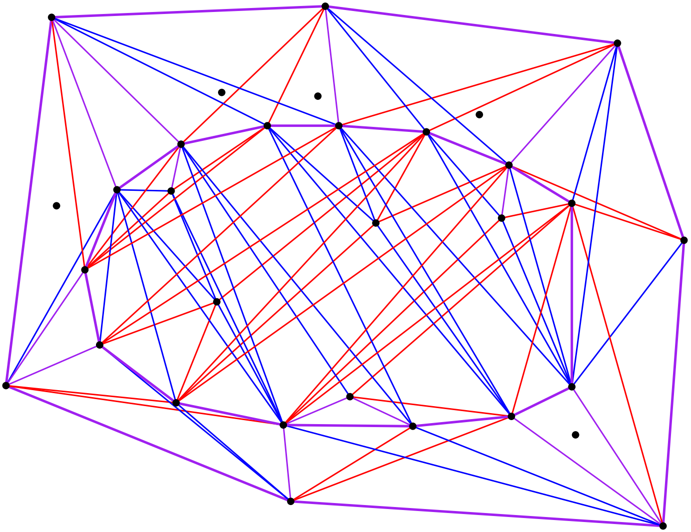

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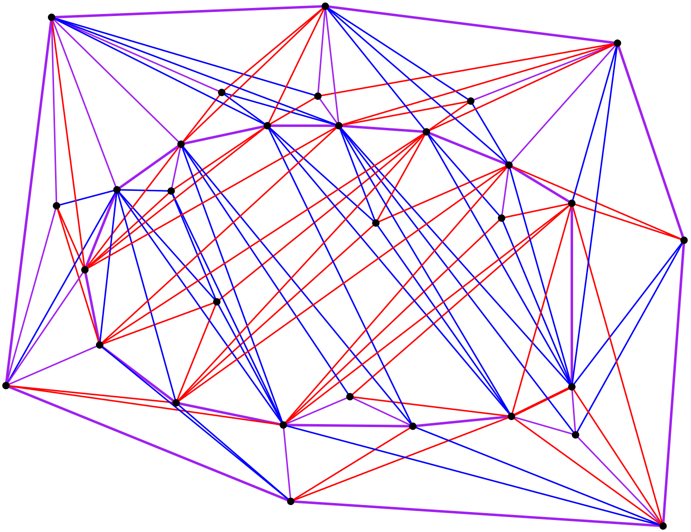
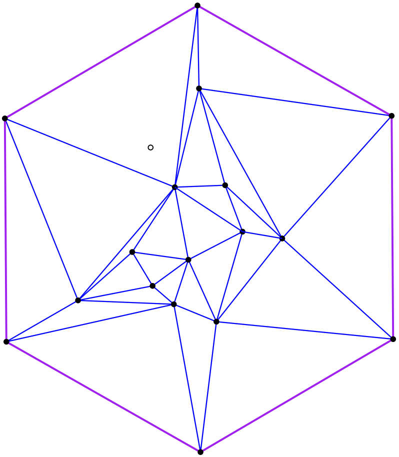


Alternative: If $|S| \geq 27$ either S contains 14 points in convex position or contains a point p such that $S - \{p\}$ admits a 4-connected plane graph.



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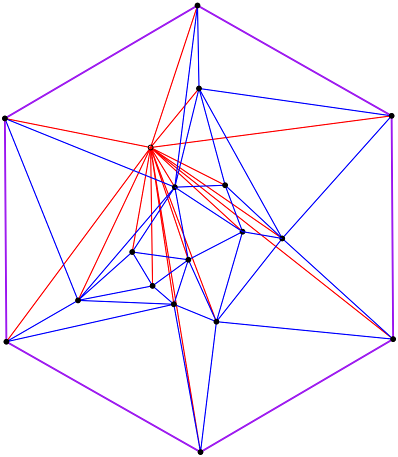
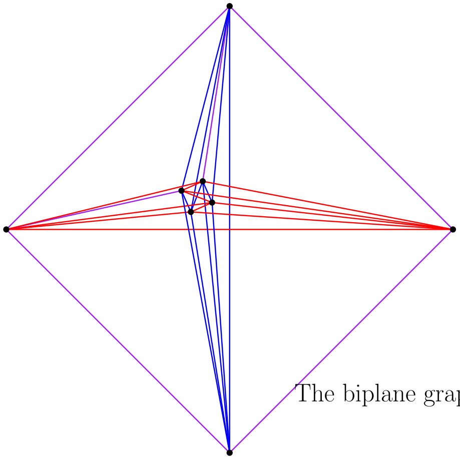


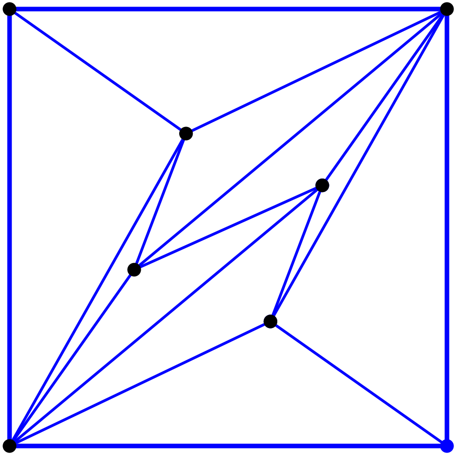
Beyond 5-connectivity



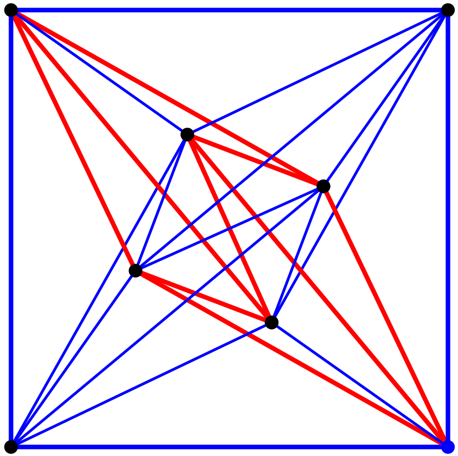
The biplane graph K_8

- 6-connected biplane graphs.
- Augmentation problems.

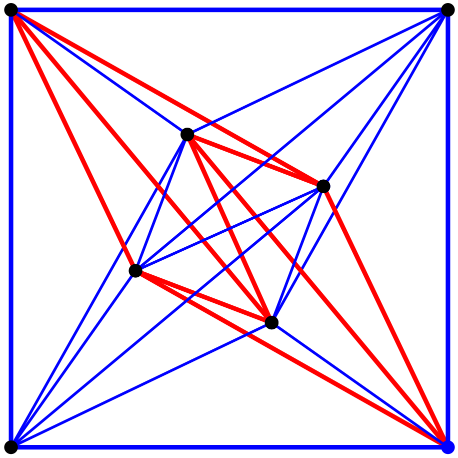
Augmentation problems



Augmentation problems



Augmentation problems



- There are 4-connected plane graphs that cannot be transformed in 5-connected biplane graphs by adding edges.

That is all.

Thanks for your kind attention!!